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A NEW WEIL COHOMOLOGY THEORY

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Abstract

There is a Weil cohomology theory with coefficients in a pseudofinite field of characteristic zero.

1. *Introduction*

This short note combines the existing theory of étale cohomology with torsion coefficients developed by the Grothendieck school with model-theoretic insight to give a new instance of a Weil cohomology theory.

The reason for developing cohomologies with coefficients in a field of zero characteristic was to be able to e.g. count fixed points of a morphism using a Lefschetz-type fixed point formula and subsequently prove the Weil conjectures. However, not all fields of zero characteristic support a Weil cohomology theory, as discussed in Serre's example in [**5**]: if we work in characteristic p, $\mathbb{Q}_p$ and $\mathbb{R}$ are excluded, but $\mathbb{Q}_l$ (for $l \neq p$) are allowed.

Hence, Grothendieck and others develop, for a separated scheme of finite type X over some field k of characteristic p and for a commutative noetherian ring Λ annihilated by an integer invertible in k, a good cohomology theory for torsion coefficients Λ with (or without) compact support, namely $H^i_c(X \otimes_k \bar{k}, \Lambda)$ (resp. $H^i(X \otimes_k \bar{k}, \Lambda)$). Then, aiming toward characteristic zero, they define $H^i(X \otimes_k \bar{k}, \mathbb{Z}_l) := \varprojlim_n H^i(X \otimes_k \bar{k}, \mathbb{Z}/l^n\mathbb{Z})$ (and similarly for compact support), followed by $H^i(X \otimes_k \bar{k}, \mathbb{Q}_l) := H^i(X \otimes_k \bar{k}, \mathbb{Z}_l) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ (and similarly for compact support), to get cohomologies with coefficients in $\mathbb{Q}_l$.

However, for a model-theorist, having a good theory for torsion coefficients, a much more natural thing to do is to take the ultraproduct over some torsion coefficients with unbounded torsion, which will give us some sort of an 'average' behaviour. In particular, for the Lefschetz fixed point formula, it does not matter if finitely many torsion coefficients get the number of fixed points wrong, as long as 'almost all' of them get it right. This approach will yield cohomologies with coefficients in pseudofinite fields of characteristic zero.

We must emphasise here that our development depends on the full power of the theory of l-adic cohomology such as Deligne's 'théorème du pgcd' and hard Lefschetz theorem ([**4**]) through work of Gabber ([**7**]), who shows that, for example, hard Lefschetz holds for all but finitely many torsion coefficients. It is unknown whether the relevant properties for torsion coefficients can be obtained directly, with no 'excursions' into the l-adic theory.

A different approach was proposed in [**12**], where it was shown that the definition of Weil cohomology is elementary (the axiomatisation is first order) and new

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cohomology theories can be obtained from the existing ones by taking ultraproducts. This is achieved by painstakingly showing certain uniformities in families of varieties. In our paper these uniformities are not relevant so technical difficulties tend to disappear. We, on the other hand, start from cohomologies with torsion coefficients which are not Weil cohomology theories themselves, and show that their ultraproduct is.

Our construction being 'natural', it is not surprising that people have thought of similar things in the past; upon circulating a preliminary version of this note I was informed that Serre used ultraproducts of cohomologies with prime torsion coefficients to obtain a representation of the absolute Galois group of the ground field (assumed to be a finite type extension of $\mathbb{Q}$) in [**15**].

2. *Model theoretic prerequisites*

The model theory used in this note is very basic, but in hope that not only model-theorists will be interested to read it, we feel obliged to keep the discussion fairly self-contained and provide all the relevant definitions.

The most common way of representing a vector space over a field k in model theory is to consider it as a structure $\langle V, +, 0, \phi_\lambda : \lambda \in k\rangle$, i.e., as an abelian group $\langle V, +\rangle$ with zero element 0 equipped with linear functionals ϕ_λ representing multiplication by the scalar λ.

Such a representation, however, does not prove itself useful for our purposes since we cannot even express finite-dimensionality of such vector spaces using a first order sentence. That is why we consider vector spaces as 2-sorted structures (to use model theoretic language) $\langle\langle V, +_V\rangle, \langle k, +_k, \cdot_k\rangle, \cdot\rangle$, where $\langle V, +_V\rangle$ is an abelian group, $\langle k, +_k, \cdot_k\rangle$ is a field, and $\cdot : k \times V \to V$ stands for multiplication by scalars. Given an integer n, it is now easy to express that V is of dimension less than n in a first order way:

$$\forall y_1, \ldots, y_n \; \exists \lambda_1, \ldots, \lambda_n \text{ not all } 0, \; \lambda_1 y_1 + \cdots + \lambda_n y_n = 0.$$

As we will mostly deal with finite-dimensional vector spaces and they can be interpreted in k^{eq} of the field k, model theorists will know what to make of this.

Let us describe the ultraproduct construction at the level of generality needed in the next section.

Definition 1.

(i) Given a collection of vector spaces $\{\langle V_i, k_i\rangle : i \in I\}$ and a non-principal ultrafilter $\mathcal{U}$ on I, we let the *ultraproduct* $\prod_{i\in I}\langle V_i, k_i\rangle/\mathcal{U}$ be the vector space $\langle V, k\rangle$, where:

(a) $V = \prod_i V_i/\sim'$ and $k = \prod_i k_i/\sim$, where $(v_i) \sim' (w_i)$ if $\{i : v_i = w_i\} \in \mathcal{U}$, and $(\lambda_i) \sim (\mu_i)$ if $\{i : \lambda_i = \mu_i\} \in \mathcal{U}$; let $[v_i]$ and $[\lambda_i]$ stand for $(v_i)/\sim'$ and $(\lambda_i)/\sim$, respectively;

(b) in k:

$$\begin{array}{lll} [\lambda_i] + [\mu_i] = [\eta_i] & \text{if} & \{i : \lambda_i + \mu_i = \eta_i \text{ in } k_i\} \in \mathcal{U}; \\ [\lambda_i][\mu_i] = [\eta_i] & \text{if} & \{i : \lambda_i\mu_i = \eta_i \text{ in } k_i\} \in \mathcal{U}; \end{array}$$

(c) in V:

$$\begin{array}{lll} [u_i] + [v_i] = [w_i] & \text{if} & \{i : u_i + v_i = w_i \text{ in } V_i\} \in \mathcal{U}; \\ [\lambda_i][v_i] = [w_i] & \text{if} & \{i : \lambda_i v_i = w_i \text{ in } \langle V_i, k_i\rangle\} \in \mathcal{U}. \end{array}$$

(ii) Given a collection of linear maps $\{V_i \xrightarrow{f_i} W_i : i \in I\}$ and a non-principal ultrafilter $\mathcal{U}$, we let their *ultraproduct* $\prod_{i\in I} f_i/\mathcal{U} : \prod_{i\in I} V_i/\mathcal{U} \to \prod_{i\in I} W_i/\mathcal{U}$ be the map f given by

$$f([v_i]) = [f_i(v_i)].$$

In particular, part (i)(b) defines the field k as the ultraproduct of k_i.

It should be clear how to form ultraproducts in the categories of groups, rings, graded vector spaces (where we have sorts for the graded components), (graded) algebras, as well as other mathematical structures and maps.

REMARK 2. For algebraists who frown at the mention of ultrafilters we give an alternative definition, following [**11**]. The set $\mathfrak{m} := \{(\lambda_i) : \{i : \lambda_i = 0\} \in \mathcal{U}\}$ is a maximal ideal in $R := \prod_i k_i$, the ultraproduct k is just $R/\mathfrak{m}$ and the ultraproduct V turns out to be $k \otimes_R \prod_i V_i$.

REMARK 3. It is a well-known fact in model theory that any first-order formula $\varphi(x_1, \ldots, x_m)$ will be satisfied by $(a_i^1), \ldots, (a_i^m)$ in the ultraproduct $\langle V, k\rangle$ if and only if $\{i : \varphi(a_i^1, \ldots, a_i^m)$ is satisfied in $\langle V_i, k_i\rangle\} \in \mathcal{U}$.

Let us recall some functorial properties of ultraproducts. The proofs are straightforward consequences of the definitions.

FACT 4. The following properties are true in any category of first-order structures where they make sense. In particular, (i)–(iii) are true in an arbitrary category, (iv) and the first part of (v) are associated with abelian categories, the second part of (v) we shall use for vector spaces, and (vi) for finite dimensional graded algebras. Let U_i, V_i, W_i be indexed by the set I and let $\mathcal{U}$ be a non-principal ultrafilter on I.

(i) If each $U_i = U$, there exists an elementary (in model-theoretic sense) embedding $U \to U^I/\mathcal{U} := \prod_i U_i/\mathcal{U}$ of U into its *ultrapower*, called the *canonical embedding*. For every map $f : U \to V$ we have the following commutative diagram:

$$\begin{tikzcd} U^I/\mathcal{U} \arrow[r, "\bar{f}"] & V^I/\mathcal{U} \\ U \arrow[u] \arrow[r, "f"'] & V \arrow[u] \end{tikzcd}$$

where the vertical arrows are canonical embeddings and $\bar{f}$ is the ultrapower of f. If f is elementary, so is $\bar{f}$.

(ii) If $\{i : U_i \to U_i$ is the identity map$\} \in \mathcal{U}$, the ultraproduct map

$$\prod_i U_i/\mathcal{U} \to \prod_i U_i/\mathcal{U}$$

is identity too.

(iii) If the set of indices i for which

$$\begin{tikzcd} U_i \arrow[rr] \arrow[dr] & & W_i \\ & V_i \arrow[ur] & \end{tikzcd}$$

commutes belongs to the ultrafilter $\mathcal{U}$, then the ultraproduct diagram is commutative:

$$\begin{tikzcd} \prod_i U_i/\mathcal{U} \arrow[rr] \arrow[dr] & & \prod_i W_i/\mathcal{U} \\ & \prod_i V_i/\mathcal{U} \arrow[ur] & \end{tikzcd}$$

(iv) If $\{i : 0 \to U_i \to V_i \to W_i \to 0$ is exact$\} \in \mathcal{U}$, the ultraproduct sequence

$$0 \to \prod_i U_i/\mathcal{U} \to \prod_i V_i/\mathcal{U} \to \prod_i W_i/\mathcal{U} \to 0$$

is exact.

(v) There is a canonical isomorphism:

$$\prod_i (U_i \times V_i)/\mathcal{U} \simeq \prod_i U_i/\mathcal{U} \times \prod_i V_i/\mathcal{U}.$$

Furthermore, if $\{i : U_i \times V_i \to k_i$ is a perfect pairing$\} \in \mathcal{U}$, the induced map

$$\prod_i U_i/\mathcal{U} \times \prod_i V_i/\mathcal{U} \to \prod_i k_i/\mathcal{U}$$

is also a perfect pairing.

(vi) If U_i and V_i are finite-dimensional (over k_i) of bounded dimension, there is a canonical isomorphism:

$$\prod_i (U_i \otimes_{k_i} V_i)/\mathcal{U} \simeq \prod_i U_i/\mathcal{U} \otimes_{\prod_i k_i/\mathcal{U}} \prod_i V_i/\mathcal{U}.$$

DEFINITION 5. An infinite field F is *pseudofinite*, if:

(i) F is perfect;
(ii) the absolute Galois group of F is $\hat{\mathbb{Z}}$;
(iii) F is *pseudo-algebraically closed (PAC)*, i.e., every absolutely irreducible variety over F has an F-rational point.

Principal examples of pseudofinite fields arise as ultraproducts of finite fields; in particular, if the fields in the ultraproduct vary in characteristic, the result will be a field of characteristic zero. For more information on ultraproducts and pseudofinite fields, see [**1**] and [**6**].

3. *The construction*

In what follows, k is a field of characteristic p and X is a separated scheme of finite type over k. We will follow the notation of [**10**] and we adopt the definition of Weil cohomology theory given there.

DEFINITION 6. Let I be an infinite set of prime numbers, $\mathcal{U}$ a non-principal ultrafilter on I and let F be the pseudofinite field $\prod_{\ell\in I} \mathbb{F}_\ell/\mathcal{U}$. Let $H_c^*(X, F) := \prod_\ell H_c^*(X \otimes_k \bar{k}, \mathbb{F}_\ell)/\mathcal{U}$ (we think of it as an ultraproduct of finite-dimensional graded $\mathbb{F}_\ell$-algebras as multi-sorted structures, as explained in the previous section).

If $f : X \to Y$ is a morphism between separated schemes of finite type over k, we let $f^* := H_c^*(f, F) : H_c^*(Y, F) \to H_c^*(X, F)$ be the ultraproduct of the induced maps f_ℓ^* in $\mathbb{F}_\ell$-cohomology. We define $H^*(X, F)$ and the corresponding maps analogously.

THEOREM 7. *The assignment $X \mapsto H^*(X) := H^*(X,F)$ is a contravariant functor from the category of irreducible, smooth, projective varieties X over an algebraically closed field to the category of graded anticommutative algebras over the pseudofinite field F of characteristic zero, with the following properties:*

(i) *(finiteness) Each $H^i(X)$ has finite dimension and vanishes unless $0 \le i \le 2d$, where $d = \dim X$.*

(ii) *(Poincaré duality) For each X of dimension d, there is a functorial 'orientation' isomorphism $H^{2d}(X) \simeq K$ which yields a non-degenerate cup-product pairing:*

$$H^i(X) \times H^{2d-i}(X) \to H^{2d}(X) \simeq K.$$

Given a Y of dimension e and a map $f : X \to Y$, we shall denote by $f_* : H^i(X) \to H^{2e-2d+i}(Y)$ the transpose of $f^* : H^{2d-i}(Y) \to H^{2d-i}(X)$.

(iii) *(Künneth formula) For each X and Y, the projections induce an isomorphism*

$$H^*(X) \otimes H^*(Y) \simeq H^*(X \times Y).$$

(iv) *(cycle map) For each X, let $C^i(X)$ denote the group of algebraic cycles of codimension i. Then there is a group homomorphism*

$$\gamma_X : C^i(X) \to H^{2i}(X),$$

called the 'cycle map', satisfying

(a) *(functoriality) for each map $X \to Y$, $f^*\gamma_Y = \gamma_X f^*$ and $f_*\gamma_X = \gamma_Y f_*$;*

(b) *(multiplicativity) $\gamma_{X\times Y}(Z \times W) = \gamma_X(Z) \otimes \gamma_Y(W)$;*

(c) *(calibration) if P is a point, then $\gamma_P : C^0(P) \to H^0(P)$ is equal to the canonical inclusion of the integers $\mathbb{Z}$ into the coefficient field K.*

(v) *(weak Lefschetz theorem) Let $h : W \to X$ be the inclusion of a smooth hyperplane section, and set $r := \dim X$. Then the induced map $h^* : H^i(X) \to H^i(W)$ is an isomorphism for $i \le r-2$ and an injection for $i = r-1$.*

(vi) *(hard Lefschetz theorem) Let W be a smooth hyperplane section of X, set $d := \dim X$, and define the Lefschetz operator $L : H^i(X) \to H^{i+2}(X)$ by $Lx := x\cdot\gamma_X(W)$. Then, for $i \le d$, the $(d-i)$-th iterate of L is an isomorphism $L^{d-i} : H^i(X) \simeq H^{2d-i}(X)$.*

Proof. The idea of the proof is straightforward: we note that every axiom holds for all except maybe finitely many prime torsion coefficients and then use Fact 4 to deduce the corresponding axiom for the ultraproduct cohomology.

Considering only smooth projective varieties over algebraically closed fields, we may disregard the distinction between H^i and H^i_c. By fixing a noncanonical isomorphism $\mathbb{F}_\ell \simeq \mathbb{F}_\ell(1)$, we can omit all Tate twists. Also, for ease of notation, we may assume that the characteristic $p \notin I$.

For (i), since all $H^i_c(X,\mathbb{F}_\ell)$ are of finite dimension (see [**8**] XIV, 1.2 and XII, 5.4 and 5.3; [**3**], Théorèmes Fin., 1.9(ii) and 1.10; [**8**] XIV, 3.2 and X, 4.2) bounded by some D (boundedness follows from [**7**]; see also Remark 2.8 in [**14**]), the sentence stating that any D vectors must be linearly dependent is true in all of them and therefore in $H^i(X)$ as well.

(ii) By [**8**] XVIII, 2.9, 2.14, 3.2.6.2, we have a nondegenerate bilinear pairing $H^i_c(X,\mathbb{F}_\ell) \times H^{2d-i}_c(X,\mathbb{F}_\ell) \to \mathbb{F}_\ell$, for all $\ell \in I$. Fact 4(v) yields the desired pairing in the F-cohomology.

(iii) If we let $U_\ell := H^*_c(X,\mathbb{F}_\ell)$, $V_\ell := H^*_c(Y,\mathbb{F}_\ell)$ and $W_\ell := H^*_c(X\times Y,\mathbb{F}_\ell)$, by [**8**] XVII, 5.4.3; [**3**], Théorèmes Fin., 1.11, we know that for each $\ell \in I$, $U_\ell \otimes_{\mathbb{F}_\ell} V_\ell \simeq W_\ell$, induced by projections. By 4(vi), projections will also induce the required isomorphism

$$\prod_\ell U_\ell/\mathcal{U} \otimes_F \prod_\ell V_\ell/\mathcal{U} \simeq \prod_\ell W_\ell/\mathcal{U}.$$

(iv)(a), (iv)(b) Even more is true, although the fact that the cycle map preserves products follows from the other properties: when $(2\dim(X)-1)!$ is invertible in $\mathbb{F}_\ell$, the map $\gamma^\ell_X : C^*(X) \to H^*(X,\mathbb{F}_\ell)$ is a homomorphism of graded rings that is functorial in X and multiplicative, see [**13**], 10.7, 9.4 and [**3**], Cycle. Everything lifts to ultraproducts in the spirit of Fact 4, the cycle map γ_X being the composite

$$C^*(X) \to C^*(X)^I/\mathcal{U} \to \prod_\ell H^*(X,\mathbb{F}_\ell)/\mathcal{U}$$

of the canonical embedding followed by the ultraproduct of the γ^ℓ_X.

(iv)(c) follows from the fact that for each n, if $n < \ell$, then for $\gamma^\ell_P : C^i(P) \to H^0(P,\mathbb{F}_\ell)$, $\gamma^\ell_P(n[P]) = 1+\cdots+1$ (n-times) in $\mathbb{F}_\ell$ (loc. cit.), so the ultraproduct cohomology will 'count' correctly.

(v) Weak Lefschetz holds in $\mathbb{F}_\ell$-cohomology for all $\ell \in I$ by [**8**] XIV, 3.3, and it will clearly hold for ultraproducts.

And lastly, (vi) follows from the previous properties and [**9**] whenever the ground field is the algebraic closure of a finite field. For the general case, we have obtained the proof of the following fact (announced in 'Complement' of [**7**]) through personal correspondence with Ofer Gabber:

Let X be a smooth projective variety of pure dimension d over an algebraically closed field k, $\mathcal{O}_X(1)$ an ample line bundle, $L = c_1(\mathcal{O}_X(1))$. Then there is an integer $n_X > 0$ such that n_X annihilates the torsion in $H^i(X,\mathbb{Z}_\ell)$ (for all i and all $\ell \neq \mathrm{char}(k)$) and the cokernels (and kernels) of the hard Lefschetz map

$$H^{d-i}(X,\mathbb{Z}_\ell) \xrightarrow{L^i\cup} H^{d+i}(X,\mathbb{Z}_\ell(i)).$$

If ℓ does not divide n_X, it follows that hard Lefschetz holds with coefficients $\mathbb{F}_\ell$. Thus, for a given X, the number of $\ell \in I$ where hard Lefschetz fails is finite, so it lifts to the ultraproduct. □

REMARK 8. Let X be a projective smooth geometrically irreducible variety over $\mathbb{F}_q$ (of characteristic p), let $\bar{X} = X \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$ and let $F \in \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ be the geometric Frobenius (inverse of $x \mapsto x^q$). Pick a non-principal ultrafilter $\mathcal{U}$ on prime numbers, determining a cohomology theory $H^*(\cdot,F)$, where $F = \prod_\ell \mathbb{F}_\ell/\mathcal{U}$.

Let $\mathcal{P}_i(T) = \det(1-TF|H^i(X,F))$, $P_i(T) = \det(1-TF|H^i(\bar{X},\mathbb{Q}_l))$ (for some $l \neq p$) and $\tilde{P}^\ell_i(T) = \det(1-TF|H^i(\bar{X},\mathbb{F}_\ell))$ be various characteristic polynomials of the Frobenius. Let $\bar{P}^\ell_i(T)$ be the reduction modulo ℓ of $P_i(T)$.

Deligne has shown in [**2**] that $P_i(T)$ has integer coefficients independent of the choice of $l \neq p$. By our Theorem 7 and [**9**], it follows that $\mathcal{P}_i(T) = P_i(T)$, independently of the choice of the ultrafilter $\mathcal{U}$. Even stronger is the result from [**7**] which shows that for all but finitely many $\ell \neq p$, $\tilde{P}^\ell_i(T) = \bar{P}^\ell_i(T)$ and thus, for a given variety, almost all prime torsion coefficients get the characteristic polynomial right.

The above remark shows that the pseudofinite cohomology is as useful as the

l-adic one when dealing with issues around the Weil conjectures. It remains to be seen whether developing the pseudofinite theory from first principles can provide an alternative route to the Weil conjectures, distinct to the l-adic theory.

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