

Theorem 1. *There exists a sequence $(b_k)_{k=1}^{\infty}$ with values in the set $\{1, 2, 3, 4, 5\}$ such that the infinite sum $\sum_{k=1}^{\infty} \frac{1}{2^k + b_k}$ is a rational number.*

Before the proof, for every integer $n \geq 0$ we denote

$$\alpha_n := \sum_{k=n+1}^{\infty} \frac{1}{2^k + 5}$$

and

$$\beta_n := \sum_{k=n+1}^{\infty} \frac{1}{2^k + 1}.$$

Lemma 2. *For every integer $n \geq 0$ we have $\alpha_n < \beta_n$.*

Proof of Lemma 2. We have

$$\begin{aligned} \alpha_n &= \frac{1}{2^{n+1} + 5} + \sum_{k=n+2}^{\infty} \frac{1}{2^k + 5} \\ &< \frac{1}{2^{n+1} + 1} + \sum_{k=n+2}^{\infty} \frac{1}{2^k + 5} \\ &\leq \frac{1}{2^{n+1} + 1} + \sum_{k=n+2}^{\infty} \frac{1}{2^k + 1} = \beta_n, \end{aligned}$$

which completes the proof. □

Lemma 3. *For every integer $n \geq 0$ we have $\beta_{n+1} \geq \alpha_{n+1} + \frac{1}{(2^{n+1}+1)(2^{n+1}+2)}$.*

Proof of Lemma 3. By Lemma 2 we have

$$\begin{aligned} \beta_{n+1} - \alpha_{n+1} &= \frac{1}{2^{n+2} + 1} - \frac{1}{2^{n+2} + 5} + \beta_{n+2} - \alpha_{n+2} \geq \frac{1}{2^{n+2} + 1} - \frac{1}{2^{n+2} + 5} \\ &= \frac{4}{(2^{n+2} + 1)(2^{n+2} + 5)} = \frac{1}{4^{n+1} + 3 \cdot 2^{n+1} + 5/4} \\ &> \frac{1}{4^{n+1} + 3 \cdot 2^{n+1} + 2} = \frac{1}{(2^{n+1} + 1)(2^{n+1} + 2)}, \end{aligned}$$

which completes the proof. □

Proof of Theorem 1. By Lemma 2 we have $\alpha_0 < \beta_0$, so there exists a rational number x such that $\alpha_0 \leq x \leq \beta_0$. We recursively define the sequence $(b_k)_{k=1}^{\infty}$ and, simultaneously, inductively verify the property

$$\sum_{k=1}^n \frac{1}{2^k + b_k} + \alpha_n \leq x \leq \sum_{k=1}^n \frac{1}{2^k + b_k} + \beta_n \quad (1)$$

for every integer $n \geq 0$. Property (1) is trivially satisfied for $n = 0$ by the assumption $\alpha_0 \leq x \leq \beta_0$ and since a sum over an empty set is interpreted as 0. Take an integer $n \geq 0$ and, if $n \geq 1$, suppose that the terms b_m for all integers m satisfying $1 \leq m \leq n$ have already been defined. By (1) we have

$$\sum_{k=1}^n \frac{1}{2^k + b_k} + \frac{1}{2^{n+1} + 5} + \alpha_{n+1} \leq x \leq \sum_{k=1}^n \frac{1}{2^k + b_k} + \frac{1}{2^{n+1} + 1} + \beta_{n+1}. \quad (2)$$

Recursively define b_{n+1} to be the largest $c \in \{1, 2, 3, 4, 5\}$ such that

$$x \leq \sum_{k=1}^n \frac{1}{2^k + b_k} + \frac{1}{2^{n+1} + c} + \beta_{n+1}.$$

By this definition we have

$$x \leq \sum_{k=1}^n \frac{1}{2^k + b_k} + \frac{1}{2^{n+1} + b_{n+1}} + \beta_{n+1}. \quad (3)$$

If $b_{n+1} = 5$, then (2) and (3) together give

$$\sum_{k=1}^{n+1} \frac{1}{2^k + b_k} + \alpha_{n+1} \leq x \leq \sum_{k=1}^{n+1} \frac{1}{2^k + b_k} + \beta_{n+1}$$

and the recursive step is complete. Otherwise, if $b_{n+1} \in \{1, 2, 3, 4\}$, then the maximality from its definition gives

$$x > \sum_{k=1}^n \frac{1}{2^k + b_k} + \frac{1}{2^{n+1} + b_{n+1} + 1} + \beta_{n+1},$$

which, in combination with Lemma 3 followed by $b_{n+1} \geq 1$, implies

$$\begin{aligned} x &> \sum_{k=1}^n \frac{1}{2^k + b_k} + \frac{1}{2^{n+1} + b_{n+1} + 1} + \alpha_{n+1} + \frac{1}{(2^{n+1} + 1)(2^{n+1} + 2)} \\ &= \sum_{k=1}^n \frac{1}{2^k + b_k} + \frac{1}{2^{n+1} + b_{n+1}} + \alpha_{n+1} - \frac{1}{(2^{n+1} + b_{n+1})(2^{n+1} + b_{n+1} + 1)} + \frac{1}{(2^{n+1} + 1)(2^{n+1} + 2)} \\ &\geq \sum_{k=1}^n \frac{1}{2^k + b_k} + \frac{1}{2^{n+1} + b_{n+1}} + \alpha_{n+1}, \end{aligned}$$

so that

$$\sum_{k=1}^{n+1} \frac{1}{2^k + b_k} + \alpha_{n+1} \leq x.$$

From the last equation and (3) we conclude

$$\sum_{k=1}^{n+1} \frac{1}{2^k + b_k} + \alpha_{n+1} \leq x \leq \sum_{k=1}^{n+1} \frac{1}{2^k + b_k} + \beta_{n+1}$$

and the recursive step is complete in this case as well.

Since the series $\sum_{k=1}^{\infty} \frac{1}{2^k + 5}$ and $\sum_{k=1}^{\infty} \frac{1}{2^k + 1}$ converge, we have that their tails converge to 0, so $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\lim_{n \rightarrow \infty} \beta_n = 0$. Using this, taking the limits of all three expressions in (1), and applying the squeeze theorem, we conclude

$$\sum_{k=1}^{\infty} \frac{1}{2^k + b_k} = x,$$

which is a rational number by our choice of x . □