5.5. TRAG

$u \in W^{1,p}(U)$
- u def. s.s.
- ako je $u \in C(\bar{U})$ $\quad u|_{\partial U}$ dobro def
- ima li smisla $u|_{\partial U}$ inače
- nap: ∂U n-dim mjere 0

TM 1 (Teorem o tragu).

Neka je $p \in [1, +\infty\rangle$, U ograničen, ∂U klase C^1. Tada $\exists$ $T \in L(W^{1,p}(U), L^p(\partial U))$ ograničen t.d.

(i)
$$Tu = u|_{\partial U}, \quad u \in W^{1,p}(U) \cap C(\bar{U})$$

(ii)
$$\|Tu\|_{L^p(\partial U)} \leq C \|u\|_{W^{1,p}(U)}, \quad u \in W^{1,p}(U)$$

C ovisi samo o p i U.

T - operator traga

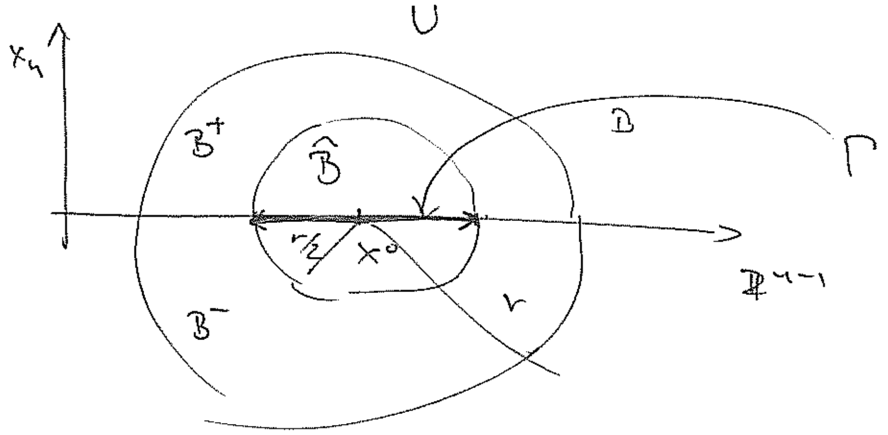
Tu - trag od u na ∂U

DOK: PRETP $u \in C^1(\bar{U})$.

1) PRETP: $x^0 \in \partial U$, ∂U "RAVAN" U OKOLINI x^0

DAN S $x_n = 0$

(KAO U TM O PROŠIRENJU)



NEKA JE: $\zeta \in C_c^\infty(B)$, $\zeta|_B \geq 0$, $\zeta|_{\hat{B}} \equiv 1$

OZNAKA: $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1} = \{x : x_n = 0\}$

$$\int_\Gamma |u|^p \, dx' \overset{\text{SV. } \zeta}{\leq} \int_{\mathbb{R}^{n-1} \cap B^+} \zeta |u|^p \, dx'$$

$$\overset{N-L}{=} -\int_{B^+} (\zeta |u|^p)_{x_n} \, dx$$

$$= -\int_{B^+} \zeta_{x_n} |u|^p + \zeta p |u|^{p-1} \operatorname{sign}(u) u_{x_n} \, dx$$

$$\leq C \int_{B^+} |u|^p \, dx + C \int_{B^+} |u|^{p-1} |u_{x_n}| \, dx$$

YOUNGOVA NEJEDNAKOST $ab \leq \frac{a^q}{q} + \frac{b^r}{r}$, $\frac{1}{q} + \frac{1}{r} = 1$

$$\leq C \left(\int_{B^+} |u|^p \, dx + \int_{B^+} |Du|^p \, dx \right)$$

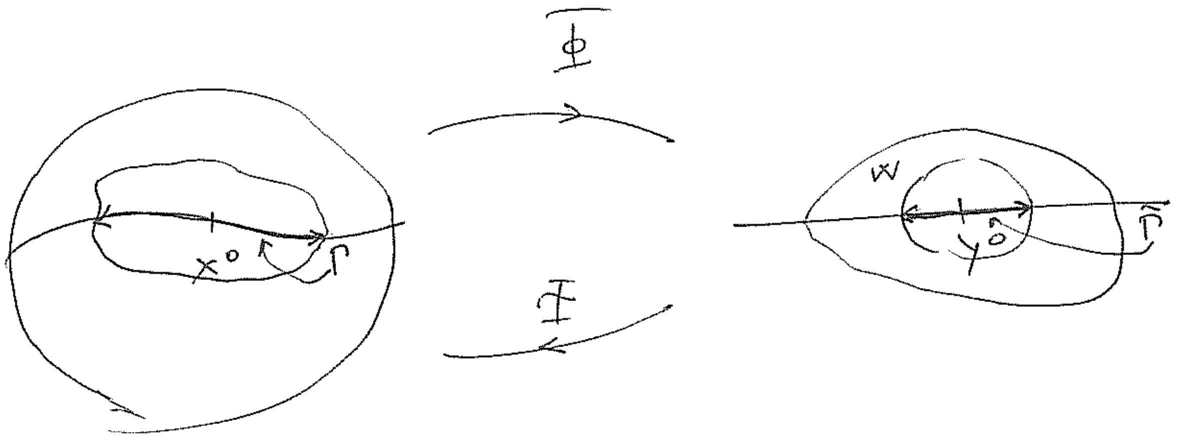
$$\leq \frac{|u|^{(p-1)q}}{q} + \frac{|u_{x_n}|^p}{p}, \quad \frac{1}{p} + \frac{1}{q} = 1$$

$p =$

$$\Rightarrow \quad \|u\|_{L^p(\Gamma)} \le C\,\|u\|_{W^{1,p}(B^+)} \le C\,\|u\|_{W^{1,p}(U)}$$

C – ODVISI OD p IN ζ

2) PREDP.: $x^0 \in \partial U$, ∂U KLASE C^1



$$\tilde u(y) := u(\Psi(y))$$

$$\Rightarrow \quad \|\tilde u\|_{L^p(\tilde\Gamma)} \le C\,\|\tilde u\|_{W^{1,p}(W)}$$

Φ – OHRANJA VOLUMEN

$$u(x) = \tilde u(\Phi(x))$$

$$\int_\Gamma |u|^p\,d\gamma = \int_\Gamma |\tilde u \circ \Phi(x)|^p\,d\gamma = \int_{\tilde\Gamma} |\tilde u|^p \,\|\gamma'\|\,d\cancel{x}'$$

$$\le C \int_{\tilde\Gamma} |\tilde u|^p\,d\cancel{x}'$$

$$\le C\,\|\tilde u\|^p_{W^{1,p}(W^+)}$$

$$= C \int_{W^+} |\tilde u|^p + |D\tilde u|^p\,d\cancel{x}$$

$$\Rightarrow \quad \|u\|^p_{L^p(\Gamma)} \leq C \int_{W^+} |u \circ \Phi|^p + |Du \circ \Phi \, D\Phi|^p \, dy$$

$\det D\Phi = 1$

$$\leq C \int_{B^+} |u|^p + |Du|^p |D\Psi|^p \, dx$$

$$\leq C \|u\|^p_{W^{1,p}(B^+)}$$

$$\leq C \|u\|^p_{W^{1,p}(B)}$$

DAKLE: $\forall \; x^0 \in \partial U \;\; \exists \; C > 0 \;;\; \Gamma$ ~~T.D~~ OTVOREN

$$\|u\|_{L^p(\Gamma)} \leq C \|u\|_{W^{1,p}(U)} \quad , \; u \in W^{1,p}(U).$$

3) ∂U KOMPAKTAN GORNJI POKRIVAČ OD ∂U SE REDUCIRA NA KONAČAN!

$i = 1, \dots, N \quad x_i^0 \in \partial U, \quad \Gamma_i \subset \partial U, \quad \bigcup_{i=1}^N \Gamma_i = \partial U$

$$\|u\|_{L^p(\Gamma_i)} \leq C \|u\|_{W^{1,p}(U)} \qquad i = 1, \dots, N$$

DEF: $Tu := u|_{\partial U}$

$$\Rightarrow \quad \|Tu\|_{L^p(\partial U)} \leq C \|u\|_{W^{1,p}(U)} \quad , \; u \in W^{1,p}(U) \cap C^1(U)$$

C OVISI O p, $\underline{\zeta_i, \Phi_i}$.

C NE OVISI O u.

4) PROŠIRENJE L. OPERATORA PO NEPREKIDNOSTI

ZA $u \in W^{1,p}(U)$ $\exists\, u_m \in C^\infty(\bar{U})$, $u_m \to u$ u $W^{1,p}(U)$

$\Rightarrow$ $\|Tu_m - Tu_l\|_{L^p(\partial U)} = \|T(u_m - u_l)\|_{L^p(\partial U)} \leq C\|u_m - u_l\|_{W^{1,p}(U)}$

(u_m) C-NIZ U $W^{1,p}(U)$

$\Downarrow$

(Tu_m) C-NIZ U $L^p(\partial U)$

$\Rightarrow$ $Tu_m \xrightarrow[L^p(\partial U)]{} * =: Tu$

NAP: NEKA JE v_m DRUGI NIZ $v_m \to v$ U $W^{1,p}(U)$

$\|Tv_m - Tu_m\|_{L^p(\partial U)} = \|T(v_m - u_m)\|_{L^p(\partial U)} \leq C\|v_m - u_m\|_{W^{1,p}(U)} \to 0$

$\Rightarrow$ Tv_m I Tu_m IMAJU ISTI LIMES

$$\|Tu_m\|_{L^p(\partial U)} \leq C\|u_m\|_{W^{1,p}(\partial U)}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\|Tu\|_{L^p(\partial U)} \leq C\|u\|_{W^{1,p}(\partial U)}$$

5) NEKA JE $u \in W^{1,p}(U) \cap C(\bar{U})$

$\exists\, u_m \in C^\infty(\bar{U})$ $u_m \to u$ $W^{1,p}(U)$ (ALI I U $C(\bar{U})$)

$\Rightarrow$ $\|u_m - u\|_{C(\bar{U})} \to 0$ IZ KONSTRUKCIJE U TM 3 U §5.3 !

$\Rightarrow$ $\|u_m - u\|_{C(\partial U)} \to 0$ $Tu_m \to u|_{\partial U}$ u $C(\partial U)$

$\Rightarrow$ $\|Tu_m - u|_{\partial U}\| \to 0$ $Tu_m \to Tu$

TM 2: NEKA JE $p \in [1, +\infty\rangle$, U OGRANIČEN I ∂U KLASE C^1.

NEKA JE $u \in W^{1,p}(U)$. TADA

$$u \in W^{1,p}_0(U) \iff Tu = 0 \text{ NA } \partial U$$

DOK: $\boxed{\Rightarrow}$ NEKA JE $u \in W^{1,p}_0(U)$

$$\Rightarrow \exists\, u_m \in C_c^\infty(U) \quad u_m \to u \quad u\ W^{1,p}(U).$$

$$Tu_m = 0$$

$$\underset{\overset{\|}{0}}{Tu_m} \longrightarrow Tu \quad u\ L^p(\partial U) \ (\text{NEPREKIDNOST OD } T)$$

$$\Rightarrow Tu = 0 \quad \text{NA } \partial U$$

$\boxed{\Leftarrow}$ NEKA JE $Tu = 0$ NA ∂U.

KORISTIMO: P. 1 & "RAVNANJE" DOMENE

DOLAZIMO DO ZADAĆE:

POLU PROSTOR

$u \in W^{1,p}(\mathbb{R}^n_+)$, u IMA KOMPAKTAN NOSAČ U $\overline{\mathbb{R}}^n_+$

$Tu = 0$ NA $\partial \mathbb{R}^n_+ = \mathbb{R}^{n-1}$

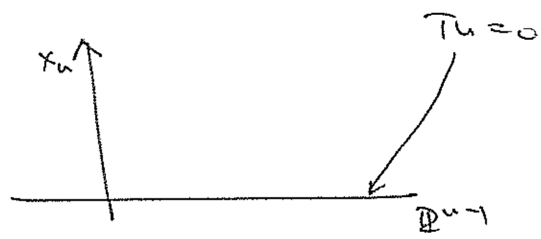
$Tu = 0$ NA $\mathbb{R}^{n-1}$

$\Downarrow$

$\exists\, u_m \in C^1(\overline{\mathbb{R}}^n_+)$

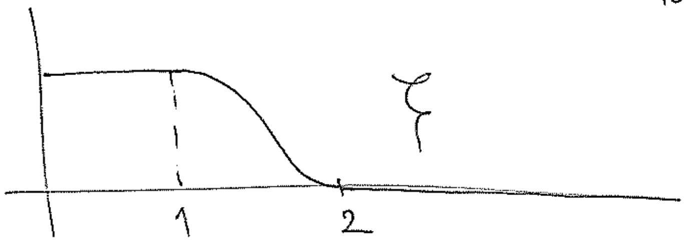
$u_m \longrightarrow u \quad W^{1,p}(\mathbb{R}^n_+)$

$u_m|_{\mathbb{R}^{n-1}} = Tu_m \longrightarrow 0 \quad L^p(\mathbb{R}^{n-1})$



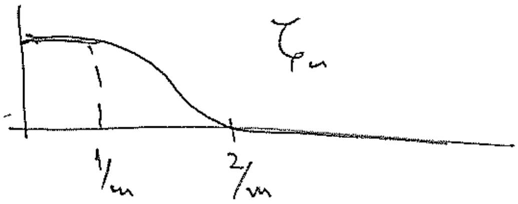
(TAKO SMO DEF T U PROŠIRENJU)

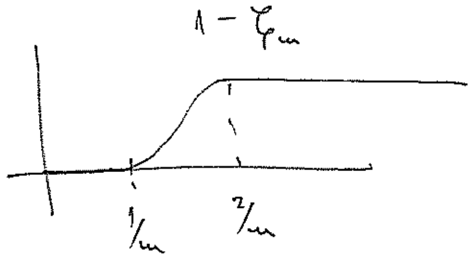
NEKA JE: $\zeta \in C^\infty([0,+\infty\rangle)$, $\zeta|_{[0,1]} \equiv 1$, $\zeta|_{[2,+\infty\rangle} \equiv 0$ $0\le\zeta\le 1$



$$\zeta_n(x) := \zeta(n x_n) \quad , \quad x \in \mathbb{R}^n_+$$

CUT-OFF FUNKCIJA





$$w_n(x) := u(x)\,(1-\zeta_n)$$

POKAŽE SE

$$w_n \longrightarrow u \quad u \quad W^{1,p}(\mathbb{R}^n_+)$$

$w_n = 0$ ZA $0 < x_n < \frac{1}{n}$ IZGLADIMO DO $C_c^\infty(\mathbb{R}^n_+)$

$$\Rightarrow \quad u \in W_0^{1,p}(\mathbb{R}^n_+)$$