CONSTRUCTIBLE SHEAVES AND DEFINABILITY

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1. Introduction

Several applications of model-theoretic methods in the theory of cohomology have appeared recently, most probably influenced by ideas of Macintyre. In his programmatic paper [10], he shows that, after expanding the language of rings by certain sorts and predicates for cohomology, the axioms of Weil cohomology theories are first order and one can form new cohomology theories as ultraproducts of already existing ones.

The present author has remarked that, although the cohomologies with torsion coefficients do not satisfy the axioms for a Weil cohomology theory individually, they do so 'on average', and one can obtain cohomologies with coefficients in pseudofinite fields of characteristic zero by taking ultraproducts ([14]). He shows that this 'pseudofinite cohomology' is at least as good as the l-adic theory when dealing with issues around the Weil conjectures. In parallel, Brunjes and Serpe developed the theory of nonstandard sheaves systematically, and they even show that the pseudofinite cohomology is better behaved than the l-adic one, the former being a derived functor cohomology ([1]).

The purpose of this short note is to clarify which aspects and invariants of the theory of (étale) constructible sheaves and cohomologies are definable *in the language of rings.* It should serve as a bridge between the algebraic-geometric and model-theoretic language and should encourage model-theorists to use the sophisticated techniques already developed by geometers. We show that, in case one needs to consider an invariant defined in terms of constructible sheaves over a finite or a pseudofinite ground field, there is a good chance that it is definable.

We use the standard notation of algebraic geometry. For a field k, $\bar{k}$ denotes the algebraic closure of k, and for a scheme X over k, $\bar{X}$ denotes $X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\bar{k})$. A *variety* over S is a reduced and separated scheme of finite type over S.

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2. Fundamental group and constructible sheaves

We recall the basic definitions and facts from algebraic geometry in an effort to keep the presentation relatively self-contained. For more details we refer the reader to [5], [11] or the widely available manuscript [12].

Given a connected scheme X and a geometric point ξ of X (i.e., a point with values in some algebraically closed field), we have the profinite étale fundamental group $\pi_1(X, \xi)$, which classifies finite étale coverings of X ([12], Section 3). This gives a covariant functor on the category of pointed schemes (X, ξ). As in topology, varying ξ just changes $\pi_1(X, \xi)$ up to an inner automorphism. Thus we shall

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usually omit the base point, writing $\pi_1(X)$, when we only require calculations up to conjugacy.

In the special case $X = \mathrm{Spec}(k)$, where k is a field, a geometric point ξ is just a choice of an algebraically closed overfield L of k, and $\pi_1(X, \xi)$ is just the Galois group $\mathrm{Gal}(k^{\mathrm{sep}}/k)$, where k^{sep} is the separable closure of k in L. Another interesting special case is that of normal X. If a connected variety X is normal, it is irreducible, say with generic point η. Its function field K is the residue field $\mathbf{k}(\eta)$. If we view an algebraic closure $\bar{K}$ of K as a geometric generic point $\bar{\eta}$ of X, the group $\pi_1(X, \bar{\eta})$ is the quotient of $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ which classifies those finite separable extensions L/K with the property that the normalisation of X in L is finite étale over X (i.e., unramified).

Given a connected scheme X, a field k and a k-valued point $x \in X(k)$, the associated morphism $\mathrm{Spec}\, k \to X$ induces a group homomorphism

$$\pi_1(\mathrm{Spec}(k)) = \mathrm{Gal}(k^{\mathrm{sep}}/k) \to \pi_1(X),$$

well-defined up to conjugacy. The image of this map is called the *Artin symbol* at x and is denoted by $\mathrm{Ar}(x)$. We shall discuss several incarnations of it in Section 3.

If X is geometrically irreducible, we have the short exact sequence for the fundamental group:

$$1 \longrightarrow \pi_1^{\mathrm{geom}}(X, \bar{\eta}) \longrightarrow \pi_1(X, \bar{\eta}) \longrightarrow \mathrm{Gal}(k^{\mathrm{sep}}/k) \longrightarrow 1,$$

where $\pi_1^{\mathrm{geom}}(X, \bar{\eta}) = \pi_1(X \times_k \bar{k}, \bar{\eta})$ is the *geometric* fundamental group. As a contrast, π_1 is sometimes called the *arithmetic* fundamental group.

Let us denote by $\acute{\mathrm{E}}(X)$ the category of étale extensions of a scheme X. An (étale) *presheaf* $\mathcal{F}$ on X in a category $\mathcal{C}$ (of sets, abelian groups, rings...) is a contravariant functor

$$\mathcal{F} : \acute{\mathrm{E}}(X) \to \mathcal{C}\,.$$

A presheaf $\mathcal{F}$ is an (étale) *sheaf,* if it satisfies these additional properties:

(1) $\mathcal{F}(U \dot{\cup} V) = \mathcal{F}(U) \times \mathcal{F}(V)$;
(2) when $U \to V$ is surjective, the following sequence is exact:

$$\mathcal{F}(V) \longrightarrow \mathcal{F}(U) \rightrightarrows \mathcal{F}(U \times_V U)$$

For an object F in $\mathcal{C}$, we can define the 'constant sheaf' F_X on X by putting

$$F_X(U) = F \times \cdots \times F = F^{n(U)}, \quad U \in \acute{\mathrm{E}}(X),$$

where $n(U)$ denotes the number of connected components of U. Any sheaf isomorphic to a sheaf of this form is called *constant.*

Definition 2.1 ([12], 6, [5], I.4)**.**
(1) A sheaf $\mathcal{F}$ on a scheme X is called *locally constant*, if there is an étale covering $U_i \to X$ such that each $\mathcal{F} \restriction U_i$ is constant. If the $\mathcal{F}(U_i)$ are all finite, we shall say that $\mathcal{F}$ is *finite locally constant.*
(2) A sheaf $\mathcal{F}$ on X is *constructible*, if X can be written as a union of finitely many locally closed subschemes $Y \subseteq X$ such that $\mathcal{F} \restriction Y$ is finite locally constant.

The following characterisation of locally constant sheaves is of great interest for our purpose.

Proposition 2.2 ([12], 6.16, [5], A I.7)**.** *Let $\mathcal{F}$ be a finite locally constant sheaf of abelian groups on a connected and normal scheme X. For any geometric point ξ of X, the stalk $\mathcal{F}_\xi$ is in a natural way a continuous $\pi_1(X,\xi)$-module. The assignment $\mathcal{F} \mapsto \mathcal{F}_\xi$ establishes an equivalence between the category of finite locally constant sheaves of abelian groups and the category of finite continuous $\pi_1(X,\xi)$-modules.*

In this context, *continuity* clearly implies that the action comes from a finite quotient of $\pi_1(X,\xi)$.

Let X be a connected and normal scheme and l a prime invertible in X. By analogy, we define a *lisse $\bar{\mathbb{Q}}_l$-sheaf* $\mathcal{F}$ of rank r on X as an r-dimensional continuous $\bar{\mathbb{Q}}_l$-representation of $\pi_1(X,\bar{\eta})$ (cf. [12], 19, [5], A I.8, [9], Appendix A). A *constructible $\bar{\mathbb{Q}}_l$-sheaf* $\mathcal{F}$ on X is given when X can be written as a union of finitely many locally closed subschemes U_i such that each $\mathcal{F} \restriction U_i$ is lisse. Note that in this case the action need not factor through a finite quotient. In our study of definability, however, we mostly consider the $\bar{\mathbb{Q}}_l$-sheaves with the *(FQ)-property*; a lisse $\bar{\mathbb{Q}}_l$-sheaf $\mathcal{F}$ on X is labelled (FQ) if it factors through a finite quotient of $\pi_1(X)$, while a constructible sheaf is called (FQ) if it is such on each piece of X where it is lisse.

Let $\mathcal{F}$ be a constructible or a constructible $\bar{\mathbb{Q}}_l$-sheaf on X, let $x : \mathrm{Spec}(k) \to X$ be a k-valued point of X and let $\bar{x} : \mathrm{Spec}(k^{\mathrm{sep}}) \to X$ be a geometric point lying over x. Since the underlying module of the *stalk* $\mathcal{F}_{\bar{x}}$ coincides with that of the pullback $x^*\mathcal{F}$, which can (tautologically) be considered as a sheaf on $\mathrm{Spec}(k)$, the stalk $\mathcal{F}_{\bar{x}}$ is naturally endowed with the action of $\mathrm{Gal}(k^{\mathrm{sep}}/k) = \pi_1(x,\bar{x})$.

3. Galois formulae vs. formulae

We start with a general consideration of Galois stratification, developed in [7], [6], but adopting the more geometric language as in [4].

Let A be an integral and normal variety. A morphism of varieties $C \to A$ is a *Galois cover*, if C is integral, h is étale, and there is a finite group $G = G(C/A)$ acting on C such that h induces the isomorphism $C/G \simeq A$.

A Galois cover $C \to A$ is *coloured*, if $G(C/A)$ is equipped with a family Con of subgroups stable by conjugation. Let S be an integral normal scheme and let $X \to S$ be a variety over S. A *normal stratification* of X,

$$\langle X, C_i/A_i : i \in I \rangle,$$

is a partition of X into a finite set of integral and normal locally closed S-subschemes A_i, each equipped with a Galois cover $C_i \to A_i$. A *Galois stratification*

$$\mathcal{A} = \langle X, C_i/A_i, \mathrm{Con}(A_i) | i \in I \rangle$$

consists of a normal stratification in which each Galois cover C_i/A_i is coloured (by $\mathrm{Con}(A_i)$).

Let S be an integral and normal scheme and let $X \to S$ be a variety over S. For a field k, a point $s \in S$, associated with a morphism $\mathrm{Spec}\, k \to S$, we denote by X_s the fibre of X over s ($X_s = X \times_S \mathbf{k}(s)$). Let $\mathcal{A} = \langle X, C_i/A_i, \mathrm{Con}(A_i) | i \in I \rangle$ be a Galois stratification of X, let $s \in S(k)$, and let $x \in A_{i,s}$. The *Artin symbol*, $\mathrm{Ar}(C_i/A_i, s, x)$ is the conjugacy class of subgroups of $G(C_i/A_i)$ consisting of the decomposition subgroups at x. More precisely, considering the map corresponding to x, $\mathrm{Spec}(k) \to A_{i,s} \to A_i$, we have the induced map

$$\mathrm{Gal}(k^{\mathrm{sep}}/k) \to \pi_1(A_{i,s}) \to \pi_1(A_i) \to G(C_i/A_i),$$

and $\mathrm{Ar}(C_i/A_i, s, x)$ is its image, defined up to conjugacy.

Let $\mathcal{A}$ be a Galois stratification over S, as above. We will call an expression of the form

$$\mathcal{A} := \{x \in X : \mathrm{Ar}(x) \subseteq \mathrm{Con}(\mathcal{A})\}$$

a *Galois formula* over S. For a field k, $s \in S(k)$ and $x \in A_{i,s}(k)$, we write $\mathrm{Ar}(x) \subseteq \mathrm{Con}(\mathcal{A})$ for $\mathrm{Ar}(C_i/A_i, s, x) \subseteq \mathrm{Con}(C_i/A_i)$, and we consider the *set of k-valued points of $\mathcal{A}_s$*,

$$\mathcal{A}_s(k) := \{x \in X_s(k) : \mathrm{Ar}(x) \subseteq \mathrm{Con}(\mathcal{A})\}.$$

Formulae as above will be referred to as *Galois formulae*. The following result shows that every Galois formula corresponds to a formula. The converse is true only under additional assumptions on the base field, cf. [7].

Proposition 3.1. *Let $Y \to X$ be a Galois cover over S with group G, and let $\mathcal{D} \subseteq G$ be a conjugacy domain of subgroups. There exists a formula $\theta^{\mathcal{D}}$ over S such that for every field k, and every $s \in S(k)$,*

$$\{x \in X_s(k) : \mathrm{Ar}(x) \subseteq \mathcal{D}\} = \{x \in X_s(k) : k \models \theta^{\mathcal{D}}_s(x)\}.$$

This result is stated under more restrictive hypotheses in [6], but they are not needed in the proof. For the convenience of the reader, we repeat the proof in our notation.

Proof. We can reduce to the case of affine varieties and we may assume that $\mathcal{D}$ is *full*, i.e., that it contains all subgroups of each group in $\mathcal{D}$. Indeed, if $\mathcal{D}$ is a single conjugacy class of groups, then $\mathcal{D} = \mathcal{D}' \setminus \mathcal{D}''$, where $\mathcal{D}'$ is the conjugacy domain of all subgroups of the groups in $\mathcal{D}$ and $\mathcal{D}''$ is the conjugacy domain of all proper subgroups of the groups in $\mathcal{D}$. Then $\theta^{\mathcal{D}} \equiv \theta^{\mathcal{D}'} \wedge \neg\theta^{\mathcal{D}''}$. In general, a conjugacy domain is a union of conjugacy classes and the required formula will be a disjunction of formulae as above.

Consider the Galois extension F/E of function fields of Y and X. For each subgroup H of G, let E_H be the fixed field of H in F and let X_H be the normalisation of X in E_H. Clearly, Y/X_H is a Galois cover with group H.

Let $x \in X_s(k)$, and let $\bar{x}$ be a geometric point over x. We claim that for every H, the image of $\pi_1(x, \bar{x}) \to \pi_1(X, \bar{x}) \to G$ is a subgroup of H if and only if there exists a $z \in X_H(k)$ that maps onto x. This will then imply (when we forget to specify the base point $\bar{x}$) that $\mathrm{Ar}(x) \subseteq \mathcal{D}$ is equivalent to

$$\bigvee_{H \text{ max. in } \mathcal{D}} \exists z \in X_H(k)\ z \mapsto x,$$

the latter condition being definable by a formula over S.

To prove the claim, let us choose geometric points $\bar{y}$ in Y and $\bar{z}$ in X_H so that $\bar{y} \mapsto \bar{z} \mapsto \bar{x}$. If the image of $\pi_1(x, \bar{x})$ is contained in H and H fixes Z, then $\mathrm{Gal}(k)$ fixes $\bar{z}$ so we have found a k-rational point of X_H mapping onto x.

Conversely, if there is such a z, we have the diagram (the last column is obtained by quotienting the middle column by $\pi_1(Y, \bar{y})$):

$$\begin{array}{ccccc}
\pi_1(z, \bar{z}) & \longrightarrow & \pi_1(X_H, \bar{z}) & \longrightarrow & H \\
\downarrow & & \downarrow & & \downarrow \\
\pi_1(x, \bar{x}) & \longrightarrow & \pi_1(X, \bar{x}) & \longrightarrow & G
\end{array}$$

Since the left vertical arrow is an isomorphism, the image of $\pi_1(x, \bar{x})$ sits in H. $\square$

Corollary 3.2. *Let $\mathcal{F}$ be a constructible sheaf (resp. a constructible $\bar{\mathbb{Q}}_l$-sheaf with (FQ)) on a variety X over S. There is a finite number of finite modules (resp. finite-dimensional vector spaces $\bar{\mathbb{Q}}_l$) M_i endowed with group actions such that for every field k, $s \in S(k)$, and $x \in X_s(k)$, the action of $\mathrm{Ar}(x)$ on $\mathcal{F}_x$ is isomorphic to some M_i. Moreover, for each M_i there is a formula θ^{M_i} over S such that*

$$\{x \in X_s(k) : \text{the action of } \mathrm{Ar}(x) \text{ on } \mathcal{F}_x \text{ is isomorphic to } M_i\} \\ = \{x \in X_s(k) : k \models \theta_s^{M_i}(x)\}.$$

Proof. Since $\mathrm{Ar}(x)$ is a conjugacy class of subgroups, it does make sense to talk about the isomorphism class of the action as above and it is clear that the isomorphism type is fixed on a union of conjugacy classes (a conjugacy domain) of subgroups. The sheaf has (FQ) using either assumption and on each stratum where it is locally constant (resp. lisse) we only need to worry about the actions obtained by restriction of the action of a finite quotient of the fundamental group to its subgroups. Definability follows directly from the proposition. □

In model theory of fields, one often studies a class of fields with a specific structure of their absolute Galois groups. Although 3.2 looks quite interesting, it cannot be used to, say, compute traces or characteristic polynomials of some distinguished element of the Galois group of the ground field on the stalk $\mathcal{F}_{\bar{x}}$, unless one expands the language with symbols which allow it to be mentioned, as in [8]. To overcome this weakness of the language of rings, one is forced to restrict the class of acceptable sheaves, as expounded in the following definition.

Definition 3.3. Let $\mathcal{K}$ be a class of fields containing a given field k_0. Let $\mathcal{F}$ be a constructible or a constructible $\bar{\mathbb{Q}}_l$-sheaf on a variety X over k_0. We shall say that $\mathcal{F}$ is *$\mathcal{K}$-invariant*, if for every $k \in \mathcal{K}$, for every k-rational point $x \in X(k)$, any two geometric points $\bar{x}_1, \bar{x}_2 \in X(k^{\mathrm{sep}})$ lying over x, and every group isomorphism $\varphi : \pi_1(x, \bar{x}_1) \to \pi_1(x, \bar{x}_2)$, we have the isomorphism of $\mathrm{Gal}(k) = \pi_1(x, \bar{x}_1)$-modules

$$\varphi^* \mathcal{F}_{\bar{x}_2} \simeq \mathcal{F}_{\bar{x}_1}.$$

Motivated by a part of the above definition, when two isomorphic groups G and G' act on modules M and M', we shall say that these actions are *strongly isomorphic* if for *every* isomorphism $\varphi : G \to G'$, the G-modules $\varphi^* M'$ and M are isomorphic.

Theorem 3.4. *Let $\mathcal{K}$ be a class of fields. Let $\mathcal{F}$ be a $\mathcal{K}$-invariant constructible sheaf (resp. constructible $\bar{\mathbb{Q}}_l$-sheaf with (FQ)) on a variety X over S. There is a finite number of finite modules (resp. finite-dimensional vector spaces over $\bar{\mathbb{Q}}_l$) M_i endowed with group actions such that for every $k \in \mathcal{K}$, $s \in S(k)$, and $x \in X_s(k)$, the action of $\mathrm{Gal}(k)$ on $\mathcal{F}_x$ is strongly isomorphic to some M_i. Moreover, for each M_i there is a formula θ^{M_i} over S such that*

$$\{x \in X_s(k) : \text{the action of } \mathrm{Gal}(k) \text{ on } \mathcal{F}_x \text{ is strongly isomorphic to } M_i\} \\ = \{x \in X_s(k) : k \models \theta_s^{M_i}(x)\}.$$

Proof. Having in mind that the action of $\mathrm{Gal}(k)$ on $\mathcal{F}_{\bar{x}}$ is the pullback of the action of $\mathrm{Ar}(x)$, we argue as in 3.2. By k-invariance, on each stratum Y where $\mathcal{F}$ is locally constant, the left hand side of the above equality is of the form

$$\{x \in Y_s(k) : Ar(x) \subseteq \mathcal{D}\},$$

for some conjugacy domain $\mathcal{D}$. In other words, this set is given by a Galois formula over S. By 3.1, there is a formula over S equivalent to it. □

Remark 3.5. At the end of Section 2, we mentioned that there exist lisse $\mathbb{Q}_l$-sheaves which do not factor through a finite quotient of the fundamental group. It is intuitively clear that such sheaves encode information contained in an infinite sequence of constructible sheaves and that we cannot expect their invariants to be definable, but only possibly ∞-definable.

For example, let k be a field of characteristic different from a prime number l. For each n, consider the group T_n of l^n-th roots of unity in $\bar{k}$, together with the action of the absolute Galois group G of k. Then, $T := \varprojlim_n T_n \simeq \mathbb{Z}_l$ is a continuous G-module in its l-adic topology, and we obtain a lisse $\bar{\mathbb{Q}}_l$-sheaf as the representation of G on $T \otimes_{\mathbb{Z}_l} \bar{\mathbb{Q}}_l$. In particular, this sheaf 'knows' which l^n-th roots of unity are contained in k for all n, and this is clearly not something expressible by a single formula.

4. Applications

4.1. **Cohomology.** Let us recall some facts about étale cohomology groups, as one of the most important classes of locally constant sheaves. Let X be connected over a finite field k and let $\mathcal{F}$ be a constructible sheaf. We have *étale cohomology groups with compact support* (cf. [12], 18, [5], I.8),

$$H^i_c(\bar{X}, \mathcal{F}),$$

which are finite continuous $\mathrm{Gal}(k^{\mathrm{sep}}/k)$-modules, and which vanish unless $i \in \{0, \ldots, 2d\}$. We consider them as locally constant sheaves on k.

Suppose now that S is a connected normal variety over $\mathbb{Z}$, and let $\pi : X \to S$ be a normal and connected variety over S. Let $\mathcal{F}$ be a constructible sheaf on X. It is shown in algebraic geometry ([12], 18.4, [5], I.8.10) that the *higher direct images of $\mathcal{F}$ with compact support,* $R^i\pi_!\mathcal{F}$, are constructible sheaves on S. Given an algebraically closed field field Ω and any geometric point $\bar{s} \in S(\Omega)$, by the proper base change theorem ([12], 17.10, [5], I.6.1, I.8.7) we have the *specialisation* property:

$$(R^i\pi_!\mathcal{F})_{\bar{s}} = H^i_c(X_{\bar{s}}, \mathcal{F}_{\bar{s}}),$$

where $X_{\bar{s}}$ is the geometric fibre over $\bar{s}$, and $\mathcal{F}_{\bar{s}}$ is the restriction (pullback) of $\mathcal{F}$ to $X_{\bar{s}}$.

Theorem 4.1. *Let $\mathcal{F}$ be a (torsion!) constructible sheaf on a variety X over S which factors through a finite quotient of $\pi_1(X)$. There are finitely many finite modules M^r_i endowed with group actions such that for every field k, $s \in S(k)$ and every integer r, there exists an i such that $H^r_c(\bar{X}_s, \mathcal{F})$ is isomorphic to M^r_i. Moreover, for all i and r, there is a formula θ^r_i such that*

$$\{s \in S(k) : H^r_c(\bar{X}_s, \mathcal{F}) \simeq M^r_i\} = \{s \in S(k) : k \models \theta^r_i(s)\}.$$

Proof. The theorem follows from the étale cohomology facts mentioned above, namely the constructibility of higher direct images with proper supports and the specialisation property, together with 3.2. □

4.2. **Finite and pseudofinite fields.** Let us consider a property related to $\mathcal{K}$-invariance, which turns out to be particularly well-behaved in the case of finite and pseudofinite fields. The absolute Galois group of a finite or a pseudofinite field being isomorphic to $\hat{\mathbb{Z}}$, our attention is drawn to the consideration of properties invariant of the choice of a topological generator of the Galois group.

Definition 4.2. Let G be a profinite group. We shall call a function f on G $\mathbb{Q}$-central, if $f(x) = f(x')$ whenever x and x' topologically generate conjugate subgroups of G.

Remark 4.3. Let $\mathcal{K}$ be the class of finite or pseudofinite fields. Suppose $\mathcal{F}$ is a lisse $\bar{\mathbb{Q}}_l$-sheaf on a scheme S. Then $\mathcal{F}$ is $\mathcal{K}$-invariant if and only if its character is $\mathbb{Q}$-central.

For a (torsion) constructible sheaf, it can be rather intricate to talk about its character. For a commutative ring Λ (usually $\Lambda = \mathbb{Z}/n\mathbb{Z}$), it is clearly possible when it is a sheaf of free Λ-modules, but it can be shown that traces can be taken even in the case of projective Λ-modules or even perfect complexes of Λ-modules ([5], II.3). By a slight abuse of notation, we will call a constructible (torsion) sheaf $\mathbb{Q}$-central if it is $\mathcal{K}$-invariant.

We have isolated the property of $\mathbb{Q}$-centrality precisely for the reason that, given a $\mathbb{Q}$-central sheaf $\mathcal{F}$, the trace or the characteristic polynomial of σ on $\mathcal{F}_{\bar{x}}$ does not depend on the choice of the topological generator σ of the absolute Galois group of the ground field. Moreover, it is *hereditary* in the sense that it is preserved under taking direct images with proper supports.

Proposition 4.4. *Let $\mathcal{F}$ be a constructible (resp. constructible $\bar{\mathbb{Q}}_l$ with (FQ)) $\mathbb{Q}$-central sheaf on a variety $X \xrightarrow{\varphi} S$. Then the direct image with compact support, $\varphi_! \mathcal{F}$, is also $\mathbb{Q}$-central (resp. $\mathbb{Q}$-central and (FQ)).*

Proof. Firstly, we reduce to the case where φ is a proper map. Stein factorisation tells us that φ can be written as a composition of a finite map and a map with geometrically connected fibres. In both cases we have an explicit description of the stalks of the direct image and it is straightforward to verify that $\mathbb{Q}$-centrality (resp. the property (FQ)) is inherited. □

The above concepts are best applied in conjunction with the Grothendieck-Lefschetz fixed point formula and Deligne's theory of weights ([3]). Using these techniques, the author shows in [13] the following conceptual improvement of the results from [6] and [2].

Theorem 4.5. *Let S be a connected normal variety over $\mathbb{Z}$ and let $\mathcal{F}$ be a constructible $\bar{\mathbb{Q}}_l$-sheaf on a variety X over S which is $\mathbb{Q}$-central and (FQ). Let χ be its character and fix a non-canonical embedding $\iota : \bar{\mathbb{Q}}_l \to \mathbb{C}$. Then there is a localisation $\mathbb{Z}[1/n]$ of $\mathbb{Z}$, a finite number of (continuous) $\mathbb{Q}$-central characters $\alpha_i : \hat{\mathbb{Z}} \to \mathbb{C}$ and a constant $C > 0$ such that, writing S' for the restriction of S to $\mathbb{Z}[1/n]$, for every finite field k, its extension k_n of degree n, and every parameter $s \in S'(k)$, there exists an i with*

$$\left| \sum_{x \in X_s(k_n)} \iota\chi_s(F_{k_n,x}) - \alpha_i(n)|k_n|^{\dim(X_s)} \right| \leq C|k_n|^{\dim(X_s)-1/2},$$

where F_k *denotes the Frobenius* $x \mapsto x^{|k|}$ *and* $F_{k,x}$ *its image in* $\mathrm{Ar}(x)$, *the* local Frobenius *element.*

Moreover, for every i *there is a formula* θ_{α_i} *in the language of rings which defines, in each finite field* k*, the set of* $s \in S'(k)$ *for which the above estimate holds.*

For further applications in pseudofinite fields, such as showing the rationality of pseudofinite L-functions, results regarding pseudofinite version of Dirichlet density, we refer the reader to [13].

Question 4.6. Is $\mathbb{Q}$-centrality preserved under *higher* direct images with proper supports? What is the most general class of fields $\mathcal{K}$ (with respect to properties of their absolute Galois groups) such that $\mathcal{K}$-invariance is preserved under taking higher direct images?

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