# TEORIJA ELASTIČNOSTI

- TIJELO $\Omega \subseteq \mathbb{R}^3$ (REFERENTNA KONFIGURACIJA)
- ELASTIČNO
- DJELUJE SILA
- TIJELO SE DEFORMIRA $\varphi : \Omega \to \mathbb{R}^3$ DEFORMACIJA
- RAVNOTEŽNI POLOŽAJ (DEFORMIRANA KONFIGURACIJA)
- GIBANJE $\varphi : \Omega \times [0,T] \to \mathbb{R}^3$

I DIO : NIŽEDIMENZIONALNI MODELI

II DIO : 3D TEORIJA

LITERATURA:


1. AGANOVIĆ, K. VESELIĆ: JEDNADŽBE MATEMATIČKE FIZIKE, ŠKOLSKA KNJIGA, 1985.

P.G. CIARLET, MATHEMATICAL ELASTICITY, VOL I: THREE-DIMENSIONAL ELASTICITY, NORTH-HOLLAND 1988.

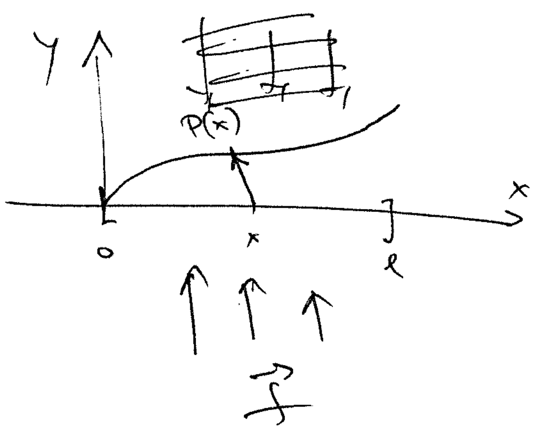
I. AGANOVIĆ, UVOD U RUBNE ZADAĆE MEHANIKE KONTINUUMA, ELEMENT, 2003.

# 1. MODEL NAPETE ŽICE

## 1.1. RAVNOTEŽA ŽICE


NEDEFORMIRANI (REFERENTNI) POLOŽAJ

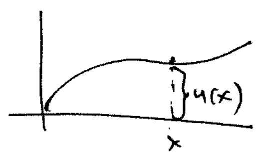
$f : [0,\ell] \to \mathbb{R}^2$ LINIJSKA GUSTOĆA SILE NA ŽICU ~~OKOMITA NA OS X~~




PRETPOSTAVKA 1:
ŽICA SE "MALO" DEFORMIRA

$P(x) \in \mathbb{R}^2$ POLOŽAJ $(x,0)$ NAKON DJELOVANJA SILE

PRETPOSTAVKA 2: $P(x) = (x, u(x))$




PRETPOSTAVKA 1: $|u'(x)| \ll 1$ (KVADRATIČNE VRIJEDNOSTI ZANEMARUJEMO)

(DEF. KRIVULJA BLISKA NEDEFORMIRANOJ)

$u'(x)$ MJERA (ELASTIČNE) DEFORMACIJE

NE ZANIMA NAS TRANSLACIJA (ROTACIJE NEMA)

$P'(x) \sim e_1$

$ds = u'(t)dt$

$x = u(t)$

$\int_{P([0,l])} ds = \int_0^l u'(t)\,dt$

MJERI RASTEZ

NAP:
$$u(x) = u(0) + \int_0^x u'(\xi)\,d\xi$$

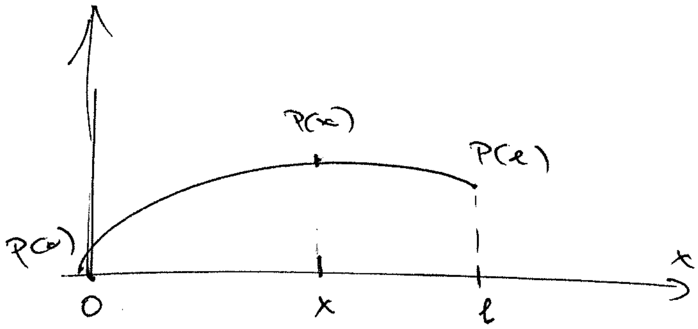
$$\Rightarrow \quad |u(x)-u(0)| \le \int_0^\ell |u'(\xi)|\,d\xi \ll \ell$$

$$\frac{|u(x)-u(0)|}{\ell} \ll 1$$ RELATIVNA OCJENA

# KONTAKTNA SILA

SILA KOJOM JEDAN KOMAD ŽICE DJELUJE NA DRUGI

$x \in [0,\ell]$  $\vec{g}(x)$ SILA KOJOM $\overline{P(x)P(\ell)}$ DJELUJE NA $\overline{P(0)P(x)}$

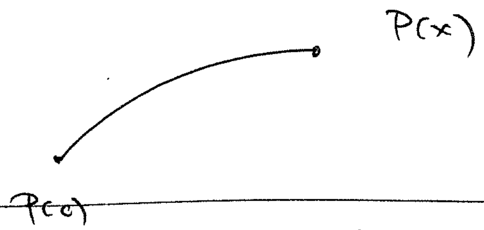



NAP: AKO UKLONIMO $\overline{P(x)P(\ell)}$ $\vec{g}(x)$ JE SILA KOJA $\overline{P(0)P(x)}$ OSTAVI ~~NA MIRU~~ U ISTOM POLOŽAJU

- $\vec{g}(\ell)$, $-\vec{g}(0)$ VANJSKE KONTAKTNE SILE
- $\vec{g}(x)$ UNUTARNJA KONTAKTNA SILA

PRINCIP RAVNOTEŽE:

ZBROJ SILA NA SVAKI KOMAD (DIO) TIJELA $= 0$




$$\boxed{\vec{g}(x) - \vec{g}(0) + \int_0^x \vec{f}(\xi)\,d\xi = 0}$$

ZAKON PONAŠANJA:

KONTAKTNA SILA JE TANGENCIJALNA NA ŽICU

$$\boxed{\vec{g}(x) = a(x)\,\vec{t}(x)}$$

NAP: $a$ MOŽE OVISITI o $u$
$\Rightarrow$ NELINEARNI MODEL

$\vec{t}(x)$ JEDINIČNA TANGENTA u $P(x)$

$a(x)$ NAPETOST ŽICE

$$\vec{t}(x) = \frac{(1, u'(x))}{\|(1, u'(x))\|} = \frac{(1, u'(x))}{\sqrt{1 + u'(x)^2}} \approx (1, u'(x))$$

IZ PR & ZP:

$$a(x)\,\vec{t}(x) - a(0)\,\vec{t}(0) + \int_0^x \vec{f}(\xi)\,d\xi = 0$$

$\Rightarrow$

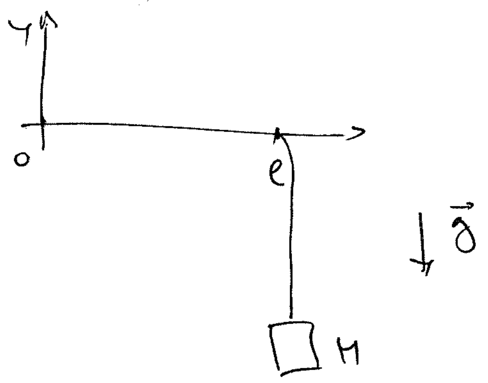
(1) $$a(x) - a(0) + \int_0^x f_x(\xi)\,d\xi = 0$$ JEDNADŽBA ZA NAPETOST

(2) $$a(x)\,u'(x) - a(0)\,u'(0) + \int_0^x f_y(\xi)\,d\xi = 0$$ JEDNADŽBA RAVNOTEŽE INTEGRALNI OBLIK

NAP: SILA JE KRIVULJNI INTEGRAL

$$\int_{P(0)P(x)} \vec{f}\,ds = \int_0^x \vec{f}(P(x))\,\underbrace{\|P'(x)\|}_{\sqrt{1+u'(x)^2}\,\approx\,1}\,dx = \int_0^x \underbrace{\vec{f}(x, u(x))}_{\approx\,\vec{f}(x)}\,dx$$

PR:



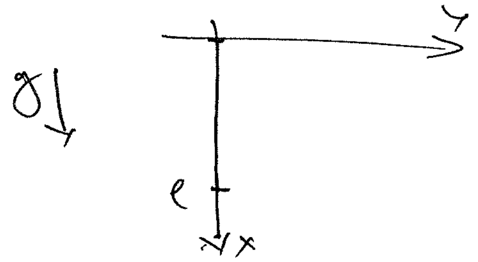

$$f_x = 0$$

$$\Rightarrow \quad a(x) = a(0)$$

$$\Rightarrow \quad a(l) = a(0) = a(x)$$

$$a(l) = g_x(l) = Mg$$

PR: TEŠKA ŽICA




$$a(l) = 0$$

$f_x(x) = \rho g$, $\rho$ - L. GUSTOĆA

$\int_0^x f_x(\eta)\,d\eta = \rho g x$ - TEŽINA KOMADA

$$a(x) = a(0) - \int_0^x f_x(\eta)\,d\eta$$

$$0 = a(l) = a(0) - \int_0^l f_x(\eta)\,d\eta \Rightarrow a(0) = \int_0^l f_x(\eta)\,d\eta = \rho g l$$

$$\Rightarrow a(x) = \rho g l - \rho g x = \underline{\rho g (l-x)} = g\,\rho(l-x)$$

MASA KOMADA $\overline{xl}$

NAP.: BIT ĆE VAŽNO ZA ANALIZU:

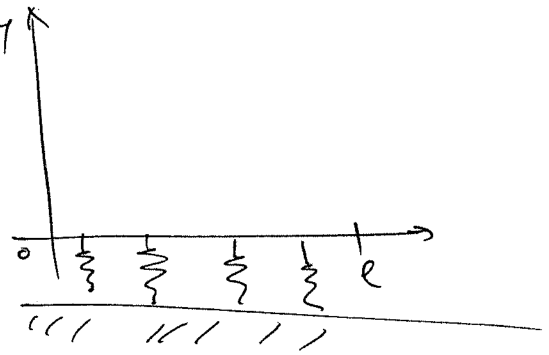
$$a(x) > 0, \quad x \in [0, l]$$

OVDJE NIJE ... SINGULARNOST

ZABORAVLJAMO NA $x$ $\quad g_y \rightsquigarrow g$

$f_y \rightsquigarrow f$

PRETPOSTAVIMO JOŠ DA JE ŽICA U ELASTIČNOM SREDSTVU



U TOČKI S PROGIBOM $u(x)$ DJELUJE I SILA S LINIJSKOM GUSTOĆOM

$$-b(x)\,u(x)$$

JEDNADŽBA RAVNOTEŽE (INTEGRALNI OBLIK):

$$a(x)u'(x) - a(0)u'(0) + \int_0^x \big(f(\xi) - b(\xi)u(\xi)\big)\,d\xi = 0$$

DERIVIRANJEM: DIFERENCIJALNI OBLIK

$$\big(a(x)u'(x)\big)' - b(x)u(x) + f(x) = 0$$

$$\boxed{-\big(a(x)u'(x)\big)' + b(x)u(x) = f(x)}, \quad x \in \langle 0, \ell\rangle$$

$a, b$ – KOEFICIJENTI, ZADANI

$f$ – DESNA STRANA, ZADANA

$u$ – PROGIB, NEPOZNANICA (OPISUJE POLOŽAJ RAVNOTEŽE)

NAP: GDJE SU ELASTIČNA SVOJSTVA MATERIJALA?

POKAŽE SE ZA CILINDAR $a \approx EA$ PRODULJENJE

# RUBNI UVJETI

- JEDNADŽBA RAVNOTEŽE: ODJ, LINEARNA, 2. REDA

  2 KONSTANTE INTEGRACIJE

- 2 UVJETA ZA DETERMINIRANOST

a) DIRICHLETOVI: $u(0) = u_0$ (UČVRŠĆEN KRAJ) ~~GEOMET~~

b) NEUMANNOVI: $g(l) = g_0$ (ZADANA KONTAKTNA SILA)

$$a(l)\,u'(l) = g_0 \Rightarrow u'(l) = \frac{g_0}{a(l)}$$

$g_0 = 0$ KRAJ SLOBODAN

c) ROBINOV: $g(l) = -\mathcal{K}\,u(l)$

$$a(l)\,u'(l) = -\mathcal{K}\,u(l)$$

$$u'(l) = -k\,u(l)$$

d) OPĆENITO:

$$\alpha\,u'(0) - \beta\,u(0) = 0$$

$$\gamma\,u'(l) + \delta\,u(l) = 0$$

$\alpha, \beta, \gamma, \delta \geq 0$ $\quad \alpha + \beta > 0$ $\quad \gamma + \delta > 0$

# RUBNA ZADAĆA

$$-(a(x)u'(x))' + b(x)u(x) = f(x) \qquad x \in \langle 0, l \rangle$$

$$\alpha u'(0) - \beta u(0) = 0$$

$$\gamma u'(l) + \delta u(l) = 0$$

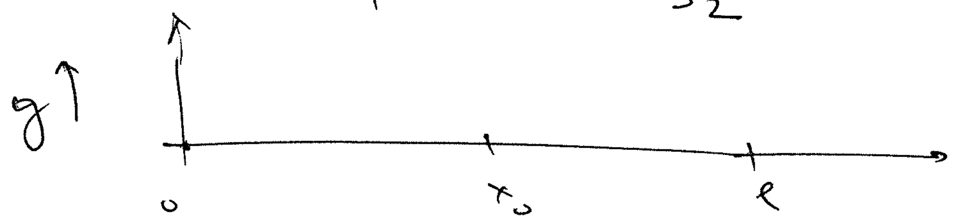
NAP: ILI JEDNADŽBA U INTEGRALNOM OBLIKU

$$a(x)u'(x) - a(0)u'(0) + \int_0^x \left(f(\xi) - b(\xi)u(\xi)\right) d\xi = 0$$

PR: TEŠKA ŽICA IZ DVA MATERIJALA

$\langle 0, x_0 \rangle \qquad \langle x_0, l \rangle$

$\rho_1 \qquad \rho_2$ – GUSTOĆE




$$f(x) = \begin{cases} \rho_1 g, & x < \\ \rho_2 g, & x > \end{cases}$$

NAPETA UTEGOM MASE M $\Rightarrow a = Mg$.

UČVRŠĆENA NA OBA KRAJA $\quad u(0) = u(l) = 0$

$b = 0$

$\Rightarrow \quad \underbrace{\int_0^x f(\xi)\,d\xi}_{\sim u'(x)}$ NEPREKIDNA (PO DIJELOVIMA AFINA) ALI NIJE SVUDA DERIVABILNA

$\Rightarrow \quad u(x)$ KLASE $C^1$ ALI NIJE SVUDA 2 PUTA DERIVABILNA

RUBNA ZADAĆA U DIF FORMULACIJI NEMA SMISLA

$$-\left(a(x)\,u'(x)\right)' = \rho_1 g \qquad x \in \langle 0, x_0 \rangle$$

$$-\left(a(x)\,u'(x)\right)' = \rho_2 g \qquad x \in \langle x_0, \ell \rangle$$

$$u(0) = u(\ell) = 0$$

$$u(x_0-) = u(x_0+)$$

$$u'(x_0-) = u'(x_0+)$$

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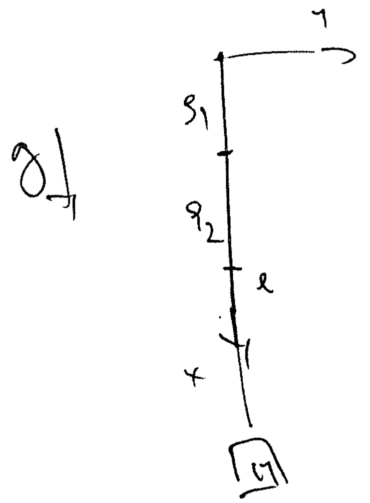
PR: $b$ IMA PREKID (3.2.1)

PR:




$$a(x) = Mg + \begin{cases} \rho_1 g(x_0 - x) + \rho_2 g(\ell - x_0), & x \\ \rho_2 g(\ell - x), & x_0 \le x \end{cases}$$

$a'$ IMA PREKID

# SLABA FORMULACIJA RUBNE ZADAĆE

NEKA JE $v \in C^1([0,\ell])$

$$-\int_0^\ell \left(a(x)u'(x)\right)' v(x)\,dx + \int_0^\ell b(x)u(x)v(x)\,dx = \int_0^\ell f(x)v(x)\,dx$$

P.I.

$$\int_0^\ell a u' v' - a u' v \Big|_0^\ell = \int_0^\ell a u' v' - a(\ell)u'(\ell)v(\ell) + a(0)u'(0)v(0)$$

$$\Rightarrow \boxed{\int_0^\ell a u' v' + b u v = \int_0^\ell f v + a(\ell)u'(\ell)v(\ell) - a(0)u'(0)v(0) \quad \forall v \in C^1([0,\ell])}$$

ZA RUBNE UVJETE

$$u(0) = 0 \qquad u'(\ell) = 0$$

ZADAĆA GLASI:

$$\boxed{\begin{array}{l}\text{NAĆI } u \text{ "DOVOLJNO GLATKO" T.D. } u(0)=0, \\ \displaystyle\int_0^\ell a u' v' + b u v = \int_0^\ell f v, \quad \forall v \in C^1([0,\ell]),\ v(0)=0\end{array}} \quad (*)$$

IZVOD SLABE IZ INTEGRALNE J. (ZA ZADAĆU)

$$\int_0^{\ell} a(x)\,u'(x)\,v'(x) - a(0)\,u'(0)\,\big(v(\ell) - v(0)\big)$$

$$+ \int_0^1 v'(x) \int_0^x \big(f(\eta) - b(\eta)u(\eta)\big)\,d\eta \,\big)\,dx = 0$$

$$\int_0^{\ell} a(x)\,u'(x)\,v'(x) - a(0)\,u'(0)\,\big(v(\ell) - v(0)\big)$$

$$- \int_0^{x} v(x)\,\big(f(x) - b(x)u(x)\big)\,dx$$

$$+ v(x) \int_0^x (f - bu) \Big|_0^{\ell} = 0$$

$$\int_0^{\ell} a u' v' - a(0)\,u'(0)\,\big(v(\ell) - v(0)\big) + \int_0^1 bu - fu$$

$$+ v(\ell) \int_0^{\ell} (f - bu) = 0$$

$$a(\ell)\,u'(\ell) - a(0)\,u'(0) + \int_0^{\ell} = 0$$

$$\int_0^{\ell} a u' v' + bu - fu - a(0)\,u'(0)\,\big(\cancel{v(\ell)} - v(0)\big)$$

$$+ v(\ell)\,\big(\cancel{a(0)\,u'(0)} - a(\ell)\,v'(\ell)\big) = 0$$

OK

~~NAP:~~

ZA R.U.

$$a(0)u'(0) = \mathcal{H}_0 u(0)$$

$$a(\ell)u'(\ell) = -\mathcal{H}_\ell u(\ell)$$

NAĆI $u$ "DOVOLJNO GLATKU" T.D.

$$\left( \int_0^\ell a u' v' + b u v \right) + \mathcal{H}_0 u(0) v(0) + \mathcal{H}_\ell u(\ell) v(\ell) = \int_0^\ell f v$$

$$\forall v \in C^1([0,\ell])$$

~~NAP:~~ UZ OZNAKU

$$B(u,v) = \left( \int_0^\ell a u' v' + b u v \right) + \mathcal{H}_0 u(0) v(0) + \mathcal{H}_\ell u(\ell) v(\ell)$$

$$L(v) = \int_0^\ell f v$$

ZADAĆA GLASI

NAĆI $u$ "DOVOLJNO GLATKU" T.D. $(u(0)=0)$

$$B(u,v) = L(v), \qquad v \in C^1([0,\ell])$$

$$(v(0)=0)$$

~~NAP:~~

$$\frac{1}{2}B(u,u) = \frac{1}{2}\left[ \left( \int_0^\ell a u' u' + b u^2 \right) + \mathcal{H}_0 u(0)^2 + \mathcal{H}_\ell u(\ell)^2 \right]$$

– ENERGIJA (ELASTIČNA)

APSTRAKTNI TEOREM EGZISTENCIJE I JEDINSTVENOSTI

TM: (LAX-MILGRAM) NEKA JE $(V, \|\cdot\|)$ BANACHOV PROSTOR, $L: V \to \mathbb{R}$ NEPREKIDAN L.F., $B: V \times V \to \mathbb{R}$ SIMETRIČNA, NEPREKIDNA, BILINEARNA FORMA KOJA JE V-ELIPTIČNA (KOERCITIVNA, POZ. DEF.):

$$\exists \beta > 0 \text{ t.d. } \forall v \in V \quad B(v,v) \geq \beta \|v\|^2$$

TADA ZADAĆA:

$$\text{NAĆI } u \in V \text{ T.D. } B(u,v) = L(v), \quad v \in V$$

IMA JEDINSTVENO RJEŠENJE.

TO RJEŠENJE JE UJEDNO I JEDINSTVENO RJEŠENJE MINIMIZACIJSKE ZADAĆE:

$$\text{NAĆI } u \in V \text{ T.D. } J(u) = \inf_{v \in V} J(v)$$

$$\left( J(v) = \frac{1}{2} B(v,v) - L(v), \quad J: V \to \mathbb{R} \right)$$

DOK: B JE V-ELIPTIČKA I NEPREKIDNA NA V:

$$\beta \|v\|^2 \leq B(v,v) \leq \|B\| \, \|v\|^2 \quad , \; v \in V$$

$\Longrightarrow$ B JE SKALARNI PRODUKT EKVIVALENTAN $\| \; \|$

$\Longrightarrow$ $(V, B)$ JE HILBERTOV PROSTOR

$\Rightarrow$ L NEPREKIDAN I NA $(V, B)$

RIESZ $\Longrightarrow$ $\exists!\, u \in V$ T.D. $L(v) = B(u,v), \; v \in V$

EGZ & JED.

Vrijedi

$$J(u+v) = \frac{1}{2} B(u+v, u+v) - L(u+v)$$

$$= \frac{1}{2}\left(B(u,u) + 2B(u,v) + B(v,v)\right) - L(u) - L(v)$$

$$= J(u) + B(u,v) + \frac{1}{2} B(v,v) - L(v)$$

Ako je $u$ rješenje $B(u,v) = L(v)$, $v \in V$

$$\Rightarrow \quad J(u+v) = J(u) + \frac{1}{2} B(v,v) \geq J(u) + \frac{\alpha}{2}\|v\|^2 \geq J(u), \quad v \in V$$

$$\Rightarrow \quad J(u) = \min_{v \in V} J(v)$$

Obratno, neka je $u$ minimizator od $J$: $v \in V$, $\theta \in \mathbb{R}$

$$\Rightarrow \quad J(u+\theta v) \geq J(u)$$

$$\Rightarrow \quad 0 \leq J(u+\theta v) - J(u) = \frac{1}{2} B(u+\theta v, u+\theta v) - L(u+\theta v) - \frac{1}{2} B(u,u) + L(u)$$

$$= \cancel{\frac{1}{2} B(u,u)} + \theta B(u,v) + \frac{1}{2}\theta^2 B(v,v) - \cancel{L(u)} - \theta L(v) - \cancel{\frac{1}{2} B(u,u)} + \cancel{L(u)}$$

$$= \theta\left(B(u,v) - L(v) + \frac{\theta}{2} B(v,v)\right),$$

$$\Rightarrow \quad 0 \leq \theta\left(B(u,v) - L(v) + \frac{\theta}{2} B(v,v)\right) \qquad \theta \in \mathbb{R}, \ v \in V$$

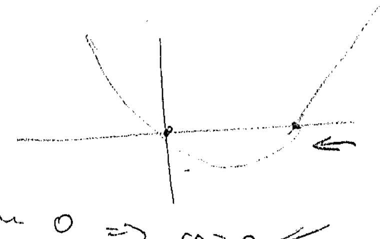
TV: $B(u,v) = L(v)$, $v \in V$

Dok:

$$f(\theta) = a\theta + b\theta^2, \quad b \geq 0$$

$f(\theta) \geq 0 \quad \theta \in \mathbb{R}$

nultočke! $0, \ -\frac{a}{b}$



jedino ako su obje nultočke $0 \Rightarrow a = 0$

NAP: I ako $B$ nije simetrična zadaća $B(u,v) = L(v)$, $v \in V$ ima jedinstveno rj.
Ova generalizacija: Lax-Milgramova lema (teorem)
Jedino tada dio s minimizacijom ne vrijedi!

# PROSTORI SOBOLJEVA

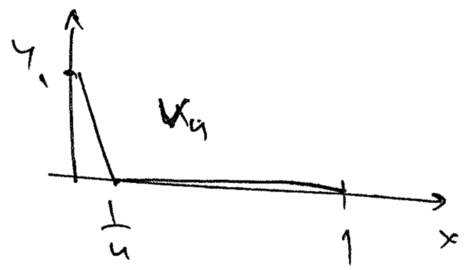
$$L^2(a,b) = \Big\{ v: \langle a,b\rangle \to \mathbb{R} : v \text{ IZMJERIVA}, \; \underbrace{\int_a^b v(x)^2\,dx}_{=\|v\|_2^2} < \infty \Big\}$$

LEBESGEOV PROSTOR

$$L^\infty(a,b) = \Big\{ v: \langle a,b\rangle \to \mathbb{R} : \underbrace{\operatorname*{ess\,sup}_{x\in\langle a,b\rangle} |v(x)|}_{=\|v\|_\infty} < \infty \Big\}$$

KLASE EKVIVALENCIJE!

PR:




$\|v_n\|_\infty = 1$

$\|v_n\|_2 \to 0$

$\Rightarrow$ NE POSTOJI $C > 0$ T.D.

$$\|v\|_2 \geq C\|v\|_\infty, \quad v \in L^\infty(a,b)$$

LEMA: $L^2(a,b)$ JE HILBERTOV UZ

$$(u,v) = \int_a^b u(x)\,v(x)\,dx$$

LEMA: (SCHWARZ-CAUCHY-BUNJAKOVSKI)

$$\int_a^b u(x)v(x)\,dx \leq \left(\int_a^b u(x)^2\,dx\right)^{1/2} \left(\int_a^b v(x)^2\,dx\right)^{1/2}$$

$$(u,v) \leq \|u\|_2 \, \|v\|_2, \quad u,v \in L^2(a,b)$$

DEF: (SLABA DERIVACIJA)

NEKA JE $u \in L^2(a,b)$. KAŽEMO DA $u$ IMA SLABO DERIVACIJU ~~AKO~~ U $L^2$ AKO POSTOJI $v \in L^2(a,b)$ TAKVA DA VRIJEDI

$$\int_a^b u(x)\,\phi'(x)\,dx = -\int_a^b v(x)\,\phi(x)\,dx$$

ZA SVAKI $\phi \in C_c^\infty(a,b)$.

TADA $v$ ZOVEMO SLABOM DERIVACIJOM OD $u$ I PIŠEMO $v = u'$.

PR:

$$u(x) = \begin{cases} x, & x \in \langle 0,1] \\ 1, & x \in \langle 1,2\rangle \end{cases}, \quad v(x) = \begin{cases} 1, & x \in \langle 0,1] \\ 0, & x \in \langle 1,2\rangle \end{cases}$$

PR:

$$u(x) = \begin{cases} x, & x \in \langle 0,1] \\ 2, & x \in \langle 1,2\rangle \end{cases}$$

NEMA SLABU DERIVACIJU

$$H^1(a,b) := \{ v \in L^2(a,b) \; : \; v' \in L^2(a,b) \}$$

$$\|v\|_{H^1} = \left( \|v\|^2_{L^2(a,b)} + \|v'\|^2_{L^2(a,b)} \right)^{1/2}$$

<u>LEMA:</u> $H^1$ JE HILBERTOV PROSTOR

$$(u,v) = \int_a^b u(x)\, v(x)\, dx + \int_a^b u'(x)\, v'(x)\, dx$$

SVOJSTVA SKALARNOG PRODUKTA ZADOVOLJENA

POTPUNOST:

$(u_k) \subseteq H^1(a,b)$ C-NIZ

POTPUNOST OD $L^2$

$\Rightarrow$ $(u_k)$ C-NIZ U $L^2(a,b)$ $\Rightarrow$ $u_k \longrightarrow u$

$(u_k')$ C-NIZ U $L^2(a,b)$ $\Rightarrow$ $u_k' \longrightarrow v$

$$\int_a^b u_k'(x)\, \varphi(x)\, dx \longrightarrow \int_a^b v(x)\, \varphi(x)\, dx$$

$$\| \qquad\qquad\qquad \| \Rightarrow v = u'$$

$$-\int_a^b u_k(x)\, \varphi'(x)\, dx \longrightarrow -\int_a^b u(x)\, \varphi'(x)\, dx$$

U SLUČAJU KADA JE $\Omega^{\text{OTV}} \subseteq \mathbb{R}$ KORISTI JE TEOREM ULAGANJA ZA PROSTORE SOBOLJEVA. NAIME ZA $n=1$, $p=2$ IZ OPĆEG TEOREMA VRIJEDI DA JE

$H^1(\Omega)$ NEPREKIDNO ULOŽENO U $C^{0,\frac{1}{2}}(\Omega) \subseteq C(\Omega)$

HÖLDEROV PROSTOR

POSLJEDICA JE TO MORREYJEVE NEJEDNAKOSTI. STOGA VRIJEDI (NEPREKIDNOST): $\exists\, C > 0 \;\; \forall u \in H^1(\Omega)$

$$\|u\|_\infty \leq C \|u\|_{H^1(\Omega)} .$$

PRI TOME ULAGANJE U $C(\Omega)$ ZNAČI DA U KLASAMA KOJI SU ELEMENTI OD $H^1(\Omega)$ POSTOJI NEPREKIDNI REPREZENTANT.

<u>NAP:</u> OCJENU NIJE TEŠKO IZVESTI I SLIJEDI IZ N-L FORMULE

$$u(x) = u(y) + \int_y^x u'(y)\,dy$$

$$\Rightarrow |u(x)| \leq |u(y)| + \int_0^\ell |u'(y)|\,dy \quad \Big| \int_0^\ell dy$$

$$\Rightarrow \ell\,|u(x)| \leq \int_0^\ell |u(y)|\,dy + \ell \int_0^\ell |u'(y)|\,dy$$

$$\leq \|u\|_{L^2(0,\ell)} \sqrt{\ell} + \ell^{3/2} \|u'\|_{L^2(0,\ell)} \quad \Big| \sup_{x \in [0,\ell]}$$

$$\Rightarrow \|u\|_{L^\infty(0,\ell)} \leq C\left(\|u\|_{L^2} + \|u'\|_{L^2}\right) \leq C \|u\|_{H^1(0,\ell)}$$

EKVIVALENCIJA 1 I 2 NORME NA $\mathbb{R}^2$!

ZA LEBESGUEOVE PROSTORE F-JE DEF. DO NA SKUP MJERE 0. NE POSTOJI VRIJEDNOST U TOČKI IPAK:

TEOREM (O TRAGU):

NEKA JE $\Omega$ LIPSCHITZOVA DOMENA. TADA PRESLIKAVANJE
$$\gamma_{\partial\Omega} : C^1(\overline{\Omega}) \longrightarrow C(\partial\Omega)$$
$$\gamma_{\partial\Omega}(u) := u|_{\partial\Omega}$$

IMA JEDINSTVENO PROŠIRENJE DO LINEARNOG I NEPREKIDNOG OPERATORA
$$\gamma_{\partial\Omega} : H^1(\Omega) \longrightarrow L^2(\partial\Omega).$$

OPERATOR $\gamma_{\partial\Omega}$ SE NAZIVA OPERATOR TRAGA.

NAP:

1) SLIKA OPERATORA TRAGA $\neq L^2(\partial\Omega)$
$$\gamma_{\partial\Omega}(H^1(\Omega)) =: H^{1/2}(\partial\Omega)$$

2) NEPREKIDNOST:
$$\exists C > 0 \quad \|\gamma_{\partial\Omega}(v)\|_{L^2(\partial\Omega)} \leq C\|v\|_{H^1(\Omega)} \qquad v \in H^1(\Omega)$$

KORISTIMO NOTACIJU: $\|v\|_{L^2(\partial\Omega)} \leq C\|v\|_{H^1(\Omega)}$

PROSTOR $V = \{ v \in H^1(a,b) : v(a) = 0 \}$

TV: $V$ JE POTPUN ($\equiv$ ZATVOREN)

$(v_k) \subseteq V \qquad v_k \longrightarrow v \quad$ u $H^1(a,b)$

$\Rightarrow v \in H^1(a,b)$

NEPREKIDNOST TRAGA:

$$\| (v_k - v)|_a \| \leq C \, \| v_k - v \|_{H^1(a,b)} \longrightarrow 0$$

$$\| (v_k - v)|_a \| = | \underbrace{v_k(a)}_{=0} - v(a) | = |v(a)|$$

$\Rightarrow v(a) = 0 \quad \Rightarrow v \in V$

LEMA (POINCARÉ):

$\exists \, C_p > 0$ T.D

$$\| v \|_{L^2(a,b)} \leq C_p \| v' \|_{L^2(a,b)}, \; v \in V.$$

DOK. NEWTON-LEIBNIZ

$$(*) \quad v(x) = v(x) - v(a) = \int_a^x v'(y)\,dy \leq \underbrace{\left( \int_a^x 1^2 dx \right)^{1/2}}_{\sqrt{b-a}} \left( \int_a^x v'(y)^2 dy \right)^{1/2}$$

$$\Rightarrow |v(x)| \leq \sqrt{b-a} \left( \int_a^b v'(y)^2 dy \right)^{1/2} \quad \Big|^2 \; \int_a^b$$

$$\| v \|^2_{L^2(a,b)} \leq (b-a) \int_a^b \int_a^b v'(y)^2 dy = \underbrace{(b-a)^2}_{C_p^2} \| v' \|^2_{L^2(a,b)}$$

NAP: NA $V$ VRIJEDI

$$\|v\|_{H^1}^2 \cancel{=} \|v\|_{L^2}^2 + \|v'\|_{L^2}^2 \leq C_p^2 \|v'\|_{L^2}^2 + \|v'\|_{L^2}^2$$

$$\leq \left(1 + C_p^2\right) \|v'\|_{L^2}^2$$

UVIJEK VRIJEDI

$$\|v'\|_{L^2} \leq \|v\|_{H^1}$$

$\Rightarrow$ $v \mapsto \|v\|_{H^1}$, $v \mapsto \|v'\|_{L^2}$ EKVIVALENTNE NA $V$

NAP: POSTOJE I NEJEDNAKOSTI TIPA

$\exists C_p > 0$ T.D.

$$\left\|v - \int_a^b v\right\|_{L^2(a,b)} \leq C_p \|v'\|_{L^2(0,\ell)}$$

DOKAZ SLIČAN. IZ (+) SE IZRAZI $\int_a^b v(x)\,dx$.

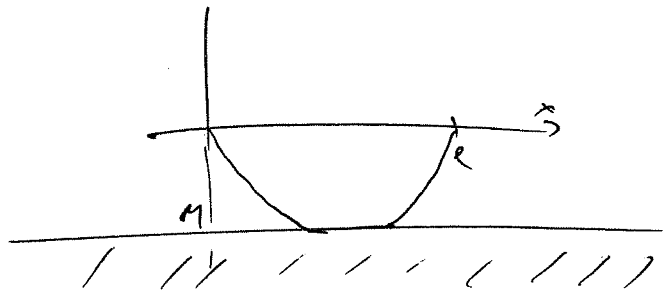
NAP: MINIMIZACIJSKU FORMULACIJU : NAĆI $u \in V$ T.D. $J(u) = \min_{v \in V} J(v)$

MOŽEMO POOPĆITI: MINIMIZIRAMO PO PODSKUPU $\mathcal{U} \subseteq V$ T.D.

NEPRAZAN, ZATVOREN, KONVEKSAN

NPR: UVJET NA $V$

NPR. $u(x) \geq m$



TEOREM EGZISTENCIJE I JEDINSTVENOSTI

- IZBOR ZA $V$? BANACHOV!

$$C^1([0,\ell]) \quad \dots \|v\|_V = \|v\|_\infty + \|v'\|_\infty$$

PROBLEM: KOERCITIVNOST

$$B(v,v) = \int_0^\ell a(v')^2 \geq m\left(\|v\|_\infty + \|v'\|_\infty\right)^2$$

PRIRODAN ODABIR ZBOG OBLIKA $B$: $V = H^1(0,\ell)$ ← HILBERTOV

$$\|v\|_V^2 = \|v\|^2_{L^2(0,\ell)} + \|v'\|^2_{L^2(0,\ell)}$$

KOERCITIVNOST: POINCARÉOVA NEJEDNAKOST

~~$\mathcal{U} = \{ v \in H^1(0,\ell)$~~

NAP:

$H^1$ JE HILBERTOV (POTPUN):

$(u_k) \subseteq H^1$ C-NIZ U $H^1$ $\Rightarrow$ $u_k$ – C-NIZ U $L^2$, $u_k'$ – C-NIZ U $L^2$ | POTPUNI $u_k \to u$ $L^2$, $u_k' \to v$ $L^2$

$$\int_0^\ell u_k'\varphi \to \int_0^\ell v\varphi$$

$$\| \qquad\qquad \|$$

$$-\int_0^\ell u_k\varphi' \to -\int_0^\ell u\varphi' \quad \Rightarrow \quad \int_0^\ell v\varphi + u\varphi' = 0 \Rightarrow v = u' \text{ u } L^2$$

TEOREM ~~[illegible]~~

NEKA JE $a, b \in C([0,\ell])$, $a>0$, $b \geq 0$, $f \in L^2(0,\ell)$

~~Z~~ADAĆA:

NAĆI $u \in V = \{ v \in H^1(0,\ell) : v(0)=0 \}$ T.D.

$$\int_0^\ell a u' v' + buv = \int_0^\ell f v \quad , v \in V$$

IMA JEDINSTVENO RJEŠENJE.

DOK:

$$B(u,v) = \int_0^\ell a u' v' + buv$$

V POTPUN! TM. O TRAGU

- BILINEARNOST, SIMETRIČNOST - OČITE
- NEPREKIDNOST:

$$|B(u,v)| = \left| \int_0^\ell a u' v' + buv \right| \leq \int_0^\ell |a|\,|u'|\,|v'| + |b|\,|u|\,|v|$$

$$\leq \|a\|_\infty \|u'\|_{L^2} \|v'\|_{L^2} + \|b\|_\infty \|u\|_{L^2} \|v\|_{L^2}$$

$$\leq \left( \|a\|_\infty + \|b\|_\infty \right) \|u\|_{H^1} \|v\|_{H^1}$$

- KOERCITIVNOST:

$$B(u,u) = \int_0^\ell a(u')^2 + \underbrace{b u^2}_{\geq 0} \geq \int_0^\ell a(u')^2 \geq \int_0^\ell a_{min} (u')^2$$

$$= a_{min} \int_0^\ell (u')^2 \geq \underline{\frac{a_{min}}{C_P^2}} \|u\|_{H^1}^2$$

$a_{min} = \min_{x \in [0,\ell]} a(x)$

$\vee$

$0$

$$L(v) = \int_0^\ell f v$$

- LINEARNOST OČITA
- NEPREKIDNOST NA V:

$$|L(v)| \leq \int_0^\ell |f|\,|v| \leq \|f\|_{L^2} \|v\|_{L^2}$$

$$\leq \|f\|_{L^2} \|v\|_{H^1}$$

NAP:

$$v(x) = v(0) + \int_0^x v'(\xi)\,d\xi \qquad (+)$$

ZA $v \in V$ $\quad (v(0)=0)$

$$\Rightarrow \quad v(x) = \int_0^x v'(\xi)\,d\xi$$

$$|v(x)| \leq \int_0^\ell 1\,|v'(\xi)|\,d\xi = \|v'\|_{L^2}\,\|1\|_{L^2} = \sqrt{\ell}\,\|v'\|_{L^2}$$

$$\Rightarrow \quad \|v\|_{L^\infty} \leq \sqrt{\ell}\,\|v'\|_{L^2}$$

$$\|v\|_{L^2}^2 = \int_0^\ell v(\xi)^2\,d\xi \leq \|v\|_{L^\infty}^2\,\ell \leq \ell^2\,\|v'\|_{L^2}^2$$

$$\Rightarrow \quad \|v\|_{L^2} \leq \ell\,\|v'\|_{L^2}$$

$$\Rightarrow \quad \|v\|_{H^1} \leq \sqrt{\|v\|_{L^2}^2 + \|v'\|_{L^2}^2} \leq \sqrt{\ell^2+1}\;\|v'\|_{L^2} \quad \forall v \in V$$

LEMA (POINCARÉ):

VRIJEDI $\exists C_p > 0\ \forall v \in H^1(0,\ell)$ T.D. $v(0)=0$

$$\|v\|_{H^1(0,\ell)} \leq C_p\,\|v'\|_{L^2(0,\ell)}.$$

NAP:

POSTOJE I NEJEDNAKOSTI TIPA

$\exists C_p > 0 \quad \forall v \in H^1(0,\ell)$ T.D.

$$\|v - \int_0^\ell v\|_{L^2(0,\ell)} \leq C_p\,\|v'\|_{L^2(0,\ell)}$$

DOKAZ SLIČAN IZ $(+)$ TE IZRAZI $\int_0^\ell v(x)\,dx$

ZAD: IZVEDITE VARIJACIJSKU FORMULACIJU DIREKTNO IZ INTEGRALNE FORMULACIJE

ZAD: DOKAŽITE TEOREM EGZISTENCIJE I JEDINSTVENOSTI ZA ROBINOVE RUBNE UVJETE

~~ZAD~~: ŠTO JE S NEUMANNOVIM RUBNIM UVJETIMA?
NEMA JEDINSTVENOSTI & NUŽAN UVJET } ZA $b=0$ !

$$\int_0^\ell a u' v' + \underbrace{\int_0^\ell b u v}_{=0} = \int_0^\ell f v + g_\ell v(\ell) + g_0 v(0)$$
$$\forall v \in C^1([0,\ell])$$

$$\begin{cases} -(au')' + \overset{=0}{bu} - f = 0 \\ a(\ell)u'(\ell) = g_\ell \\ -a(0)u'(0) = g_0 \end{cases}$$

STAVIM $v = 1$

$$0 = \int_0^\ell f + g_\ell + g_0$$ NUŽAN UVJET EGZISTENCIJE

NEKA JE $u$ - RJ. $\Rightarrow$ $u + \mathrm{CONST}$ RJEŠENJE !

ZAD: DOKAŽITE EGZISTENCIJU I JEDINSTVENOST / NEJEDINSTVENOST U PUTA: $H^1(0,\ell)/\mathbb{R}$ ILI $V = \{ v \in H^1(0,\ell) : \int_0^\ell v = 0 \}$
(PURE TRACTION PROBLEM)

ZAD: MOŽE LI $b$ BITI NEGATIVAN A DA JOŠ UVIJEK IMAMO EGZISTENCIJU RJEŠENJA ISTOM TEHNIKOM?

KOREKTNOST ZADAĆE

$\Rightarrow$ NEPREKIDNA OVISNOST RJEŠENJA O ZADANIM PODACIMA $(f)$

NAĆI $u \in V = \{ v \in H^1(0,l) : v(0) = 0 \}$

$$\int_0^l a u' v' + b u v = \int_0^l f v \quad , \ v \in V$$

$v = u$

$$\int_0^l a (u')^2 + b u^2 = \int_0^l f u \leq \|f\|_{L^2} \|u\|_{L^2}$$

$$\vee\!\!/$$

$$a_{min} \int_0^l (u')^2 \geq C_p^2 a_{min} \|u\|^2_{H^1}$$

$$\Rightarrow \quad \|u\|^2_{H^1} \leq \frac{\|f\|_{L^2}}{C_p^2 a_{min}} \|u\|_{H^1}$$

$$\Rightarrow \quad \|u\|_{H^1} \leq \frac{1}{C_p^2 a_{min}} \|f\|_{L^2}$$

KOREKTNOST RJEŠENJA

ZAD: DOKAŽITE KOREKTNOST ZA ROBINOVE R.U.

NAP: JEDNADŽBA RAVNOTEŽE $\overline{P(x_1)P(x_2)}$, INTEGRALNI OBLIK

$$a(x_2)u'(x_2) - a(x_1)u'(x_1) - \int_{x_1}^{x_2} b(\xi)u(\xi)d\xi + \int_{x_1}^{x_2} f(\xi)d\xi = 0$$

$$F(x) = \int_0^x f(\xi)\,d\xi$$ — SILA NA $\overline{P(0)P(x)}$ ZADANA GUSTOĆOM

UKUPNA SILA

$$a(x_2)u'(x_2) - a(x_1)u'(x_1) - \int_{x_1}^{x_2} b(\xi)u(\xi)d\xi + F(x_2) - F(x_1) = 0$$

NE MORA BITI IZ GUSTOĆE

NPR: $$F(x) = \begin{cases} 0 & x < x_0 \\ F_0 & x_0 < x \end{cases}$$

"KONCENTRIRANA SILA" $\forall\, x_1 < x_2 \quad x_0 \notin \langle x_1, x_2\rangle$

$F(x_2) - F(x_1) = 0$

NEMA GUSTOĆU!

$\forall\, x_1 < x_2 \quad x_0 \in \langle x_1, x_2 \rangle$

$F(x_2) - F(x_1) = F_0$

OSNOVNA LEMA VARIJACIJSKOG RAČUNA

$\Rightarrow \quad f(x) = 0 \quad , \; x \neq x_0$

$$\int_0^\ell f = F(\ell) - F(0) = F_0 \neq 0 \quad \Rightarrow\!\Leftarrow$$

ZADAĆA:
$$\left(a(x)u'(x)\right)' - b(x)u(x) = 0 \qquad x \in \langle 0, x_0\rangle \cup \langle x_0, \ell\rangle$$

$$g(x_2) - g(x_1) - \int_{x_1}^{x_2} bu + F_0 = 0 \qquad x_1 < x_0 < x_2$$

$\lim_{x_1 \to x_0-} \quad \lim_{x_2 \to x_0+}$

$$a(x_0+)u'(x_0+) - a(x_0-)u'(x_0-) + F_0 = 0$$ UVJET TRANSMISIJE

KAO I PRIJE: $u(x_0+) = u(x_0-)$

+ RUBNI UVJET

## 1.2. EVOLUCIJSKA JEDNADŽBA

$u : [0,\ell] \times [0,T] \to \mathbb{R}$

$f : [0,\ell] \times [0,T] \to \mathbb{R}$

JEDNADŽBA GIBANJA (2. NEWTONOV ZAKON)

INT: $$\int_0^x \rho u_{tt}(\xi,t)\,d\xi = a(x)u'(x) - a(0)u'(0) + \int_0^x f(\xi)\,d\xi - \int_0^x b(\xi)u(\xi)\,d\xi$$

DIF: $$\rho u_{tt} - (au')' + bu = f$$ ~~JEDNADŽBA G~~

INICIJALNO RUBNA ZADAĆA ~~N~~APETE ŽICE

$$\begin{cases} \rho u_{tt} - (au_x)_x + bu = f \\ u(0,t) = 0 \\ u'(\ell,t) = 0 \\ u(x,0) = u_0 \\ u_t(x,0) = u_1 \end{cases}$$

PDJ

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NAP: POSTOJI I ZA OVO HILBERTOVA TEORIJA

SLABA FORMULACIJA:

$$\int_0^\ell \rho u_{tt} v + \int_0^\ell a u'v' + buv = \int_0^\ell fv \qquad v \in V = \{v \in H^1(0,\ell) : v(0) = 0\}$$

SVE BAZIRAMO NA ENERGETSKOJ JEDNAKOSTI

STAVIMO $v = u_t$

$$\frac{d}{dt}\frac{1}{2}\int_0^\ell \rho u_t^2 + \frac{d}{dt}\frac{1}{2}\int_0^\ell au' + \frac{d}{dt}\frac{1}{2}\int_0^\ell bu^2 = \int_0^\ell fu_t$$

$$\frac{d}{dt}E(t) = \int_0^\ell fu_t$$

TRAŽIMO RJ. U SEPARIRANIM VARIJABLAMA ($\overset{\text{HOM.}}{\text{JDBA}}$ + R.U.)

$$u(x,t) = X(x)\,T(t)$$

$$\rho T'' X - (aX')' T + bXT = 0 \qquad \Big| : \rho X T$$

$$\left.\begin{aligned} X(0)\,T(t) &= 0 \\ X'(\ell)\,T(t) &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} X(0) &= 0 \\ X'(\ell) &= 0 \end{aligned}$$

$$\frac{T''}{T} = \frac{(aX')'}{\rho X} - \frac{b}{\rho} = -\lambda$$

$$T'' + \lambda T = 0$$

$$\boxed{\begin{aligned} -(aX')' + bX &= \lambda \rho X \\ X(0) = X'(\ell) &= 0 \end{aligned}}$$

NAĆI $\lambda \in \mathbb{R}$

I $X \neq 0$

SVOJSTVENA ZADAĆA!

$\lambda$ – SV. VRIJEDNOST

$X$ – SV. VEKTOR (SV. FUNKCIJA)

STURM – LIOUVILLEOVA ZADAĆA

NAP:

RUBNI UVJETI – MOGU BITI I DRUKČIJI (NPR $X(0) = X(\ell) = 0$)

– BITNO: HOMOGENI

NAP:

$$LX = -(aX')' + bX$$

$$\begin{cases} LX = \lambda \rho X \\ X(0) = X'(\ell) = 0 \end{cases}$$

GENERALIZIRANI SVOJSTVENI PROBLEM

$K : L^2(0,\ell) \to L^2(0,\ell)$

$f \xmapsto{K} u$ rj. $\begin{cases} -(au')' + bu = f \\ u(0) = u'(\ell) = 0 \end{cases}$

$f \in L^2(0,\ell)$, $u \in H^1(0,\ell)$

SVOJSTVA OD $K$:

1) LINEARAN

$$\alpha f_1 + \beta f_2 \xmapsto{?} \alpha u_1 + \beta u_2$$

$$f_1 \xmapsto{K} u_1$$

$$f_2 \xmapsto{K} u_2$$

$$\begin{cases} -(au')' + bu = \alpha f_1 + \beta f_2 \\ u(0) = u'(\ell) = 0 \end{cases}$$

TV: $u = \alpha u_1 + \beta u_2$ JEDINSTVENOST!

2) SIMETRIČAN NA $L^2(0,\ell)$, $Kf = u$, $Kg = v$

$$(Kf, g) = (u, g) = \int_0^\ell av'u' + bvu = (v, f)$$
$$= (Kg, f)$$

3) NEPREKIDAN

$$\int_0^\ell au'u' + buu = \int_0^\ell fu \leq \|f\|_{L^2} \|u\|_{L^2} \leq \|f\|_{L^2} \|u\|_{H^1}$$

$$\geq$$

$$a_{min} C_p^2 \|u\|_{H^1}^2$$

$$\Rightarrow \quad \|u\|_{H^1} \leq \frac{\|f\|_{L^2}}{a_{min} C_p^2} \quad *$$

$$\Rightarrow \quad \|Kf\| \leq \frac{1}{a_{min} C_p^2} \|f\|_{L^2}$$ NEPREKIDNOST (KOREKTNOST)

4) KOMPAKTAN : $K\left(L^2(0,\ell)\right) \subseteq H^1(0,\ell) \xhookrightarrow{c} L^2(0,\ell)$

KOMPAKTNO ULAGANJE

OGRANIČEN NIZ PRESLIKA U RELATIVNO KOMPAKTAN

S-L ZADAĆA POMOĆU K:

$$x = K(\lambda \rho x)$$

$\rho \geq 0$, DEF: $Rx = \sqrt{\rho}\, x = R^T x$

$$Rx = \lambda \underbrace{R K R^T}_{=\tilde{K}} \underbrace{Rx}_{=y}$$

$$\tilde{K} y = \frac{1}{\lambda} y$$

K SIMETRIČAN KOMPAKTAN $\Rightarrow \tilde{K} = R K R^T$ SIMETRIČAN KOMPAKTAN

<u>TEOREM</u> (SPEKTRALNI TEOREM ZA SIMETRIČNE KOMPAKTNE OPERAT

NEKA JE $\tilde{K}: H \to H$ KOMPAKTAN, $\dim H = \infty$, H SEPARABILA

TADA JE

(i) $0 \in \sigma(\tilde{K})$

(ii) $\sigma(\tilde{K}) \setminus \{0\} = \sigma_p(\tilde{K}) \setminus \{0\}$

(iii) $\sigma(\tilde{K}) \setminus \{0\}$ JE KONAČAN

ILI

$\sigma(\tilde{K}) \setminus \{0\}$ JE NIZ KOJI KONVERGIRA K 0

AKO JE $\tilde{K}$ I SIMETRIČAN

(IV) $\sigma(\tilde{K}) \subseteq \mathbb{R}$

(V) SVOJSNENE VRIJEDNOSTI SU KONAČNE KRATNOSTI

(VI) POSTOJI ORTONORMIRANA BAZA SVOJSTVENIH VEKTORA

NEKA SU $(\mu_k) \subseteq \mathbb{R}$ SV. VR. OD $\tilde{K}$

$$\mu_k \to 0 \qquad \lambda_k = \frac{1}{\mu_k}$$

NAP.:

$$\int_0^\ell a(u')^2 + bu^2 = \lambda \int_0^\ell \varrho u^2$$

ZA $(\lambda, u)$ SVOJSNENI PAR

$$\lambda = \frac{\int_0^\ell a(u')^2 + bu^2}{\int_0^\ell \varrho u^2} \geq 0$$

$\lambda \neq 0$ JER $\int_0^\ell a(u')^2 + bu^2 = 0 \Rightarrow u = 0$

$\Rightarrow \lambda > 0 \qquad \sigma_p(\tilde{K}) \subseteq \langle 0, +\infty \rangle$

$\Rightarrow \mu_k > 0, \ k \in \mathbb{N}$

MOGU IH SORTIRATI U PADAJUĆI NIZ $\mu_k$

$\lambda_k \nearrow +\infty$ RASTUĆI NIZ BEZ KONAČNIH GOMILIŠTA!

$0 < \lambda_1 \leq \lambda_2 \leq \ldots$

$(\mu_k, w_k)$ SV. PAR $\qquad (\psi_k)$ OHB OD $H = L^2(0,\ell)$

$$\tilde{K} \psi_k = \mu_k \psi_k = \frac{1}{\lambda_k} \psi_k$$

$$R K R^T w_k = \frac{1}{\lambda_k} \psi_k$$

$$\psi_k = R \alpha_k \,, \quad k \in \mathbb{N}$$

$$\Rightarrow \quad \lambda_k K(\rho u_k) = u_k \,, \quad k \in \mathbb{N}$$

$$\lambda_k \rho u_k = -(a u_k')' + b u_k \,, \quad k \in \mathbb{N}$$

$\psi_k$ OHB U $L^2(0,\ell)$ $\qquad \int_0^\ell \psi_k \psi_j = \delta_{kj}$

$$\Updownarrow$$

$$\int_0^\ell \sqrt{\rho}\, u_k \sqrt{\rho}\, u_j = \int_0^\ell \rho\, u_k u_j$$

$(u,v) \longmapsto \int_0^\ell \rho u v$ JE SKALARNI PRODUKT!

$(u_k)$ OHB U $L^2(0,\ell)$ S TIM SKALARNIM PRODUKTOM!

NAP: POGLEDATI 2.11 U I. AGANOVIĆ, K. VESELIĆ, LINEARNE DIFERENCIJALNE JEDNADŽBE ZA DIREKTNU ANALIZU

JER JE $\sqrt{\rho} u_k$ OHB U $L^2$ ZA $g \in L^2(0,\ell)$ $\exists\, a_k \in \mathbb{R}$ T.D.

$$\sqrt{\rho}\, g = \sum_{k=1}^{\infty} a_k \sqrt{\rho}\, u_k \qquad \Big| \cdot \sqrt{\rho}\, u_j \quad \int_0^\ell$$

$$\int_0^\ell \rho\, g u_j = a_k$$

**PR:**

$$u'' + \lambda u = 0 \qquad (a=1,\ b=0,\ g=1)$$
$$u(0)=0,\ u(1)=0$$

ZNAMO: $\lambda > 0$

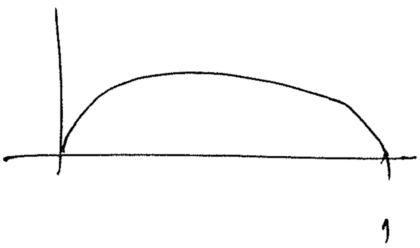
$$u(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

$$0 = u(0) = A$$

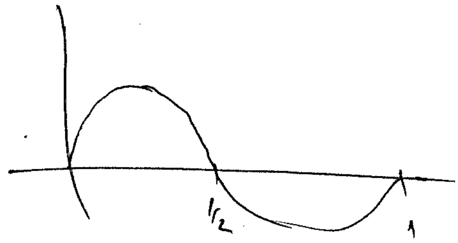
$$0 = u(1) = B\sin\sqrt{\lambda} \Rightarrow \sin\sqrt{\lambda} = 0 \rightarrow \sqrt{\lambda} = k\pi,\ k\in\mathbb{N}$$
$$\Rightarrow \lambda_k = (k\pi)^2,\ k\in\mathbb{N}$$

$$u_k(x) = \sin k\pi x, \quad x\in[0,1]$$

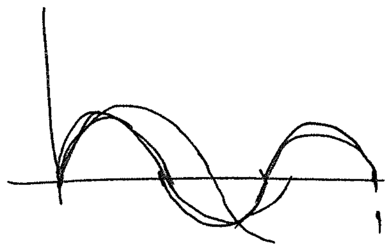
$k=1$




$k=2$




$k=3$




ITD.

ORTOGONALNOST

$$\int_0^1 \sin k\pi x \sin j\pi x = \delta_{kj}$$

**PR:**

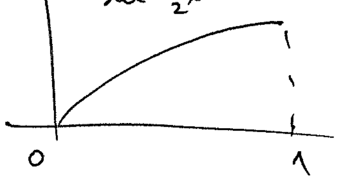
$$u'' + \lambda u = 0$$
$$u(0) = u'(1) = 0$$

$\lambda > 0$, $u(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$

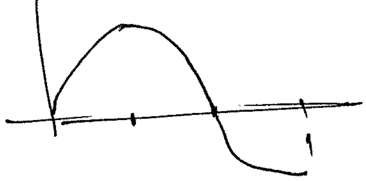
$$0 = u(0) = A$$
$$0 = u'(1) = B\sqrt{\lambda}\cos\sqrt{\lambda} \Rightarrow \sqrt{\lambda} = \frac{\pi}{2} + k\pi$$
$k\in\mathbb{N}\cup\{0\}$

$$\lambda_k = \pi^2\left(\frac{1}{2}+k\right)^2 \qquad u_k(x) = \sin\left(\frac{\pi}{2}+k\pi\right)x$$

$k=0$ $\quad \sin\frac{\pi}{2}x$




$k=1$ $\quad \sin\frac{3\pi}{2}x$



VRAĆAMO SE NA GIBANJE

$(\lambda_k, X_k)$ SVOJSTVENA FAMILIJA ORTONORMIRANA

$$T_k'' + \lambda_k T_k = 0 \qquad T_k(t) = A_k \cos\sqrt{\lambda_k}t + B_k \sin\sqrt{\lambda_k}t$$
$$\lambda_k > 0$$

SEPARIRANA RJ.

$$T_k(t)X_k(x) = A_k \cos\sqrt{\lambda_k}t + B_k \sin\sqrt{\lambda_k}t$$

$$u(x,t) = \sum_{k=1}^{\infty} T_k(t)X_k(x) = \sum_{k=1}^{\infty}\left(A_k\cos\sqrt{\lambda_k}t + B_k\sin\sqrt{\lambda_k}t\right)X_k(x)$$

ZADOVOLJAVA R.U.

P.U.:

$$u_0(x) = u(x,0) = \sum_{k=1}^{\infty} A_k \underbrace{X_k(x)}_{\text{ONB}}$$

KOEF

$$\boxed{A_k = \int_0^{\ell} \rho\, u_0(x) X_k(x)\, dx}$$

$$u_1(x) = u_t(x,0) = \sum_{k=1}^{\infty} T_k'(0) X_k(x) = \sum_{k=1}^{\infty} B_k\sqrt{\lambda_k}X_k(x)$$

$$\boxed{B_k = \frac{1}{\sqrt{\lambda_k}} \int^{\ell} \rho(x)\, u_1(x)\, X_k(x)}$$

IMAMO GIBANJE : SUPERPOZICIJA $\underline{T_k(t)X_k(x)}$

$\frac{2\pi}{\sqrt{\lambda_k}}$ - PERIOD $\qquad \sqrt{\lambda_k}$ - KRUŽNA FREKVENCIJA

$X_k$ - MODOVI GIBANJA

PR:

$$u'' + \lambda u = 0 \qquad a = 1,\ b = 0,\ \rho = 1$$
$$u(0) = u(\ell) = 0$$

$$u(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

$$0 = u(0) = A$$
$$0 = u(\ell) = B\sin\sqrt{\lambda}\ell \quad \Rightarrow \quad \sqrt{\lambda}\ell = k\pi$$
$$\lambda_k = \left(\frac{k\pi}{\ell}\right)^2,\ k \in \mathbb{Z}$$

ŽICA NA GITARI — FREKVENCIJA OVISI O $\ell$!

$\frac{k\pi}{\ell}$ – KRUŽNA FREKVENCIJA

$\frac{2\ell}{k}$ – PERIOD OSCILACIJA

NEHOMOGENA SILA $\quad f \neq 0$

$$u_k(x,t) = T_k(t)X_k(x), \qquad X_k \text{ – VEĆ IZRAČUNATI}$$

$T_k$ – NEODREĐENI! NE $T_k'' + \lambda_k T_k = 0$

UVRSTIMO $u(x,t) = \sum_{k=1}^{\infty} T_k(t)X_k(x)$ U JEDNADŽBU

$$\sum_{k=1}^{\infty} \rho T_k'' X_k \underbrace{- \sum_{k=1}^{\infty} (a X_k')' T_k + \sum_{k=1}^{\infty} b T_k X_k}_{} = f \qquad \cdot X_n$$

$$\sum_{k=1}^{\infty} \rho T_k'' X_k + \sum_{k=1}^{\infty} \lambda_k \rho T_k X_k = f$$

$$\rho \sum_{k=1}^{\infty} (T_k'' + \lambda_k T_k) X_k = f \ \Big|\cdot X_n \int_0^{\ell}$$

$$T_n'' + \lambda_n T_n = \int_0^{\ell} f \cdot X_n =: f_n(t)$$

RAZVIJEMO U RED POČETNE UVJETE (FUNKCIJE)

$$u_0(x) = \sum_{k=1}^{\infty} C_k X_k(x) \quad \Rightarrow \quad T_k(0) = C_k$$

$$u_1(x) = \sum_{k=1}^{\infty} D_k X_k(x) \quad \Rightarrow \quad \dot{T}_k(0) = D_k$$

$$\Rightarrow \quad \left| \begin{array}{l} \ddot{T}_k + \lambda_k T_k = f_k \\ T_k(0) = C_k \\ \dot{T}_k(0) = D_k \end{array} \right.$$

C – ZADAĆA

$$\Rightarrow \quad T_k(t) = \frac{1}{\sqrt{\lambda_k}} \int_0^t f_k(\tau) \sin \sqrt{\lambda_k}(t-\tau)\, d\tau + C_k \cos \sqrt{\lambda_k}\, t + \frac{D_k}{\sqrt{\lambda_k}} \sin \sqrt{\lambda_k}\, t$$

$$\Rightarrow \quad u(x,t) = \sum_{k=1}^{\infty} T_k(t) X_k(x)$$

Ako je $\quad f(x,t) = \varphi(x) \sin \omega t$

$$f_k(t) = \sin \omega t \int_0^{\ell} \varphi(x) X_k(x)\, dx$$

UVRSTIMO GORE:

1) Ako je $\omega^2 \neq \lambda_k$

$$T_k(t) = \frac{\int_0^{\ell} \varphi(x) X_k(x)\, dx}{\omega^2 - \lambda_k} \left( \frac{\omega}{\sqrt{\lambda_k}} \sin \sqrt{\lambda_k}\, t - \sin \omega t \right)$$

2) Ako je $\omega^2 = \lambda_k$

$$T_k(t) = \frac{\int_0^{\ell} \varphi(x) X_k(x)\, dx}{2 \sqrt{\lambda_k}} \left( \frac{1}{\sqrt{\lambda_k}} \sin \sqrt{\lambda_k} - t \cos \sqrt{\lambda_k}\, t \right)$$

REZONANCIJA !!!

# METODA KONAČNIH ELEMENATA (FEM)

APSTRAKTNA FORMULACIJA GALERKINOVE METODE:

$H$ - HILBERTOV, $B$ BILINEARNA, SIMETRIČNA, NEPREKIDNA KOERCITIVNA FORMA

$L$ LINEARAN FUNKCIONAL NEPREKIDAN

ZADAĆA:

NAĆI $u \in H$ T.D.

$$B(u,v) = L(v), \quad v \in H$$

NEKA JE $H_n \leq H$ $\dim H_n = n$

APROKSIMACIJSKA ZADAĆA:

NAĆI $u_n \in H_n$ T.D.

$$B(u_n, v_n) = L(v_n), \quad v_n \in H_n$$

LAX-MILGRAM $\Rightarrow$ $\exists!\, u_n \in H_n$

POSTOJI TEORIJA ZA $\|u - u_n\| \to 0$ NRPDJ

I.A., K.V. JEDNADŽBE MATEMATIČKE FIZIKE

N.J. NPM2

НЕКА ЈЕ $\varphi_1, \dots, \varphi_n$ БАЗА ЗА $H_n$

$$\Rightarrow \quad u_n = \sum_{i=1}^{n} \alpha_i \varphi_i$$

$$\sum_{i=1}^{n} \alpha_i \, B(\varphi_i, \varphi_j) = L(\varphi_j), \qquad j = 1, \dots, n$$

DEF:

$$A = \left( B(\varphi_i, \varphi_j) \right)_{j,i=1,\dots,n}, \qquad F = \left( L(\varphi_j) \right)_{j=1,\dots,n}$$

$$x = (\alpha_i)_{i=1,\dots,n}$$

$$(AZ) \Leftrightarrow \qquad A \cdot x = F \qquad (\exists! \, x)$$

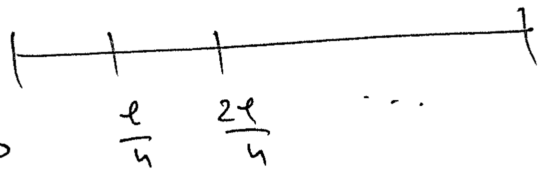
~~ZA~~ ZICU ($u(0)=0,\ u'(0)=1$)

$$B(u,v)=\int_0^\ell a u'v' + b u v$$

$$L(v)=\int_0^\ell f v$$

$$V=\{v\in H^1(0,\ell):\ v(0)=0\}$$

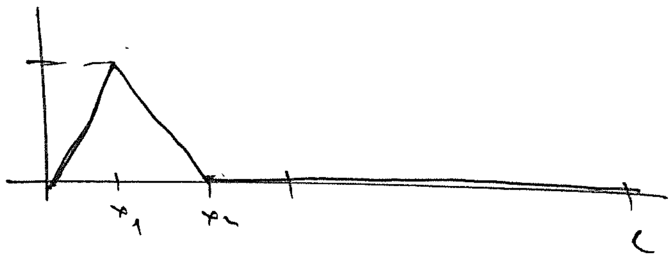
FIX $n\in\mathbb{N}$, $h=\frac{\ell}{n}$



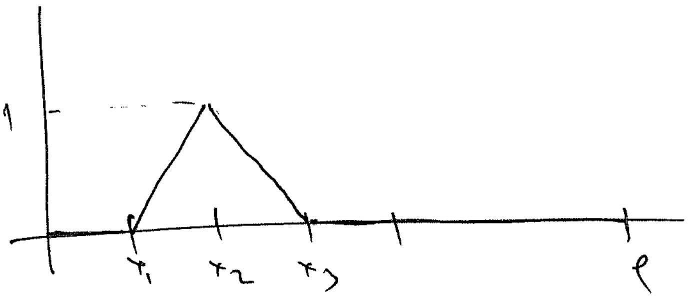

$x_i = i\frac{\ell}{n}$

$i=0,\dots,n$

$\varphi_1$




$\varphi_2$



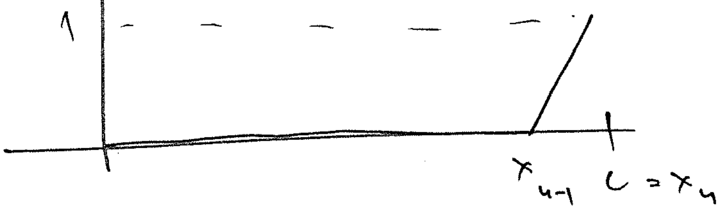

$\varphi_i\in H^1(0,\ell)$, $\varphi_i(0)=0$

$H_n=\mathrm{span}\{\varphi_1,\dots,\varphi_n\}$

⋮

$\varphi_i(x_j)=\delta_{ij}$

$\varphi_n$




$$u_n(x_j)=\sum_{i=1}^n \alpha_i \varphi_i(x_j)=\alpha_j \quad !!$$

$$B_1(u,v) = \int_0^\ell a u' v' \quad , \quad B_2(u,v) = \int_0^\ell b u v$$

$$B_1(\varphi_i, \varphi_j) \neq 0 \qquad |i-j| \leq 1$$

$$B_1(\varphi_i, \varphi_i) = \int_0^\ell a(\varphi_i')^2 = \int_{x_{i-1}}^{x_{i+1}} a(\varphi_i')^2 = \int_{x_{i-1}}^{x_i} a \frac{1}{h^2} + \int_{x_i}^{x_{i+1}} a \frac{1}{(-h)^2}$$

$$= \frac{1}{h^2} \int_{x_{i-1}}^{x_i} a + \frac{1}{h^2} \int_{x_i}^{x_{i+1}} a$$

TRAPEZNA FORMULA (PAZI NA RED!)

$$\approx \frac{1}{h^2} \left( \frac{a(x_i) + a(x_{i-1})}{2} h + \frac{a(x_{i+1}) + a(x_i)}{2} h \right)$$

$$= \frac{1}{2h} \left( a(x_{i-1}) + 2a(x_i) + a(x_{i+1}) \right)$$

$$B_1(\varphi_i, \varphi_{i+1}) = \int_0^\ell a \varphi_i' \varphi_{i+1}' = \int_{x_i}^{x_{i+1}} a \varphi_i' \varphi_{i+1}' = \int_{x_i}^{x_{i+1}} a \left(-\frac{1}{h}\right) \frac{1}{h}$$

$$\approx -\frac{1}{h^2} \left( \frac{a(x_i) + a(x_{i+1})}{2} h \right) = -\frac{1}{2h} (a(x_i) + a(x_{i+1}))$$

$$B_2(\varphi_i, \varphi_i) = \int_0^\ell b \varphi_i^2 = \int_{x_{i-1}}^{x_i} b\varphi_i^2 + \int_{x_i}^{x_{i+1}} b\varphi_i^2$$

$$\approx \frac{1}{2} \Big( \cancel{b(x_{i-1}) \underbrace{\varphi_i(x_{i-1})}_{0}{}^2} + b(x_i) \overbrace{\varphi_i(x_i)}^{=1}{}^2 \Big) h$$

$$+ \frac{1}{2} \Big( b(x_i) \underbrace{\varphi_i(x_i)}_{1}{}^2 + \cancel{b(x_{i+1}) \underbrace{\varphi_i(x_{i+1})}_{0}{}^2} \Big) h = h\, b(x_i)$$

$$B_2(\varphi_i, \varphi_{i+1}) = \int_0^\ell b \varphi_i \varphi_{i+1} = \int_{x_i}^{x_{i+1}} b \varphi_i \varphi_{i+1} \approx 0$$

$$L(\varphi_i) = \int_0^\ell f\varphi_i = \int_{x_{i-1}}^{x_i} f\varphi_i + \int_{x_i}^{x_{i+1}} f\varphi_i = \frac{f(x_i)}{2} \cdot h + \frac{f(x_i)}{2} \cdot h = h f(x_i)$$

SVOJSTVENA ZADAĆA: NAĆI $\lambda \in \mathbb{R}$ i $u \neq 0$

$$\int_0^\ell a u'v' + buv = \lambda \underbrace{\int_0^\ell g u v}_{= C(u,v)} \quad , v \in V$$

$\vdots$

$$\sum_{i=1}^n \alpha_i B(\varphi_i, \varphi_j) = \lambda \sum_{i=1}^n \alpha_i C(\varphi_i, \varphi_j) \quad , j=1,\dots,n$$

$$A = \big(B(\varphi_i,\varphi_j)\big)_{j,i} \quad , \quad M = \big(C(\varphi_i,\varphi_j)\big)_{j,i} \quad , \quad X = (\alpha_i)_{i=1,\dots,n}$$

NAĆI $\lambda \in \mathbb{R}$ i $0 \neq X \in \mathbb{R}^n$ T.D.

$$AX = \lambda M X$$

GENERALIZIRANA SVOJSTVENA ZADAĆA

EVOLUCIJSKA ZADAĆA (FEM + VREMENSKA DISKRETIZACIJA)

$$\rho u_{tt} - (au')' + bu = f$$

+ P.U. + R.U.

SLABA:

$$\int_0^l \rho u_{tt} v + \int_0^l a u' v' + b u v = \int_0^l f v \quad , v \in V \subseteq H^1(0,l)$$

+R.U.

+ P.U.

~~PROSTORNA DISKRETIZACIJA (FEM)~~

ILI

$$\frac{d^2}{dt^2} \int_0^l \rho u v + B(u,v) = L(v), \quad v \in V$$

$$\int_0^l \rho u v = C(u,v)$$

+ P.U.

PROSTORNA DISKRETIZACIJA (FEM)

$u_n(t) \in V_n$ - KON. DIM.

$$u_n(t) = \sum_{i=1}^n \alpha_i(t) \varphi_i$$

$$A = \left(B(\varphi_i, \varphi_j)\right)_{j,i}, \quad F(t) = \left(L(\varphi_j)\right)_j, \quad X(t) = \left(\alpha_j(t)\right)_j$$

$$M = \left(C(\varphi_i, \varphi_j)\right)_{j,i}$$

$$\frac{d^2}{dt^2}\left(M X(t)\right) + A X(t) = F(t)$$

+P.U.

$$M \ddot{X}(t) + A X(t) = F(t)$$

$$X(0) = X_0$$

$$\dot{X}(0) = X_1$$

SUSTAV: $Y_1(t) = X(t)$ $\quad\left.\begin{array}{l} Y_1(0) = X_0 \\ Y_2(0) = X_1 \end{array}\right\}$ P.U.

$Y_2(t) = \dot{X}(t)$

$$\begin{cases} M\,\dot{Y}_2(t) + A\,Y_1(t) = F(t) \\ \dot{Y}_1(t) - Y_2(t) = 0 \end{cases} \qquad Y(t) = \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}}_{\tilde{M}} \dot{Y}(t) + \underbrace{\begin{bmatrix} 0 & -I \\ A & 0 \end{bmatrix}}_{\tilde{K}} Y(t) = \underbrace{\begin{bmatrix} 0 \\ F(t) \end{bmatrix}}_{\tilde{F}(t)}$$

$$\begin{cases} \tilde{M}\,\dot{Y}(t) + \tilde{K}\,Y(t) = \tilde{F}(t) \\ Y(0) = \begin{bmatrix} X_0 \\ X_1 \end{bmatrix} =: Y_0 \end{cases}$$

IMPLICIT MID-POINT RULE $\qquad Y^n \approx Y(t^n),\ t^n = n\Delta t$

$$\tilde{M}\,\frac{Y^{n+1} - Y^n}{\Delta t} + \tilde{K}\left(\frac{Y^{n+1} + Y^n}{2}\right) = \frac{\tilde{F}^{n+1} + \tilde{F}^n}{2} \quad \Big| \cdot \Delta t$$

$$\left(\tilde{M} + \frac{1}{2}\Delta t\,\tilde{K}\right) Y^{n+1} = M Y^n - \frac{1}{2}\Delta t\,\tilde{K} Y^n + \left(\frac{1}{2}\tilde{F}^{n+1} + \frac{1}{2}\tilde{F}^n\right)$$

$$\begin{bmatrix} I & -\frac{1}{2}\Delta t\,I \\ \frac{1}{2}\Delta t\,A & M \end{bmatrix} Y^{n+1} = \begin{bmatrix} I & +\frac{1}{2}\Delta t\,I \\ -\frac{1}{2}\Delta t\,A & M \end{bmatrix} Y^n + \frac{1}{2}\Delta t\left(\tilde{F}^{n+1} + \tilde{F}^n\right)$$

$$\begin{bmatrix} I & -\frac{1}{2}\Delta t\,I \\ 0 & M + \frac{1}{4}(\Delta t)^2 A \end{bmatrix} Y^{n+1} = \begin{bmatrix} I & \frac{1}{2}\Delta t\,I \\ -\Delta t\,A & M - \frac{1}{4}(\Delta t)^2 A \end{bmatrix} Y^n + \frac{1}{2}\Delta t \begin{bmatrix} 0 \\ F^{n+1} + F^n \end{bmatrix}$$

$$\left(M + \frac{1}{4}(\Delta t)^2 A\right) Y_2^{n+1} = \left(M - \frac{1}{4}(\Delta t)^2 A\right) Y_2^n - \Delta t A Y_1^n + \frac{1}{2}\Delta t \left(F^n + F^{n+1}\right)$$

1. JDBA

$$Y_1^{n+1} = \frac{1}{2}\Delta t\, Y_2^{n+1} + Y_1^n + \frac{1}{2}\Delta t\, Y_2^n$$
$$= Y_1^n + \frac{1}{2}\Delta t \left( Y_2^n + Y_2^{n+1} \right)$$

---

$Y_1^{n+1} = X^{n+1}$ – POLOŽAJ

$Y_2^{n+1} = \dot{X}^{n+1}$ – BRZINA

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RADI RUBNIH UVJETA BOLJE JE RJEŠAVATI SUSTAV

2. JDBA

$$K Y_1^{n+1} = K \left( Y_1^n + \frac{1}{2}\Delta t \left( Y_2^n + Y_2^{n+1} \right) \right)$$