Math 33A, Section 3, Fall 2009

Quiz #1

1. *(1 point each)*
Which of the following matrices are in reduced row-echelon form (RREF):

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$ YES NO

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ YES NO

$$\begin{bmatrix} 0 & 1 & 0 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ YES NO

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ YES NO

2. *(1 point each)*
Which of the following matrix products are defined (i.e. make sense):

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$ YES NO

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$ YES NO

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$ YES NO

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$ YES NO

(continued on the other side)

3. *(2 points each)*
Determine the number of solutions for each of the following linear systems. Some systems are given by the augmented coefficient matrix, and some systems are given in the matrix form $A\vec{x} = \vec{b}$.

0	=	there are no solutions, i.e. the system is inconsistent
1	=	there is exactly one solution (i.e. a unique solution)
∞	=	there are infinitely many solutions

$$\begin{cases} x_1 + + 2x_3 = 4 \\ x_2 + 3x_3 = 5 \end{cases}$$

0 1 ∞

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 4 \\ x_2 + 2x_3 = 5 \\ x_3 = 6 \end{cases}$$

0 1 ∞

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

0 1 ∞

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & 7 \\ 0 & 0 & 1 & 3 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & 9 \end{array}\right]$$

0 1 ∞

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

0 1 ∞

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

0 1 ∞

♣ V. K.

Math 33A, Section 3, Fall 2009

Quiz #1 solutions

1. *(1 point each)*
Which of the following matrices are in reduced row-echelon form (RREF):

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$
YES $\boxed{\text{NO}}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
YES $\boxed{\text{NO}}$

$$\begin{bmatrix} 0 & 1 & 0 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$\boxed{\text{YES}}$ NO

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$\boxed{\text{YES}}$ NO

2. *(1 point each)*
Which of the following matrix products are defined (i.e. make sense):

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$
$\boxed{\text{YES}}$ NO

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$
$\boxed{\text{YES}}$ NO

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$
$\boxed{\text{YES}}$ NO

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$
YES $\boxed{\text{NO}}$

(continued on the other side)

3. *(2 points each)*
Determine the number of solutions for each of the following linear systems. Some systems are given by the augmented coefficient matrix, and some systems are given in the matrix form $A\vec{x} = \vec{b}$.

0	=	there are no solutions, i.e. the system is inconsistent
1	=	there is exactly one solution (i.e. a unique solution)
∞	=	there are infinitely many solutions

$$\begin{cases} x_1 + \quad\quad + 2x_3 = 4 \\ \quad\quad x_2 + 3x_3 = 5 \end{cases}$$

0 1 $\boxed{\infty}$

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 4 \\ \quad\quad x_2 + 2x_3 = 5 \\ \quad\quad\quad\quad x_3 = 6 \end{cases}$$

0 $\boxed{1}$ ∞

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

0 $\boxed{1}$ ∞

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & 7 \\ 0 & 0 & 1 & 3 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & 9 \end{array}\right]$$

0 1 $\boxed{\infty}$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$$

$\boxed{0}$ 1 ∞

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

0 $\boxed{1}$ ∞

♣ V. K.

Easy problems:

1. Solve the following linear system:

$$\begin{cases} x_1 + 2x_2 - x_3 + 2x_4 = 3 \\ 3x_1 + 6x_2 - x_3 = 5 \end{cases}$$

2. For $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}$ compute AB and BA.

3. Find the matrix of the rotation about x-axis through an angle of $90°$ counterclockwise as viewed from the positive x-axis in the space.

Medium problems:

1. Depending on the real parameter k determine the number of solutions of the linear system

$$\begin{cases} x + y + z = 0 \\ x + 2y + kz = 1 \\ x + 4y + k^2 z = 2 \end{cases}$$

2. Find all 2×2 matrices A that commute with the given matrix $B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, i.e. $AB = BA$.

3. Suppose that the matrix product AB is defined and that A has a row consisting only of zeros. Does it necessarily imply that AB also has a row of zeros only?

4. Let T be a linear transformation on the plane $\mathbb{R}^2$ obtained as a composition of the reflection about the x-axis followed by the reflection about the line $y = x$. Find the matrix of T and then interpret T geometrically as a rotation by some angle. Find that angle.

5. Find the 2×2 matrix of the composition of the orthogonal projection to the line $y = 3x$ followed by the orthogonal projection to the line $y = -\frac{1}{3}x$. You can reason either algebraically or geometrically.

6. Find a matrix A such that

$$A \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \quad A \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad A \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

7. For which real numbers a does the vector $\begin{bmatrix} 1 \\ a \\ a^2 \end{bmatrix}$ belong to the linear span of vectors $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$?

8. Let W be the kernel of the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ x_2 - 3x_3 \end{bmatrix}.$$

Find a vector that spans W. Then give an example of a linear transformation $S\colon \mathbb{R}^2 \to \mathbb{R}^3$ whose image is W.

Hard problems:

1. For a square matrix M we denote $M^n = \underbrace{MM\dots M}_{n}$. If A, B are $n \times n$ matrices such that $(AB)^2 = \mathbf{0}$, show that $(BA)^3 = \mathbf{0}$.

2. Find all matrices X such that

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 0 \end{bmatrix} X = \begin{bmatrix} 6 & 2 \\ 3 & 0 \\ 0 & 4 \end{bmatrix}.$$

3. Find all 2×2 matrices A such that the corresponding linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ has the following property: vectors $A\overrightarrow{v}$ and $\overrightarrow{v}$ are perpendicular (i.e. orthogonal) for all $\overrightarrow{v}$ in $\mathbb{R}^2$.

4. Suppose that A, B, C are invertible $n \times n$ matrices. Express the unknown $n \times n$ matrix X from the following equation:

$$(BA^{-1})^{-1} X B C^{-1} = AC(CB^{-1})^{-1}.$$

5. Suppose that A is an invertible 2×2 matrix such that both rows of A add up to 5. Show that both rows of A^{-1} add up to $\frac{1}{5}$.

Mathematics 33A, First Midterm, October 16, 2009.

Calculators, books, or notes of any kind are not allowed on this exam. Do not use any paper other than that provided. (You may write on the back if you need more space, but indicate this clearly on the front.)

There are 11 items on this exam altogether, and they all have equal value. Answer as many of them as you can. The questions are not always arranged in order of difficulty. Look through them when you start so you get an idea of the time you'll need. If you're not sure what to do on an item then move onward and return to it later. (Some of the questions can be solved with geometric reasoning instead of lengthy computations; this may save you some time.) **Good luck.**

Family Name: ______________________

Given names: ______________________

Discussion section: ______________________

UID#: ______________________

Email address: ______________________

Signature: ______________________

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Total: ______________________

Question 1. Solve the system of equations $\left\{ \begin{array}{rcl} x + 2y - z + 2w & = & 3 \\ 3x + 6y - z & = & 5 \end{array} \right.$.

Question 2. Compute the product $\begin{pmatrix} -1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}$.

Question 3. Let $A = \begin{pmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{pmatrix}$. Find all matrices $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ so that $B \cdot A = A \cdot B$. Interpret your answer geometrically.

Question 4. Find the inverse of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Question 5. Let A be the matrix of orthogonal projection to the line $y = 3x$ in $\mathbb{R}^2$, and let B be the matrix of orthogonal projection to the line $y = -\frac{1}{3}x$. (The two lines are perpendicular.) Find $B \cdot A$.

Question 6. Let A be the matrix of reflection about the line $y = 3x$ in $\mathbb{R}^2$, and let B be the matrix of reflection about the line $y = -\frac{1}{3}x$. Find $B \cdot A$.

Question 7. For each of the following, give an example if there is one, and otherwise write "none".

(a) A 2×2 matrix A so that kernel(A) = image(A).

(b) A linear transformation T whose kernel(T) is the span of the vector $\begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$.

Question 8 Use Gauss-Jordan elimination (row operations) to determine the number of solutions of the following system. For which k is there a unique solution? infinitely many? none?

$$\begin{cases} x + y + z = 1 \\ x + 2y + kz = 2 \\ x + 4y + k^2 z = 3 \end{cases}$$

Question 9. Find a matrix A so that $A \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, $A \cdot \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, and $A \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$.

Question 10. Find the matrix of orthogonal projection to the line $y = 5x$ in $\mathbb{R}^2$.

Math 33A, Midterm 1 solutions

1. We first write the augmented coefficient matrix and then perform Gauss-Jordan eliminations (row operations):

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 2 & 3 \\ 3 & 6 & -1 & 0 & 5 \end{array}\right)$$

subtract 3 times row I from row II

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 2 & 3 \\ 0 & 0 & 2 & -6 & -4 \end{array}\right)$$

divide row II by 2

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 2 & 3 \\ 0 & 0 & 1 & -3 & -2 \end{array}\right)$$

add row II to row I

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 1 \\ 0 & 0 & 1 & -3 & -2 \end{array}\right)$$

From the RREF we see that variables y and w are going to be arbitrary parameters, while x and z are going to be expressed in terms of these parameters. We successively write:

$$w = s, \quad z = 3s - 2, \quad y = t, \quad x = -2t + s + 1,$$

for arbitrary real parameters s and t. We can also write the solution in the form

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2t + s + 1 \\ t \\ 3s - 2 \\ s \end{pmatrix}.$$

2. The result is:

$$\begin{pmatrix} 0 & -1 & -2 \\ 1 & 2 & 3 \\ 2 & 5 & 8 \end{pmatrix}$$

3. First observe that

$$A = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$$

although this is not crucial and we could have left A in the trigonometric form. Now we compute both products:

$$BA = \frac{1}{2}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} a + b\sqrt{3} & -a\sqrt{3} + b \\ c + d\sqrt{3} & -c\sqrt{3} + d \end{pmatrix}$$

$$AB = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{2}\begin{pmatrix} a - c\sqrt{3} & b - d\sqrt{3} \\ a\sqrt{3} + c & b\sqrt{3} + d \end{pmatrix}$$

Comparing corresponding entries in the first column, we obtain $a+b\sqrt{3} = a - c\sqrt{3}$ and $c + d\sqrt{3} = a\sqrt{3} + c$, which gives $b = -c$ and $d = a$. In that case entries in the second column are automatically equal. We conclude that B has the form

$$B = \begin{pmatrix} a & -c \\ c & a \end{pmatrix}$$

for arbitrary numbers a and c.

As in class we conclude that this matrix represents the composition of a rotation and a dilation. To see this, it is enough to take $r = \sqrt{a^2+c^2}$, and find an angle θ so that $a = r\cos\theta$, $c = r\sin\theta$. Then we have:

$$B = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

4. We perform the algorithm given in class:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

subtract 3 times row I from row II

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -3 & -3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

multiply row II by -1

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

subtract row II from row I

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & -2 & 1 & 0 \\ 0 & 1 & 3 & 3 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

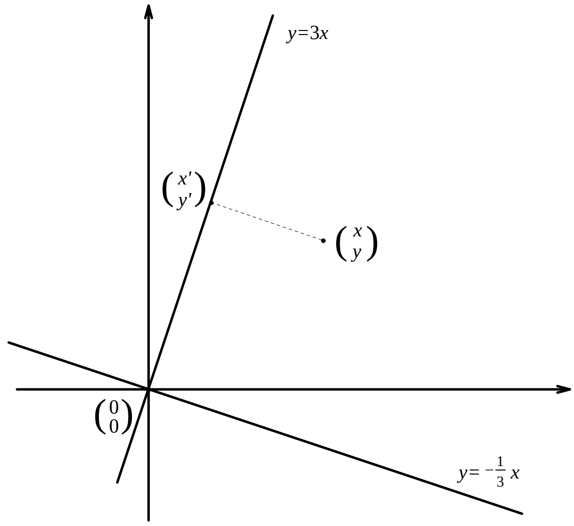
add 2 times row III to row I
subtract 3 times row III from row II

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 2 \\ 0 & 1 & 0 & 3 & -1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Therefore the inverse is:

$$\begin{pmatrix} -2 & 1 & 2 \\ 3 & -1 & -3 \\ 0 & 0 & 1 \end{pmatrix}.$$

5. The transformation A maps an arbitrary point $\begin{pmatrix} x \\ y \end{pmatrix}$ to a point $\begin{pmatrix} x' \\ y' \end{pmatrix}$ on the line $y = 3x$. (We do not need the actual formula for x' and y'.) Since the two lines are perpendicular, B maps every point from the line $y = 3x$ to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and in particular it maps $\begin{pmatrix} x' \\ y' \end{pmatrix}$ to the origin $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.



In short, we can write

$$\begin{pmatrix} x \\ y \end{pmatrix} \overset{A}{\mapsto} \begin{pmatrix} x' \\ y' \end{pmatrix} \overset{B}{\mapsto} \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

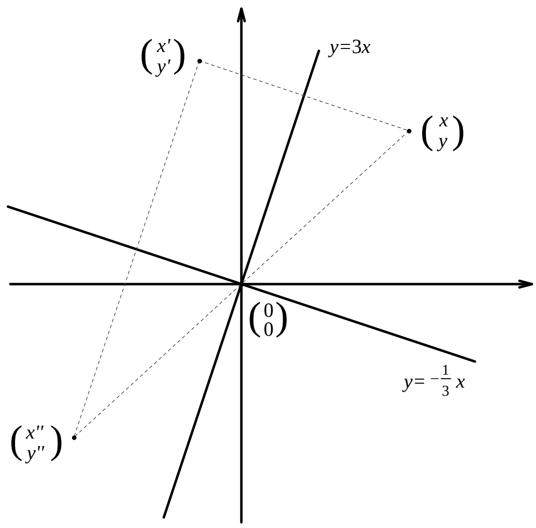
Since the matrix product BA corresponds to the composition of A followed by B, we conclude

$$BA\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and BA must be the zero-matrix $\mathbf{0}$, i.e.

$$BA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

6. The transformation A maps an arbitrary point $\begin{pmatrix} x \\ y \end{pmatrix}$ to some point $\begin{pmatrix} x' \\ y' \end{pmatrix}$, and then B maps it further to some point $\begin{pmatrix} x'' \\ y'' \end{pmatrix}$.



In short, we can write

$$\begin{pmatrix} x \\ y \end{pmatrix} \overset{A}{\mapsto} \begin{pmatrix} x' \\ y' \end{pmatrix} \overset{B}{\mapsto} \begin{pmatrix} x'' \\ y'' \end{pmatrix}.$$

Since the two lines are perpendicular, we see from the picture that these 3 points are vertices of a right-angled triangle and that the origin is at the midpoint of its hypotenuse. Thus

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}.$$

Since the matrix product BA corresponds to the composition of A followed by B, we conclude

$$BA\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix},$$

so BA is the rotation by 180°. Now we can write the matrix:

$$BA = \begin{pmatrix} \cos(180°) & -\sin(180°) \\ \sin(180°) & \cos(180°) \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This can also be seen from

$$BA\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

7. (a) An example of such matrix is $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. To verify the property we first find the kernel by solving the linear system

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Its solution can be read off immediately:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} t \\ 0 \end{pmatrix} = t\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

so

$$\text{kernel}(A) = \text{span}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

On the other hand

$$A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1\begin{pmatrix} 0 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = x_2\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

so also

$$\text{image}(A) = \text{span}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(b) Here we have to find a linear system whose solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5t \\ 2t \\ 3t \end{pmatrix}.$$

From the last row we read off $t = \frac{1}{3}x_3$ so that $x_1 = 5t = \frac{5}{3}x_3$, and $x_2 = 2t = \frac{2}{3}x_3$. This system can be written more nicely as

$$\begin{cases} 3x_1 \qquad\quad - 5x_3 = 0 \\ \qquad 3x_2 - 2x_3 = 0 \end{cases}$$

and corresponds to the matrix (i.e. linear transformation)

$$T = \begin{pmatrix} 3 & 0 & -5 \\ 0 & 3 & -2 \end{pmatrix}.$$

8. We first write the augmented coefficient matrix and then perform Gauss-Jordan eliminations (row operations):

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & k & 2 \\ 1 & 4 & k^2 & 3 \end{array}\right)$$

subtract row I from row II
subtract row I from row III

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & k-1 & 1 \\ 0 & 3 & k^2-1 & 2 \end{array}\right)$$

subtract row II from row I
subtract 3 times row II from row III

$$\left(\begin{array}{ccc|c} 1 & 0 & -k+2 & 0 \\ 0 & 1 & k-1 & 1 \\ 0 & 0 & k^2-3k+2 & -1 \end{array}\right)$$

Let us observe that $k^2 - 3k + 2 = 0$ has the solutions $k = 1$ and $k = 2$.

Case 1. $k \neq 1, 2$
In this case we can divide the third row by $k^2 - 3k + 2$, and then use the obtained 1 to annihilate all other elements in the third column. The first 3 columns of the RREF are thus the identity 3×3 matrix, and so the system has a **unique solution**.

Case 2. $k = 1$ or $k = 2$
For both of these values of k the last row of RREF reads

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & -1 \end{array}\right),$$

which shows that the system is inconsistent, i.e. has **no solutions**.

9. Let $\overrightarrow{v_1}$, $\overrightarrow{v_2}$, $\overrightarrow{v_3}$ be columns of A, i.e. $A = [\overrightarrow{v_1}\ \overrightarrow{v_2}\ \overrightarrow{v_3}]$.

Since $A\begin{pmatrix}0\\0\\1\end{pmatrix}$ is just $\overrightarrow{v_3}$, from the first equation we get $\overrightarrow{v_3} = \begin{pmatrix}2\\1\\0\end{pmatrix}$.

After that, since $A\begin{pmatrix}3\\0\\1\end{pmatrix} = 3\overrightarrow{v_1} + \overrightarrow{v_3}$, we obtain from the second equation

$$\overrightarrow{v_1} = \frac{1}{3}\begin{pmatrix}0\\0\\1\end{pmatrix} - \frac{1}{3}\overrightarrow{v_3} = \frac{1}{3}\begin{pmatrix}0\\0\\1\end{pmatrix} - \frac{1}{3}\begin{pmatrix}2\\1\\0\end{pmatrix} = \begin{pmatrix}-\frac{2}{3}\\-\frac{1}{3}\\\frac{1}{3}\end{pmatrix}.$$

Finally, from $A\begin{pmatrix}2\\1\\0\end{pmatrix} = 2\overrightarrow{v_1} + \overrightarrow{v_2}$, and the third equation we get:

$$\overrightarrow{v_2} = \begin{pmatrix}3\\0\\1\end{pmatrix} - 2\overrightarrow{v_1} = \begin{pmatrix}3\\0\\1\end{pmatrix} - 2\begin{pmatrix}-\frac{2}{3}\\-\frac{1}{3}\\\frac{1}{3}\end{pmatrix} = \begin{pmatrix}\frac{13}{3}\\\frac{2}{3}\\\frac{1}{3}\end{pmatrix}.$$

Therefore

$$A = \begin{pmatrix}-\frac{2}{3} & \frac{13}{3} & 2\\-\frac{1}{3} & \frac{2}{3} & 1\\\frac{1}{3} & \frac{1}{3} & 0\end{pmatrix}.$$

10. The line $y = 5x$ is spanned (determined) for instance by the vector $\overrightarrow{w} = \begin{pmatrix}w_1\\w_2\end{pmatrix} = \begin{pmatrix}1\\5\end{pmatrix}$. The general formula for the matrix of the orthogonal projection onto the line spanned by $\overrightarrow{w}$ is

$$\frac{1}{w_1^2 + w_2^2}\begin{pmatrix}w_1^2 & w_1w_2\\w_1w_2 & w_2^2\end{pmatrix},$$

so in our particular case the matrix becomes

$$\frac{1}{26}\begin{pmatrix}1 & 5\\5 & 25\end{pmatrix} = \begin{pmatrix}\frac{1}{26} & \frac{5}{26}\\\frac{5}{26} & \frac{25}{26}\end{pmatrix}.$$

This can also be derived using the formula for the orthogonal projection:

$$\text{proj}_{\overrightarrow{w}}(\overrightarrow{v}) = \frac{1}{|\overrightarrow{w}|^2}(\overrightarrow{v}\cdot\overrightarrow{w})\overrightarrow{w}.$$

V. K.

Name: ________________________________ UID: ________________

Quiz #2

1. *(1 point each)*
Which of the following matrices or linear transformations are invertible?

YES = invertible
NO = not invertible

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$	**[YES]**	NO
$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ for some real number θ	**[YES]**	NO
$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ when $ad = bc$	YES	**[NO]**
vertical shear in $\mathbb{R}^2$	**[YES]**	NO
$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	**[YES]**	NO
$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$	YES	**[NO]**
orthogonal projection onto the plane $3x - 2y + 5z = 0$ in $\mathbb{R}^3$	YES	**[NO]**
rotation about the y-axis in $\mathbb{R}^3$	**[YES]**	NO
$\begin{bmatrix} 1 & 2 & 4 & 7 \\ 0 & 3 & 5 & 8 \\ 0 & 0 & 6 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix}$	**[YES]**	NO
$\begin{bmatrix} 5 & 1 & 10 & 4 \\ 6 & 2 & 3 & 12 \\ 0 & 0 & 0 & 0 \\ 8 & 7 & 11 & 9 \end{bmatrix}$	YES	**[NO]**

(continued on the other side)

2. *(2 points each)*
Determine if the following vectors are linearly independent.

YES	=	linearly independent
NO	=	not linearly independent (i.e. linearly dependent)

$$\begin{bmatrix} -2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$ YES $\boxed{\text{NO}}$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$ YES $\boxed{\text{NO}}$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 9 \\ 1 \\ 0 \end{bmatrix}$$ $\boxed{\text{YES}}$ NO

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$ $\boxed{\text{YES}}$ NO

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$ YES $\boxed{\text{NO}}$

◇ V. K.

Easy problems:

1. Find an orthonormal basis for the plane $2x_1 - x_2 + x_3 = 0$ in $\mathbb{R}^3$.

2. Determine the parameter k so that the angle between vectors $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ k \end{pmatrix}$ is $60°$.

3. Find the parameter k so that the matrices $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & -2 \\ 1 & 1 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 & 0 \\ 1 & -1 & -4 \\ 2 & 0 & k \end{pmatrix}$ have the same image.

4. Find an orthonormal basis of the kernel of the matrix $A = \begin{pmatrix} -1 & 2 & -3 & 0 \\ 1 & 0 & 2 & -1 \end{pmatrix}$.

Medium problems:

1. Let V be the subspace
$$\text{span}\{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix}\}$$
and consider its orthogonal complement $V^\perp$.
   (a) Find a basis for $V^\perp$.
   (b) Using the Gram-Schmidt process find an orthonormal basis for $V^\perp$ starting with the basis from part (a).

2. Let T be the orthogonal projection onto the subspace $V = \text{span}\{\vec{v_1}, \vec{v_2}\}$, where
$$\vec{v_1} = \begin{pmatrix} 3/5 \\ 4/5 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{v_2} = \begin{pmatrix} -4/5 \\ 3/5 \\ 0 \\ 0 \end{pmatrix}.$$
   (a) Find the matrix of T (in the standard basis for $\mathbb{R}^4$).
   (b) Find vectors $\vec{v_3}, \vec{v_4}$ such that $\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}$ is an orthonormal basis for $\mathbb{R}^4$
   (c) Find the matrix of T in the orthonormal basis from part (b).

3. Find an orthogonal matrix of the form $\begin{pmatrix} 1/2 & 1/3 & a \\ 1/4 & b & 1/4 \\ c & 0 & 0 \end{pmatrix}$.

Hard problems:

1. Find k such that the matrices $A = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ 0 & k \end{pmatrix}$ are similar.

2. Find all non-invertible matrices A of the form $A = \begin{pmatrix} a & b \\ 1 & 2 \end{pmatrix}$ such that the orthogonal complement of $\text{im}(A)$ is precisely $\ker(A)$.

Mathematics 33A, Practice Midterm, November 2, 2009.

Calculators, books, or notes of any kind are not allowed on the exam.

There are 10 items on this practice exam altogether, and they all have equal value. Answer as many of them as you can in **50 minutes**. You must show your work in all questions.

The questions are not always arranged in order of difficulty. Look through them when you start so you get an idea of the time you'll need. If you're not sure what to do on an item then move onward and return to it later.

Good luck.

Question 1. Find the coordinates of $\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ with respect to the basis $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$.

Question 2. Find an orthonormal basis for $span\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}\}$.

Question 3. Find a basis for the plane $x_1 + 2x_2 + 3x_3 = 0$ in $\mathbb{R}^3$.

Question 4. Let A be a 4×4 matrix with columns $\vec{v}_1, \ldots, \vec{v}_4$. We are told that $\vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3 + 4\vec{v}_4 = 0$. What are the possible values of $rank(A)$? Explain why.

Question 5. Working in $\mathbb{R}^5$ let W be the subspace of all $\vec{x}$ so that $x_1 + x_2 + x_3 + x_4 + x_5 = 0$ and $2x_1 + x_2 + 2x_3 + x_4 + 2x_5 = 0$. Find the dimension of W.

Question 6. Give an example of a basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ of $\mathbb{R}^2$ so that $\left[\begin{pmatrix} 2 \\ 3 \end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Question 7. Let $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and let $\vec{v}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$. Let L be the line through the origin in the direction of $\vec{v}_1$. Let T be the orthogonal projection to L.

(a) Find $T(\vec{v}_1)$ and $T(\vec{v}_2)$. (You can reason geometrically.) Draw $\vec{v}_1$, $\vec{v}_2$, L, $T(\vec{v}_1)$, and $T(\vec{v}_2)$.

(b) Find the matrix of T with respect to the basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$.

Question 8. Let A be an invertible 2×2 matrix and let $B = \begin{pmatrix} 3 & 2 & 1 \\ 4 & 2 & 2 \end{pmatrix}$. Find the kernel of B. Then find the kernel of AB (and explain how you got it).

Question 9. You are told that the 3×3 matrix $A = (\vec{v}_1 \vec{v}_2 \vec{v}_3)$ is orthogonal, and that $\vec{v}_1$ and $\vec{v}_2$ both lie on the plane $x_1 + 2x_2 + 3x_3 = 0$. Find $\vec{v}_3$.

Mathematics 33A, Second Midterm, November 6, 2009.

Calculators, books, or notes of any kind are not allowed on this exam. Do not use any paper other than that provided. (You may write on the back if you need more space, but indicate this clearly on the front.)

There are 9 items on this exam altogether, and they all have equal value. Answer as many of them as you can. Show your work and explain your reasoning.

The questions are not always arranged in order of difficulty. Look through them when you start so you get an idea of the time you'll need. If you're not sure what to do on an item then move onward and return to it later.

Good luck.

Family Name: ______________________

Given names: ______________________

Discussion section: ______________________

UID#: ______________________

Email address: ______________________

Signature: ______________________

Total: ________ out of 45 points

Question 1. (5 points) Find a basis for the plane $2x_1 - x_2 + x_3 = 0$ in $\mathbb{R}^3$.

Question 2. (5 points) Find k so that the matrices $\begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & -2 \\ 1 & 1 & -1 \end{pmatrix}$ and $\begin{pmatrix} 3 & 1 & 0 \\ 1 & -1 & -4 \\ 2 & 0 & k \end{pmatrix}$ have the same image.

Question 3. (5 points) Let T from $\mathbb{R}^5$ to $\mathbb{R}^5$ be the orthogonal projection to span$\{\begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}\}$. Find the dimension of kernel(T).

Question 4. Let $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$, and $\vec{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$. Let $V = \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

(a) (5 points) Using Gram-Schmidt, find an orthonormal basis $\vec{u}_1, \vec{u}_2, \vec{u}_3$ for V.

(b) (5 points)

Write each $\vec{v}_i$ as a linear combination of $\vec{u}_1, \vec{u}_2, \vec{u}_3$, and find the QR decomposition of $\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix}$.

(c) (5 points) Find $V^\perp$ and an orthonormal basis for $V^\perp$.

Question 5. In this question, $\vec{v}_1$ and $\vec{v}_2$ are vectors in $\mathbb{R}^2$, and you are told that $\vec{v}_i \cdot \vec{v}_j$ is the entry a_{ij} of the matrix $\begin{pmatrix} 3 & 5 \\ 5 & 7 \end{pmatrix}$. L is the line spanned by $\vec{v}_1$.

(a) (5 points) Find the $\{\vec{v}_1, \vec{v}_2\}$ coordinates of $\text{proj}_L(\vec{v}_2)$, the orthogonal projection of $\vec{v}_2$ onto the line L.

(b) (5 points) Find the matrix of the orthogonal projection onto L in the basis $\{\vec{v_1}, \vec{v}_2\}$.

Question 6. (5 points) Find the orthogonal projection of $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ to the plane in $\mathbb{R}^3$ spanned by $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix}$.

Math 33A, Midterm 2 solutions

1. We have to solve the linear system of 1 equation in 3 variables. Its augmented coefficient matrix is
$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & 0 \end{array}\right),$$
and after dividing by 2 we get its RREF
$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \end{array}\right).$$
Alternatively, we can say that we are finding the kernel of the matrix
$$\begin{pmatrix} 2 & -1 & 1 \end{pmatrix}.$$
The solution is
$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = s\begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$$
where s, t are arbitrary parameters, i.e. the subspace (the plane) can be written as
$$\text{span}\{\begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix}\}.$$
The above two vectors are linearly independent (which is always the case whenever we compute the kernel from the RREF), and thus
$$\begin{pmatrix} \frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$$
is a basis for the given subspace.

2. Denote the matrices by A and B respectively.
$$\text{image}(A) = \text{span}\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix}\} = \text{span}\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}\},$$
because $\begin{pmatrix} 0 \\ -2 \\ -1 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$. Also
$$\text{image}(B) = \text{span}\{\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -4 \\ k \end{pmatrix}\} = \text{span}\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -4 \\ k \end{pmatrix}\}.$$
In the last equality we have used that
$$\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = -\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

and also

$$\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}3\\1\\2\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\-1\\0\end{pmatrix}, \quad \begin{pmatrix}2\\0\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}3\\1\\2\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\-1\\0\end{pmatrix}$$

so that $\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}$ and $\begin{pmatrix}3\\1\\2\end{pmatrix}, \begin{pmatrix}1\\-1\\0\end{pmatrix}$ span the same subspace.

Therefore we conclude that $\begin{pmatrix}0\\-4\\k\end{pmatrix}$ must be a linear combination of $\begin{pmatrix}1\\1\\1\end{pmatrix}$ and $\begin{pmatrix}2\\0\\1\end{pmatrix}$. We solve the linear system

$$\begin{pmatrix}1 & 2 & | & 0\\1 & 0 & | & -4\\1 & 1 & | & k\end{pmatrix}$$

using Gauss-Jordan elimination to obtain the RREF

$$\begin{pmatrix}1 & 0 & | & -2\\0 & 1 & | & 2\\0 & 0 & | & -k-2\end{pmatrix}$$

Now we see that the system has a unique solution precisely when $-k-2=0$, i.e. $k=-2$.

3. The vectors $\begin{pmatrix}1\\2\\1\\2\\0\end{pmatrix}, \begin{pmatrix}0\\1\\2\\1\\2\end{pmatrix}$ are linearly independent because they are not scalar multiples of each other. Their linear span is precisely image(T), so rank(T) = dim(image(T)) = 2. By the rank-nullity theorem we conclude

$$\text{nullity(T)} = 5 - \text{rank}(T) = 5 - 2 = 3,$$

so dim(kernel(T)) = 3.

4. (a)

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \begin{pmatrix}1/2\\1/2\\1/2\\1/2\end{pmatrix}$$

$$\vec{v}_2^{\,||} = (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1 = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\vec{v}_2^{\,\perp} = \vec{v}_2 - \vec{v}_2^{\,||} = \begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}$$

$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \begin{pmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{pmatrix}$$

$$\vec{v}_3^{\|} = (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 + (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{v}_3^{\perp} = \vec{v}_3 - \vec{v}_3^{\|} = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

$$\vec{u}_3 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \begin{pmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$$

(b)

$$\begin{aligned} \vec{v}_1 &= 2\vec{u}_1 \\ \vec{v}_2 &= 0\vec{u}_1 + 2\vec{u}_2 \\ \vec{v}_3 &= 1\vec{u}_1 + (-1)\vec{u}_2 + 2\vec{u}_3 \end{aligned}$$

The QR decomposition of the given matrix is:

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

(c) The easiest way is to guess a vector $\vec{u}_4 = \begin{pmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{pmatrix}$ so that $\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4$ form an orthonormal basis for $\mathbb{R}^4$. Then $\vec{u}_4$ is an orthonormal basis for $V^{\perp}$ and $V^{\perp} = \text{span}\{\vec{u}_4\}$.

Alternatively we could take $\vec{u}_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and continue the Gram-Schmidt process.

5. (a) From the matrix we read of

$$\vec{v}_1 \cdot \vec{v}_2 = 5, \quad \|\vec{v}_1\|^2 = \vec{v}_1 \cdot \vec{v}_1 = 3, \quad \|\vec{v}_2\|^2 = \vec{v}_2 \cdot \vec{v}_2 = 7.$$

We have the formula for proj_L:

$$\text{proj}_L(\vec{x}) = \frac{1}{\|\vec{v}_1\|^2}(\vec{x} \cdot \vec{v}_1)\vec{v}_1.$$

Since

$$\text{proj}_L(\vec{v}_2) = \frac{1}{\|\vec{v}_1\|^2}(\vec{v}_2 \cdot \vec{v}_1)\vec{v}_1 = \frac{5}{3}\vec{v}_1 = \frac{5}{3}\vec{v}_1 + 0\vec{v}_2,$$

we have

$$[\text{proj}_L(\vec{v}_2)]_{\{\vec{v}_1,\vec{v}_2\}} = \begin{pmatrix} 5/3 \\ 0 \end{pmatrix}$$

(b) Since

$$\text{proj}_L(\vec{v}_1) = \vec{v}_1, \quad \text{proj}_L(\vec{v}_2) = \frac{5}{3}\vec{v}_1,$$

we conclude that the matrix of proj_L in the basis $\vec{v}_1, \vec{v}_2$ is

$$[\text{proj}_L]_{\{\vec{v}_1,\vec{v}_2\}} = \begin{pmatrix} 1 & 5/3 \\ 0 & 0 \end{pmatrix}$$

6. Let us first "orhonormalize" the vectors

$$\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix},$$

i.e. find an orthonormal basis of the plane. The Gram-Schmidt process gives

$$\vec{u}_1 = \begin{pmatrix} 2/3 \\ 1/3 \\ 2/3 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} -2/3 \\ 2/3 \\ 1/3 \end{pmatrix}.$$

Finally we use the formula:

$$\text{proj}_{\text{span}(\vec{v}_1,\vec{v}_2)}(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}) = \text{proj}_{\text{span}(\vec{u}_1,\vec{u}_2)}(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix})$$

$$= (\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \vec{u}_1)\,\vec{u}_1 + (\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \vec{u}_2)\,\vec{u}_2$$

$$= \frac{10}{3}\begin{pmatrix} 2/3 \\ 1/3 \\ 2/3 \end{pmatrix} + \frac{5}{3}\begin{pmatrix} -2/3 \\ 2/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 10/9 \\ 20/9 \\ 25/9 \end{pmatrix}$$

Name: ______________________ UID: ______________

Quiz #3

$$A = \begin{pmatrix} -1 & 2 & 0 & 0 \\ 1 & 0 & 2 & -1 \end{pmatrix}$$

(a) (5 points)
Find image(A) and write it as a linear span of as few vectors as possible, i.e. without any redundant vectors.

(b) (5 points)
Find kernel(A) and write it as a linear span of as few vectors as possible.

(c) (5 points)
Find rank(A) and nullity(A).

(d) (5 points)
Find an orthonormal basis for kernel(A).

QUIZ #3 – SOLUTIONS

$$A = \begin{pmatrix} -1 & 2 & 0 & 0 \\ 1 & 0 & 2 & -1 \end{pmatrix}$$

(a) (5 points)
Find image(A) and write it as a linear span of as few vectors as possible, i.e. without any redundant vectors.

Solution.

$$\text{image}(A) = \text{span}\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}\} = \text{span}\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}\}$$

(b) (5 points)
Find kernel(A) and write it as a linear span of as few vectors as possible.

Solution. We have to solve the linear system:

$$\left(\begin{array}{cccc|c} -1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 2 & -1 & 0 \end{array}\right).$$

Using Gauss-Jordan elimination we get the following RREF:

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} & 0 \end{array}\right).$$

By taking $x_4 = t$, $x_3 = s$, $x_2 = -s + \frac{1}{2}t$, $x_1 = -2s + t$, we obtain:

$$\text{kernel}(A) = \text{span}\{\begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{1}{2} \\ 0 \\ 1 \end{pmatrix}\}$$

(c) (5 points)
Find rank(A) and nullity(A).

Solution. From parts (a) and (b) we have: $\text{rank}(A) = 2$, $\text{nullity}(A) = 2$.

(d) (5 points)
Find an orthonormal basis for kernel(A).

Solution. We apply the Gram-Schmidt process to vectors:

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1/2 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \begin{pmatrix} 2/3 \\ 1/3 \\ 0 \\ 2/3 \end{pmatrix}$$

$$\vec{v}_2^{\|} = (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1 = \begin{pmatrix} -10/9 \\ -5/9 \\ 0 \\ -10/9 \end{pmatrix}$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - \vec{v}_2^{\|} = \begin{pmatrix} -8/9 \\ -4/9 \\ 1 \\ 10/9 \end{pmatrix}$$

$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{1}{\sqrt{261}} \begin{pmatrix} -8 \\ -4 \\ 9 \\ 10 \end{pmatrix}$$

Math 33A, Section 3, Fall 2009

Name: ______________________________ UID: ______________

Quiz #4

Compute the following determinants. You can use any methods/properties you want. Each problem is worth 2 points. Working time is 20 minutes.

1. $\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} =$

2. $\begin{vmatrix} -9 & 4 \\ -2 & 1 \end{vmatrix} =$

3. $\begin{vmatrix} 0 & 0 & 2 \\ 0 & 3 & 0 \\ 5 & 0 & 0 \end{vmatrix} =$

4. $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 3 & 6 \end{vmatrix} =$

5. $\begin{vmatrix} \frac{2}{3} & \frac{7}{3} & 5 \\ 7 & 8 & 9 \\ \frac{2}{5} & \frac{7}{5} & 3 \end{vmatrix} =$

6. $\begin{vmatrix} 3 & 2 & 4 & 1 \\ 2 & 3 & 1 & 4 \\ 5 & 5 & 5 & 5 \\ 6 & 7 & 8 & 9 \end{vmatrix} =$

(continued on the other side)

7. $$\begin{vmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{vmatrix} =$$

8. $$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 \\ 3 & -2 & 1 & 0 & 0 \\ 4 & -3 & 2 & -1 & 0 \\ 5 & -4 & 3 & -2 & 1 \end{vmatrix} =$$

9. $$\begin{vmatrix} 0 & 0 & 5 & 9 & 2 \\ 8 & 3 & 8 & 6 & 1 \\ 1 & 0 & 9 & 4 & 3 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 7 & 1 & 0 \end{vmatrix} =$$

10. $$\begin{vmatrix} 1 & 0 & 0 & 5 & -7 & 9 \\ 5 & -2 & 0 & 8 & 6 & -3 \\ 4 & 6 & 2 & 4 & 5 & 7 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 9 & 1 & 0 \\ 0 & 0 & 0 & -9 & 8 & -1 \end{vmatrix} =$$

V. K.

Math 33A, Section 3, Fall 2009

Quiz #4 Solutions

1. $\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 2 \cdot 6 - 3 \cdot 4 = 12 - 12 = 0$

2. $\begin{vmatrix} -9 & 4 \\ -2 & 1 \end{vmatrix} = (-9) \cdot 1 - 4 \cdot (-2) = -9 + 8 = -1$

3. There is only one non-zero pattern, and it has 3 inversions.
$$\begin{vmatrix} 0 & 0 & 2 \\ 0 & 3 & 0 \\ 5 & 0 & 0 \end{vmatrix} = -2 \cdot 3 \cdot 5 = -30$$

4. We subtract 2 times row I from row II, and 3 times row I from row III.
$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 3 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{vmatrix} = 1 \cdot 1 \cdot 3 = 3$$

5. We first factor out $\frac{1}{3}$ and $\frac{1}{5}$ from rows I and III respectively, and then realize that the obtained determinant has two equal rows.
$$\begin{vmatrix} \frac{2}{3} & \frac{7}{3} & 5 \\ 7 & 8 & 9 \\ \frac{2}{5} & \frac{7}{5} & 3 \end{vmatrix} = \frac{1}{3} \cdot \frac{1}{5} \cdot \begin{vmatrix} 2 & 7 & 15 \\ 7 & 8 & 9 \\ 2 & 7 & 15 \end{vmatrix} = 0$$

6. If we subtract row I from row III, we get precisely row II.
$$\begin{vmatrix} 3 & 2 & 4 & 1 \\ 2 & 3 & 1 & 4 \\ 5 & 5 & 5 & 5 \\ 6 & 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 3 & 2 & 4 & 1 \\ 2 & 3 & 1 & 4 \\ 2 & 3 & 1 & 4 \\ 6 & 7 & 8 & 9 \end{vmatrix} = 0$$

7. There is only one non-zero pattern, and it has 6 inversions.
$$\begin{vmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{vmatrix} = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1$$

8. The matrix is lower-triangular.

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 \\ 3 & -2 & 1 & 0 & 0 \\ 4 & -3 & 2 & -1 & 0 \\ 5 & -4 & 3 & -2 & 1 \end{vmatrix} = 1 \cdot (-1) \cdot 1 \cdot (-1) \cdot 1 = 1$$

9. There is only one non-zero pattern, and it has 5 inversions.

$$\begin{vmatrix} 0 & 0 & 5 & 9 & 2 \\ 8 & 3 & 8 & 6 & 1 \\ 1 & 0 & 9 & 4 & 3 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 7 & 1 & 0 \end{vmatrix} = -2 \cdot 3 \cdot 1 \cdot 4 \cdot 1 = -24$$

10. The matrix is block-upper-triangular and each of the blocks is lower-triangular.

$$\begin{vmatrix} 1 & 0 & 0 & 5 & -7 & 9 \\ 5 & -2 & 0 & 8 & 6 & -3 \\ 4 & 6 & 2 & 4 & 5 & 7 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 9 & 1 & 0 \\ 0 & 0 & 0 & -9 & 8 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 5 & -2 & 0 \\ 4 & 6 & 2 \end{vmatrix} \cdot \begin{vmatrix} 3 & 0 & 0 \\ 9 & 1 & 0 \\ -9 & 8 & -1 \end{vmatrix} = (1 \cdot (-2) \cdot 2) \cdot (3 \cdot 1 \cdot (-1)) = 12$$

V. K.

Mathematics 33A, Section 3, Practice Final

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There are 10 problems (20 items) on this exam altogether, and they all have equal value. Answer as many of them as you can. Show your work and explain your reasoning.

The questions are not always arranged in order of difficulty. Look through them when you start so you get an idea of the time you'll need. If you're not sure what to do on an item then move onward and return to it later.

Working time is 3 hours.

Good luck.

Name: ______________________

UID#: ______________________

Signature: ______________________

Problem 1		(out of 10 points)
Problem 2		(out of 10 points)
Problem 3		(out of 10 points)
Problem 4		(out of 10 points)
Problem 5		(out of 10 points)
Problem 6		(out of 10 points)
Problem 7		(out of 10 points)
Problem 8		(out of 10 points)
Problem 9		(out of 10 points)
Problem 10		(out of 10 points)
Total		(out of 100 points)

Problem 1.
(a) [5 points] Solve the linear system

$$\begin{cases} 2x_1 - x_2 = 2 \\ x_1 + 2x_3 = 3 \\ x_2 + 4x_3 = 4 \end{cases}$$

(b) [5 points] Write the system in the matrix form $A\vec{x} = \vec{b}$, and find rank of the coefficient matrix A.

Problem 2.

Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$.

(a) [5 points] Compute AB and BA.

(b) [5 points] Compute $(B^T B)^3$.

Problem 3.
Let A be matrix of rotation by 120° counterclockwise, and let B be the matrix of reflection about the line $y = x$, both in $\mathbb{R}^2$.

(a) [5 points] Find the matrix product A^3B^2. You can reason either geometrically or algebraically.

(b) [5 points] Find all 2×2 matrices C that commute with B.

Problem 4.

(a) [5 points] Compute: $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 6 & 2 \\ 3 & 0 \end{bmatrix}$.

(b) [5 points] Find all matrices X such that

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 0 \end{bmatrix} X = \begin{bmatrix} 6 & 2 \\ 3 & 0 \\ 0 & 4 \end{bmatrix}.$$

Problem 5.
Let W be the kernel of the linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ x_2 - 3x_3 \end{bmatrix}.$$

(a) [5 points] Find a vector that spans W.

(b) [5 points] Give an example of a linear transformation $S\colon \mathbb{R}^2 \to \mathbb{R}^3$ whose image is W.

Problem 6.
Let V be the subspace

$$\text{span}\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}\}$$

and consider its orthogonal complement $V^{\perp}$.

(a) [5 points]   Find a basis for $V^{\perp}$.

(b) [5 points]   Using the Gram-Schmidt process find an orthonormal basis for $V^{\perp}$ starting with the basis from part (a).

Problem 7.

(a) [5 points] Suppose that A is a 3×3 matrix with QR factorization:

$$A = QR, \quad \text{where } R = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & -3 \\ 0 & 0 & 1 \end{bmatrix}.$$

Find $\det(A^4)$.

(b) [5 points] Fit a quadratic function to the data points $(-1, 2)$, $(0, 0)$, $(0, 2)$, $(1, 4)$, using least squares. Sketch the solution.

Problem 8.

(a) [5 points] Compute the determinant:

$$\begin{vmatrix} 0 & 2 & 0 & 0 & 4 \\ 0 & 1 & 7 & -3 & 5 \\ 0 & 0 & 0 & 0 & -2 \\ 4 & 0 & 1 & 0 & 1 \\ 3 & 0 & 0 & 0 & 3 \end{vmatrix}$$

(b) [5 points] Let A and B be two square matrices of the same size. If $\det A = 2$ and $\det B = 3$, find

$$\det(AB^T A^T B^{-1} A^{-1} B).$$

Problem 9.

Let $A = \begin{bmatrix} -2 & -1 & -1 \\ 2 & -1 & 2 \\ 0 & 1 & -1 \end{bmatrix}$.

(a) [5 points] Find all eigenvalues of A, with their algebraic multiplicities.

(b) [5 points] Find a basis for each eigenspace of A. Find geometric multiplicities of its eigenvalues.

Problem 10. Let $A = \begin{bmatrix} 3 & -4 \\ -4 & -3 \end{bmatrix}$.

(a) [5 points] Is A orthogonally diagonalizable, i.e. is there an orthonormal basis for $\mathbb{R}^2$ consisting of its eigenvectors? If so, find one such orthonormal basis.

(b) [5 points] Compute $(\frac{1}{5}A)^{2009}$.

Mathematics 33A, Section 3, Final Exam

Monday, December 7, 2009

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There are 10 problems (20 items) on this exam altogether, and they all have equal value. Answer as many of them as you can. Show your work and explain your reasoning.

The questions are not always arranged in order of difficulty. Look through them when you start so you get an idea of the time you'll need. If you're not sure what to do on an item then move onward and return to it later.

Working time is 3 hours.

Good luck.

Name: ______________________

UID#: ______________________

Signature: ______________________

Problem 1		(out of 10 points)
Problem 2		(out of 10 points)
Problem 3		(out of 10 points)
Problem 4		(out of 10 points)
Problem 5		(out of 10 points)
Problem 6		(out of 10 points)
Problem 7		(out of 10 points)
Problem 8		(out of 10 points)
Problem 9		(out of 10 points)
Problem 10		(out of 10 points)
Total		(out of 100 points)

Problem 1.

(a) [5 points] Solve the linear system

$$\begin{cases} x_1 + 2x_2 + 2x_3 = -1 \\ x_1 + x_2 - x_3 = 1 \\ 2x_1 + x_2 - 5x_3 = 4 \end{cases}$$

(b) [5 points] Write the system in the matrix form $A\vec{x} = \vec{b}$, and find a basis for the kernel of the coefficient matrix A.

Problem 2.

Let R be the matrix of rotation by $20°$ counterclockwise in $\mathbb{R}^2$ and let $\vec{v} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$.

(a) [5 points] Find the matrix power R^6.

(b) [5 points]

What is the angle between vectors $R^6\vec{v}$ and $\vec{v}$? Compute the dot product $R^6\vec{v} \cdot \vec{v}$.

Problem 3.

Denote $A = \begin{bmatrix} 1 & a & a \\ 0 & 1 & a \\ 0 & 0 & a \end{bmatrix}$.

(a) [5 points]
Compute $\det A$. For which values of the parameter $a \in \mathbb{R}$ is the matrix A invertible?

(b) [5 points] Find A^{-1} for those values of a.

Problem 4.

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

(a) [5 points] Find image(A) and kernel(A) by describing them as linear spans of their basis vectors. Find rank(A) and nullity(A).

(b) [5 points] Find an orthonormal basis for the orthogonal complement $V^{\perp}$ of the subspace $V = \text{kernel}(A^T)$.

Problem 5.

(a) [5 points] Give an example of a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ whose kernel is the line $2x - y = 0$ and image is the line $x + 2y = 0$. You can both reason and describe it geometrically.

(b) [5 points] Let T be a linear transformation from $\mathbb{R}^5$ to $\mathbb{R}^2$. What are the possible values of nullity(T)? Explain. Give an example of T for each of the possible values of nullity(T).

Problem 6.
Let V be a subspace of $\mathbb{R}^4$ defined by the equation $x_1 - x_2 - x_3 + x_4 = 0$.

(a) [5 points] Find a basis for V and $\dim(V)$.

(b) [5 points] Using the Gram-Schmidt process find an orthonormal basis for V starting with the basis from part (a).

Problem 7.

(a) [5 points] Suppose that the following matrix is orthogonal:

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{2}{3} & a \\ -\frac{1}{\sqrt{2}} & \frac{2}{3} & b \\ c & d & e \end{bmatrix}.$$

for some parameters $a, b, c, d, e \in \mathbb{R}$.
What must be the value of c? What are the possible values of d? Explain.

(b) [5 points] Show that the linear system $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 1 \\ 1 & -2 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix},$$

has no exact solutions. Find the least-squares solution $\vec{x}^*$.

Problem 8.
(a) [5 points] Compute the determinant:

$$\begin{vmatrix} 1 & 3 & 1 & 2 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 3 & -1 & 2 & 1 & -3 & 2 \\ -1 & 0 & 2 & 3 & 0 & -1 \end{vmatrix}$$

(b) [5 points] Suppose that matrices A and B are such that

$$\det(A^2B) = 40, \qquad \det(A^{-1}B^2) = -50.$$

Find $\det A$ and $\det B$.

Problem 9.

Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

(a) [5 points] Find all eigenvalues of A and the corresponding eigenspaces. State explicitly algebraic and geometric multiplicities of the eigenvalues.

(b) [5 points] Is A diagonalizable? If so, write it in the form $A = SDS^{-1}$, with D diagonal. Find a formula for A^t, where t is an arbitrary positive integer.

Problem 10.

Denote $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

(a) [5 points] Find an orthonormal eigenbasis for A.

(b) [5 points] Find a matrix of the form

$$B = \begin{bmatrix} 2 & 3 & 0 & 0 \\ a & b & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & c & d \end{bmatrix},$$

for some parameters $a, b, c, d \in \mathbb{R}$, such that A and B are similar.

Mathematics 33A, Section 3, Final Exam Solutions

1. **(a)** We write the augmented coefficient matrix and then perform Gauss-Jordan elimination (i.e. row operations):

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & -1 \\ 1 & 1 & -1 & 1 \\ 2 & 1 & -5 & 4 \end{array}\right]$$

subtract 3 times row I from row II, and subtract 2 times row I from row III

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & -1 \\ 0 & -1 & -3 & 2 \\ 0 & -3 & -9 & 6 \end{array}\right]$$

divide row II by -1, and divide row III by -3

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & -1 \\ 0 & 1 & 3 & -2 \\ 0 & 1 & 3 & -2 \end{array}\right]$$

subtract 2 times row II from row I, and subtract row II from row III

$$\left[\begin{array}{ccc|c} 1 & 0 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From the reduced row echelon form we see that x_3 is arbitrarily parametrized, while x_1 and x_2 are expressed in terms of that parameter. We successively write:

$$x_3 = t, \quad x_2 = -3t - 2, \quad x_1 = 4t + 3,$$

for arbitrary real parameters t. We can also write the solution in the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4t + 3 \\ -3t - 2 \\ t \end{bmatrix}.$$

(b) The coefficient matrix is

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & -1 \\ 2 & 1 & -5 \end{bmatrix}$$

and from the computation in part (a) we see its reduced row echelon form:

$$\text{RREF}(A) = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

As before, we simply read off that the kernel is the set of all:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4t \\ -3t \\ t \end{bmatrix} = t \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$$

Therefore

$$\text{kernel}(A) = \text{span}(\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix})$$

and thus one basis for kernel(A) is: $\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$.

2. **(a)** If R is the rotation by $20°$, then R^6 is a composition of 6 rotations, each by $20°$. We conclude that R^6 is the rotation by $2 \cdot 20° = 120°$, and thus its matrix is

$$R = \begin{bmatrix} \cos 120° & -\sin 120° \\ \sin 120° & \cos 120° \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}.$$

(b) **Answer:** $\angle(R^6\vec{v}, \vec{v}) = 120°$. $R^6\vec{v} \cdot \vec{v} = -\frac{25}{2}$.

The angle between $R^6\vec{v}$ and $\vec{v}$ is $120°$, because $R^6\vec{v}$ is obtained by rotating $\vec{v}$ by $120°$. Also notice that $\|R^6\vec{v}\| = \|\vec{v}\| = \sqrt{3^2 + (-4)^2} = 5$. Therefore we get

$$R^6\vec{v} \cdot \vec{v} = \|R^6\vec{v}\|\|\vec{v}\| \cos 120° = 5 \cdot 5 \cdot \frac{-1}{2} = -\frac{25}{2}.$$

3. **(a)** **Answer:** $\det A = a$. For $a \neq 0$.

Since A is upper-triangular, we can easily compute its determinant as the product of diagonal entries: $\det A = 1 \cdot 1 \cdot a = a$. We see that A is invertible precisely when $a \neq 0$.

(b) We use the usual algorithm for finding the inverse: write the identity matrix to the right and then perform Gauss-Jordan elimination until we obtain the identity matrix on the left side.

$$\left[\begin{array}{ccc|ccc} 1 & a & a & 1 & 0 & 0 \\ 0 & 1 & a & 0 & 1 & 0 \\ 0 & 0 & a & 0 & 0 & 1 \end{array}\right]$$

subtract a times row II from row I

$$\left[\begin{array}{ccc|ccc} 1 & 0 & a-a^2 & 1 & -a & 0 \\ 0 & 1 & a & 0 & 1 & 0 \\ 0 & 0 & a & 0 & 0 & 1 \end{array}\right]$$

divide row III by a, which we can since $a \neq 0$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & a-a^2 & 1 & -a & 0 \\ 0 & 1 & a & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{a} \end{array}\right]$$

subtract $(a - a^2)$ times row III from row I, and subtract a times row III from row II

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -a & a-1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{a} \end{array}\right]$$

From the last augmented matrix we simply read off:

$$A^{-1} = \begin{bmatrix} 1 & -a & a-1 \\ 0 & 1 & -1 \\ 0 & 0 & \frac{1}{a} \end{bmatrix}.$$

4. **(a)**

$$\text{image}(A) = \text{span}(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}) = \text{span}(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}).$$

The third vector above is redundant (as the sum of previous two vectors), and so was omitted. The remaining two vectors are linearly independent, and so a basis for image(A). Consequently, rank(A) = dim(image(A)) = 2.

We find kernel(A) by solving the linear system:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

subtract row I from row III, and subtract row II from row IV

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From here we read off:

$$\text{kernel}(A) = \text{span}(\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}).$$

Consequently, nullity(A) = dim(kernel(A)) = 1.

(b) Computations can be shortened if one uses the following result (see Theorem 5.4.1 in the textbook): image(A) and kernel(A^T) are orthogonal complements of each other. This immediately gives us:

$$V^{\perp} = \text{image}(A) = \text{span}(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}).$$

Therefore $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ is a basis for $V^{\perp}$. Since $\vec{v}_1 \cdot \vec{v}_2 = 0$, we can orthonormalize $\vec{v}_1$, $\vec{v}_2$ simply dividing by their length to obtain an orthonomal basis $\vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

* * *

Alternatively, we could have first found the kernel of A^T:

$$V = \text{kernel}(A^T) = \text{span}(\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix})$$

and then the set of vectors orthogonal to the above two vectors. This would lead us to the system (given by its augmented coefficient matrix):

$$\left[\begin{array}{cccc|c} -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \end{array}\right]$$

divide both rows by -1

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{array}\right]$$

with the solution

$$V^\perp = \text{span}(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}).$$

Finally we normalize as before.

5. **(a)** Observe that the two lines are orthogonal, because they are determined by vectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$, which are easily seen to be orthogonal:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} = 1 \cdot (-2) + 2 \cdot 1 = 0.$$

Therefore the orthogonal projection onto the line $x + 2y = 0$, is projecting along the line $2x - y = 0$. In particular its image is the line $x + 2y = 0$, and its kernel is the line $2x - y = 0$.

(b) **Answer:** $\text{nullity}(T) = 3, 4, 5$.

By the rank-nullity theorem applied to T we get $\text{rank}(T) + \text{nullity}(T) = 5$. Since $\text{image}(T)$ is contained in $\mathbb{R}^2$, its dimension must satisfy $0 \le \text{rank}(T) \le 2$. Therefore $3 \le \text{nullity}(T) \le 5$, and so the possible values are $\text{nullity}(T) = 3, 4, 5$. To see that all of these are actually possible it is enough to list the following examples (in the same order):

$$T_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad T_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad T_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

6. **(a)** We have to solve (the system of) one linear equation in 4 variables. The augmented coefficient matrix is:

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \end{array}\right].$$

It is already in the reduced row echelon form, and we see that x_2, x_3, x_4 are parametrized, while x_1 can be expressed in terms of these parameters. The solution is:

$$V = \text{span}(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}).$$

Therefore one possible basis for V is

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

and consequently $\dim V = 3$.

(b) We perform the Gram-Schmidt process on vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$. with the usual notation.

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{v}_2^{||} = (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - \vec{v}_2^{||} = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$$

$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$$

$$\vec{v}_3^{||} = (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 + (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2 = \frac{1}{3} \begin{bmatrix} -2 \\ -1 \\ -1 \\ 0 \end{bmatrix}$$

$$\vec{v}_3^{\perp} = \vec{v}_3 - \vec{v}_3^{||} = \frac{1}{3} \begin{bmatrix} -1 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

$$\vec{u}_3 = \frac{\vec{v}_3^{\perp}}{\|\vec{v}_3^{\perp}\|} = \frac{1}{\sqrt{12}} \begin{bmatrix} -1 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

The desired orthonormal basis is

$$\vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \quad \vec{u}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-1\\2\\0\end{bmatrix}, \quad \vec{u}_3 = \frac{1}{\sqrt{12}}\begin{bmatrix}-1\\1\\1\\3\end{bmatrix}.$$

7. (a) Answer: $c = 0$, $d = \pm\frac{1}{3}$.

A matrix is orthogonal precisely when its columns are orthonormal vectors.

The first column must be a unit vector (i.e. must have length 1), so $(\frac{1}{\sqrt{2}})^2 + (-\frac{1}{\sqrt{2}})^2 + c^2 = 1$, which gives $c^2 = 0$, i.e. $c = 0$.

The second column must be a unit vector, so $(\frac{2}{3})^2 + (\frac{2}{3})^2 + d^2 = 1$, which gives $d^2 = \frac{1}{9}$, i.e. $d = -\frac{1}{3}$ or $d = \frac{1}{3}$.

We must argue that these values are actually possible, i.e. that there exist orthogonal matrices with $c = 0$, $d = -\frac{1}{3}$, and with $c = 0$, $d = \frac{1}{3}$. The two columns are orthogonal (and thus orthonormal): $\begin{bmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\\0\end{bmatrix} \cdot \begin{bmatrix}\frac{2}{3}\\\frac{2}{3}\\\pm\frac{1}{3}\end{bmatrix} = 0$. Thus we could find the third column by completing the first two columns to the orthonormal basis for $\mathbb{R}^3$, but we don't have to do that explicitly.

(b) Answer: $\vec{x}^* = \begin{bmatrix}\frac{1}{7}\\-\frac{3}{7}\end{bmatrix}$.

The system has the following augmented coefficient matrix:

$$\left[\begin{array}{cc|c}1 & 1 & -2\\1 & -1 & 0\\2 & 1 & 1\\1 & -2 & 1\end{array}\right].$$

Its reduced row echelon form is

$$\left[\begin{array}{cc|c}1 & 0 & -1\\0 & 1 & -1\\0 & 0 & 1\\0 & 0 & 0\end{array}\right].$$

The third row reads $0 = 1$, so the system is inconsistent, i.e. has no exact solutions.

Now we turn to the corresponding normal equation: $A^TA\,\vec{x}^* = A^T\vec{b}$.

$$A^TA = \begin{bmatrix}1 & 1 & 2 & 1\\1 & -1 & 1 & -2\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\\2 & 1\\1 & -2\end{bmatrix} = \begin{bmatrix}7 & 0\\0 & 7\end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix}1 & 1 & 2 & 1\\1 & -1 & 1 & -2\end{bmatrix}\begin{bmatrix}-2\\0\\1\\1\end{bmatrix} = \begin{bmatrix}1\\-3\end{bmatrix}$$

The least squares solution $\vec{x}^*$ is obtained by solving the linear system:

$$\left[\begin{array}{cc|c} 7 & 0 & 1 \\ 0 & 7 & -3 \end{array}\right]$$

divide both rows by 7

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{1}{7} \\ 0 & 1 & -\frac{3}{7} \end{array}\right]$$

and it is $\vec{x}^* = \begin{bmatrix} \frac{1}{7} \\ -\frac{3}{7} \end{bmatrix}$.

* * *

Alternatively, we can compute $\vec{x}^*$ as:

$$\vec{x}^* = (A^TA)^{-1}(A^T\vec{b}) = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} \frac{1}{7} & 0 \\ 0 & \frac{1}{7} \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} \frac{1}{7} \\ -\frac{3}{7} \end{bmatrix}.$$

8. (a) Answer: 12.

Notice that the matrix is block-lower-triangular with blocks of size 4×4 and 2×2 on the diagonal. Furthermore, each of those blocks is upper-triangular, so its determinant is simply the product of diagonal entries. Therefore we can compute:

$$\begin{vmatrix} 1 & 3 & 1 & 2 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 3 & -1 & 2 & 1 & -3 & 2 \\ -1 & 0 & 2 & 3 & 0 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & -2 \end{vmatrix} \cdot \begin{vmatrix} -3 & 2 \\ 0 & -1 \end{vmatrix}$$

$$= 1 \cdot 2 \cdot (-1) \cdot (-2) \cdot (-3) \cdot (-1) = 12.$$

* * *

Alternatively, by inspecting rows in the following order: IV, III, II, I, VI, V, we see that there is only one pattern that leads to a nonzero term. The number of inversions is 0, so its signature is +1, and we have:

$$\begin{vmatrix} \boxed{1} & 3 & 1 & 2 & 0 & 0 \\ 0 & \boxed{2} & 1 & 0 & 0 & 0 \\ 0 & 0 & \boxed{-1} & 2 & 0 & 0 \\ 0 & 0 & 0 & \boxed{-2} & 0 & 0 \\ 3 & -1 & 2 & 1 & \boxed{-3} & 2 \\ -1 & 0 & 2 & 3 & 0 & \boxed{-1} \end{vmatrix} = 1 \cdot 2 \cdot (-1) \cdot (-2) \cdot (-3) \cdot (-1) = 12.$$

(b) Answer: $\det A = -2$, $\det B = 10$.

Denote $a = \det A$, $b = \det B$. Using the product rule for determinants we have

$$\det(A^2B) = (\det A)^2(\det B), \qquad \det(A^{-1}B^2) = \frac{1}{\det A}(\det B)^2,$$

so we obtain the system for a and b:

$$a^2 b = 40 = 2^3 \cdot 5$$
$$\frac{b^2}{a} = -50 = -2 \cdot 5^2$$

Squaring the second equation any multiplying with the first one we get:

$$b^5 = (a^2 b)\left(\frac{b^2}{a}\right)^2 = (2^3 \cdot 5)(-2 \cdot 5^2)^2 = 2^5 \cdot 5^5 = 10^5$$

and thus $b = 10$. By plugging into the second equation we also get $a = \frac{b^2}{-2 \cdot 5^2} = -2$.

9. (a) The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 0 \\ 0 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2(2-\lambda)$$

so the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$.
The eigenspace E_1 is found by solving the linear system

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

and the result is

$$E_1 = \text{span}(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}).$$

The eigenspace E_2 is found by solving the linear system

$$\left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

and the result is

$$E_2 = \text{span}(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}).$$

In particular, $\lambda_1 = 1$ has both algebraic and geometric multiplicity 2, while $\lambda_2 = 2$ has both algebraic and geometric multiplicity 1.

(b) Since the sum of geometric multiplicities is equal to the dimension of the space (order of the matrix A), we conclude that A is diagonalizable. We can write it in the form $A = SDS^{-1}$ by taking D to be the diagonal matrix with eigenvalues on the diagonal, and S to be the matrix whose columns are corresponding basis eigenvectors:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \qquad S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

$$A = SDS^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

For any positive integer t we have:

$$A^t = (SDS^{-1})^t = SD^tS^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^t \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2^t-1 & 0 \\ 0 & 2^t & 0 \\ 0 & 2^t-1 & 1 \end{bmatrix}.$$

10. (a) The characteristic polynomial of A is

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 1 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 1 \\ 0 & 0 & 1 & -\lambda \end{vmatrix} = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} \cdot \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix}$$
$$= (\lambda^2 - 1)^2 = (\lambda + 1)^2(\lambda - 1)^2$$

so the eigenvalues are $\lambda_1 = -1$, $\lambda_2 = 1$.

The eigenspace E_{-1} is found by solving the linear system

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

and the result is

$$E_{-1} = \text{span}(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}).$$

The eigenspace E_1 is found by solving the linear system

$$\left[\begin{array}{cccc|c} -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

and the result is

$$E_1 = \text{span}(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}).$$

Thus we have obtained an eigenbasis:

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix},$$

but it is not orthonormal. Since the vectors are indeed mutually orthogonal (but not unit), orthonormalization is done simply dividing by their lengths:

$$\vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{u}_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{u}_4 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

(b) Since A is diagonalizable with the diagonal form $D = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, we know that B will be similar to A precisely when it is also diagonalizable and has the same diagonal form, i.e. the same characteristic polynomial.

The characteristic polynomial of B is

$$\det(B - \lambda I) = \begin{vmatrix} 2-\lambda & 3 & 0 & 0 \\ a & b-\lambda & 0 & 0 \\ 0 & 0 & 2-\lambda & 3 \\ 0 & 0 & c & d-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 3 \\ a & b-\lambda \end{vmatrix} \cdot \begin{vmatrix} 2-\lambda & 3 \\ c & d-\lambda \end{vmatrix}$$
$$= \Big(\lambda^2 - (b+2)\lambda + (2b-3a)\Big)\Big(\lambda^2 - (d+2)\lambda + (2d-3c)\Big).$$

In order to have $\det(A - \lambda I) = \det(B - \lambda I)$, we can simply choose the coefficients in the above expression:

$$b + 2 = 0, \qquad 2b - 3a = -1, \qquad d + 2 = 0, \qquad 2d - 3c = -1.$$

The solution of that system is:

$$a = -1, \qquad b = -2, \qquad c = -1, \qquad d = -2,$$

so one such matrix is:

$$B = \begin{bmatrix} 2 & 3 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & -1 & -2 \end{bmatrix}.$$

However, we still have to verify that B is diagonalizable. It is most easily seen by computing its eigenspaces:

$$\tilde{E}_{-1} = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}\right)$$

$$\tilde{E}_1 = \text{span}(\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -3 \\ 1 \end{bmatrix})$$

The sum of geometric multiplicities is $2+2=4$, which means that B is diagonalizable. Since we have chosen a, b, c, d so that characteristic polynomials of A and B are the same, their diagonal forms are the same too, and thus A and B are similar.

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It would have also been completely legitimate to immediately guess B in some other way, and then verify that it is really similar to A.

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