Theorem 1. *It is possible to partition $\mathbb{R}^2$ into 25 color classes such that none of them contains the vertices of a rectangle of area 1.*

Proof. Every rectangle of area 1 is a special case of a parallelogram such that the product of lengths of its two consecutive sides equals 1. Thus, it is sufficient to give a coloring of $\mathbb{R}^2$ that uses 25 colors and has a slightly stronger property: no color class will contain the vertices of a parallelogram such that the product of lengths of its two consecutive sides equals 1.

Let $\mathbf{i}$ denote the complex imaginary unit. Identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. Let us place a parallelogram $P = abcd$ in the complex plane, so that its vertices a, b, c, d are respectively coordinatized by the complex numbers z_a, z_b, z_c, z_d. Consider a complex quantity $I(P)$ defined as $I(P) := z_a^2 - z_b^2 + z_c^2 - z_d^2$. In this definition we specify the vertex a to be the one with the smallest coordinate z_a in the lexicographic ordering of $\mathbb{C} \equiv \mathbb{R}^2$.

There exist $u, v, z \in \mathbb{C}$ such that the vertices of P have complex coordinates

$$z_a = z, \quad z_b = z + u, \quad z_c = z + u + v, \quad z_d = z + v.$$

The quantity $I(P)$ now simplifies as

$$I(P) = z^2 - (z+u)^2 + (z+u+v)^2 - (z+v)^2 = 2uv.$$

Consecutive side lengths of P are $|u|$ and $|v|$, so we have

$$|I(P)| = 2$$

whenever their product equals 1. Therefore, it remains to find a coloring of $\mathbb{C}$ such that, if all vertices of P are assigned the same color, then the complex number $I(P)$ does not lie on the circle $\{w \in \mathbb{C} : |w| = 2\}$.

For each pair $(j, k) \in \{0, 1, 2, 3, 4\}^2$ define a color class $C_{j,k}$ as

$$C_{j,k} := \left\{ z \in \mathbb{C} : z^2 \in \frac{10}{3}\left(\mathbb{Z} + \mathbf{i}\mathbb{Z} + \frac{j + \mathbf{i}k}{5} + \left[0, \frac{1}{5}\right) + \mathbf{i}\left[0, \frac{1}{5}\right) \right) \right\}.$$

If the four vertices of $P = abcd$ belonged to the same color class, then, by the definition of $I(P)$, we would clearly have

$$I(P) \in \frac{10}{3}\left(\mathbb{Z} + \mathbf{i}\mathbb{Z} + \left(-\frac{2}{5}, \frac{2}{5}\right) + \mathbf{i}\left(-\frac{2}{5}, \frac{2}{5}\right) \right).$$

The above set does not intersect the circle $\{w \in \mathbb{C} : |w| = 2\}$. Indeed, the central square lies fully inside the circle $\{w \in \mathbb{C} : |w| = 2\}$ because of $4\sqrt{2}/3 < 2$, while all remaining open squares lie fully outside of that circle. □