2.2.5. ENERGETSKE METODE

PRINCIP MAKSIMUMA $\rightsquigarrow$ JEDINSTVENOST
U OTVOREM, ∂U KLASE C^1.

TEOREM 16 (JEDINSTVENOST)
POSTOJI NAJVIŠE JEDNO RJEŠENJE $u \in C^2(\bar{U})$ OD

$$\begin{cases} -\Delta u = f & u\ U \\ u = g & na\ \partial U \end{cases}$$

DOK: PRETPOSTAVIMO DA POSTOJE DVA $u, \hat{u}$.
DEF:
$$w := u - \hat{u}$$

$$\Rightarrow \quad \Delta w = 0 \quad \Big|\cdot w \quad \int_U$$
$$w = 0 \quad na\ \partial U$$

$$0 = \int_U w \Delta w = \int_U w \operatorname{div}(Dw)$$

$$= \int_U \operatorname{div}(w Dw) - Dw \cdot Dw$$

$$= \int_{\partial U} \underbrace{w}_{=0} Dw \cdot \nu - \int_U Dw \cdot Dw$$

$$= -\int_U |Dw|^2 \qquad \Rightarrow \quad Dw = 0 \quad u\ U$$

$$\Rightarrow \quad w = \text{const} \quad u\ U$$

$$w\big|_{\partial U} = 0 \Rightarrow \text{const} = 0$$

$$\Rightarrow \quad w = 0$$

$$\Rightarrow \quad u = \hat{u}$$

FUNKCIONAL ENERGIJE

$$I(w) = \int_U \left(\frac{1}{2} |Dw|^2 - wf \right) dx$$

NA SKUPU $A = \{ w \in C^2(\bar{U}) : w = g \text{ na } \partial U \}$

TM 17 (DIRICHLETOV PRINCIP)

NEKA JE $u \in C^2(\bar{U})$ RJEŠENJE

$$\left.\begin{aligned} -\Delta u &= f \quad \text{u } U \\ u &= g \quad \text{na } \partial U \end{aligned}\right\} \quad (*)$$

TADA, ~~AKO~~

$$I(u) = \min_{w \in A} I(w)$$

OBRATNO, AKO JE u MINIMUM OD I NA A

TADA u ZADOVOLJAVA $(*)$.

DOK: $\boxed{\Rightarrow}$

$$-\Delta u = f \quad , \quad w \in A \quad (w = g \text{ na } \partial U)$$

$$\Big| \cdot (u - w) \quad \int_U$$

$$\int_U -\Delta u \, (u-w) = \int_U f(u-w)$$

P.I.

$$=$$

$$(u-w)\big|_{\partial U} = 0 \rightarrow \int_U Du \cdot D(u-w)$$

$$\int_U Du \cdot Du - \int_U fu = \int_U Du \cdot Dw - fw$$

$$\leq \int_U |Du| \cdot |Dw| - fw$$

$$\leq \frac{1}{2} \int_U |Du|^2 + |Dw|^2 - f\cancel{w}$$

$$\frac{1}{2} \int_U |Du|^2 - \int_U fu \leq \frac{1}{2} \int_U |Dw|^2 - \int_U fw \quad , \quad w \in A$$

$u \in \mathcal{A}$ & $I(u) \le I(w)$, $w \in \mathcal{A}$

$\boxed{\Leftarrow}$ $I(u) = \min_{w \in \mathcal{A}} I(w)$

$v \in C_c^\infty(U)$ & $j(t) := I(u + tv)$, $t \in \mathbb{R}$

$$j(t) = \int_U \tfrac{1}{2}|Du + tDv|^2 - f(u+tv)$$

$$= \int_U \tfrac{1}{2}|Du|^2 + tDu\cdot Dv + \tfrac{1}{2}t^2|Dv|^2 - fu - tfv$$

$$= \int_U \tfrac{1}{2}|Du|^2 - fu + t\int_U Du\cdot Dv - fv + \tfrac{1}{2}t^2\int_U |Dv|^2$$

KVADRATIČNA FUNKCIJA U t

U 0 IMA MINIMUM

$$\Rightarrow 0 = j'(0) = \int_U Du\cdot Dv - fv$$

DAKLE:

$$\int_U Du\cdot Dv = \int_U fv, \quad v \in C_c^\infty(U)$$

P.I.

$$\parallel$$

$$-\int_U \Delta u \cdot v$$

$$\int_U (\Delta u + f)v = 0, \quad v \in C_c^\infty(U)$$

$$\Rightarrow -\Delta u = f \text{ u } U$$

$u \in \mathcal{A} \Rightarrow$ $u = g$ na ∂U

2.3. JEDNADŽBA PROVOĐENJA

$\subseteq \mathbb{R}^n$ OTVOREN

$u_t - \Delta u = 0$ HOMOGENA

$u_t - \Delta u = f$ NEHOMOGENA

$(x,t) \in U \times \langle 0, +\infty \rangle$

RJEŠENJE : $u : \bar{U} \times [0, +\infty\rangle \to \mathbb{R}$

$u(x, t)$

PRISTUP SLIJEDI PRISTUP KOD HARMONIJSKIH FUNKCIJA

MOTIVACIJA: EVOLUCIJA (U VREMENU) GUSTOĆE u

$V \subset U$

$$\underbrace{\frac{d}{dt} \int_V u \, dx}_{\text{BRZINA PROMJENE KOLIČINE U } V} = \underbrace{- \int_{\partial V} F \cdot \nu \, ds}_{\text{UKUPAN PROTOK KROZ } \partial V}$$

TM. O DIVERGENCIJI

$$u_t = - \operatorname{div} F$$

ZAKON PONAŠANJA : $F = - a Du \quad , a > 0$

$$u_t = a \operatorname{div} Du = a \Delta u$$

2.3.1. FUNDAMENTALNA RJEŠENJA

SIMETRIJE RJ $\longrightarrow$ SPECIJALNA RJ.

u RJ $\longrightarrow$ DEF: $\bar{u}(x,t) = u(\lambda x, \lambda^2 t)$, $\lambda \in \mathbb{R}$

$$\bar{u}_t - \Delta_x \bar{u} = \lambda^2 u_t(\lambda x, \lambda^2 t) - \lambda^2 \Delta u(\lambda x, \lambda^2 t) = 0$$

$\Rightarrow$ $\bar{u}$ JE RJEŠENJE

SUGERIRA DA IMA SMISLA TRAŽITI RJ. OBLIKA

$$u(x,t) = v\left(\frac{|x|^2}{t}\right)$$

MALO DRUKČIJE:

$$u(x,t) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right) \quad , \alpha, \beta \in \mathbb{R}$$

TRAŽIMO v!

$$0 = u_t - \Delta_x u = -\alpha \frac{1}{t^{\alpha+1}} v\left(\frac{x}{t^\beta}\right) + \frac{1}{t^\alpha} Dv\left(\frac{x}{t^\beta}\right) \cdot (-\beta) x \frac{1}{t^{\beta+1}}$$

$$- \frac{1}{t^\alpha} \frac{1}{t^{2\beta}} \Delta v\left(\frac{x}{t^\beta}\right)$$

ZAMJENA VARIJABLI:

$$y = \frac{x}{t^\beta}$$

$$\alpha v(y) + \beta Dv(y) \cdot y + t^{1-2\beta} \Delta v(y) = 0$$

BIRAM $\beta = \frac{1}{2}$ DA t NESTANE

$$\alpha v(y) + \frac{1}{2} Dv(y) \cdot y + \Delta v(y) = 0$$

I OVA JDBA INVARIJANTNA NA ROTACIJE

TRAŽIMO RADIJALNA RJEŠENJA:

$$v(y) = w(|y|) \quad , w: \mathbb{R} \to \mathbb{R}$$

$$\alpha w + \frac{1}{2} r w' + w'' + \frac{n-1}{r} w' = 0 \quad \Big| \, r^{n-1}$$

$$r^{n-1} w'' + (n-1) r^{n-2} w' + \frac{1}{2} r^n w' + \alpha r^{n-1} w = 0$$

$$\left(r^{n-1} w'\right)' + \frac{1}{2}\left(r^n w\right)' + \left(\alpha - \frac{n}{2}\right) r^{n-1} w = 0$$

BIRAM: $\alpha = \frac{n}{2}$

$$\left(r^{n-1} w'\right)' + \frac{1}{2}\left(r^n w\right)' = 0$$

$$\Rightarrow \quad r^{n-1} w' + \frac{1}{2} r^n w = a = \text{const}$$

ZA $a \neq 0$ w ILI w' NEOGRANIČENO ~~~~ u $+\infty$

ZATO $a = 0$

$$w' = -\frac{1}{2} r w$$

$$\frac{w'}{w} = -\frac{1}{2} r$$

$$\ln w = -\frac{r^2}{4} + C$$

$$w(r) = b\, e^{-\frac{r^2}{4}}$$

<u>DAKLE:</u>

$$u(x,t) = \frac{b}{t^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$$

<u>DEF:</u> FUNDAMENTALNO RJEŠENJE JEDNADŽBE PROVOĐENJA

$$\Phi(x,t) := \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, & x \in \mathbb{R}^n, t > 0 \\ 0, & x \in \mathbb{R}^n, t < 0 \end{cases}$$

$$\lim_{t \to 0+} \Phi(x,t) = \begin{cases} 0 & , \quad x \neq 0 \\ +\infty & , \quad x \neq 0 \end{cases}$$

$(0,0)$ SINGULARNA TOČKA

<u>LEMA</u> : $\forall t > 0$

$$\int_{\mathbb{R}^n} \Phi(x,t)\, dx = 1$$

<u>DOK:</u>

$$\int_{\mathbb{R}^n} \Phi(x,t)\, dx = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}}\, dx = \left| \begin{array}{l} z = \frac{x}{2\sqrt{t}} \\ dz = \frac{dx}{(2\sqrt{t})^n} \end{array} \right|$$

$$= \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{-|z|^2}\, dz$$

$$= \left(\frac{1}{\pi^{1/2}} \int_{-\infty}^{\infty} e^{-z_i^2}\, dz_i \right)^n = 1 .$$

OBJAŠNJAVA IZBOR KONSTANTE b!

CAUCHYJEVA ZADAĆA:

$$u_t - \Delta_x u = 0 \quad u \quad \mathbb{R}^n \times \langle 0, +\infty \rangle$$
$$u = g \quad na \quad \mathbb{R}^n \times \{0\}$$

$\forall$ ZADAN $\quad (x,t) \longmapsto \Phi(x-y,t) \quad$ Rj. JDBE

$(x,t) \longmapsto \Phi(x-y,t)\, g(y) \quad$ Rj. JDBG

$(x,t) \longmapsto \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\, dy \quad$ KANDIDAT

TM 1 (RJEŠENJE C.Z.)

NEKA JE $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ I

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy \quad , x \in \mathbb{R}^n, t > 0$$

TADA:

(i) $u \in C^\infty(\mathbb{R}^n \times \langle 0, +\infty \rangle)$

(ii) $u_t - \Delta u = 0 \quad u \quad \mathbb{R}^n \times \langle 0, +\infty \rangle$

(iii) $\displaystyle\lim_{\substack{(x,t) \to (x^0, 0) \\ \mathbb{R}^n \times \langle 0, +\infty \rangle}} u(x,t) = g(x^0), \quad x^0 \in \mathbb{R}^n.$

DOK: (i) NEKA JE $t>0$

— JER JE $\quad |u(x,t)| \le \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}}\, dy\, \|g\|_{L^\infty(\mathbb{R}^n)} = \|g\|_{L^\infty(\mathbb{R}^n)}$

INTEGRAL JE DOBRO DEFINIRAN

—

$$\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}} \quad \text{je} \quad C^\infty\left(\mathbb{R}^n \times \langle 0, +\infty\rangle\right)$$

SVE DERIVACIJE SU INTEGRABILNE

$$\Rightarrow \quad u \in C^\infty\left(\mathbb{R}^n \times \langle 0, +\infty\rangle\right)$$

& SMIJEMO ZAMIJENITI DERIVACIJE I INTEGRAL

(ii) $t>0$

$$u_t(x,t) - \Delta_x u(x,t) = \int_{\mathbb{R}^n} \underbrace{\left(\Phi_t - \Delta_x \Phi\right)}_{=0}(x-y,t)\, g(y)\, dy = 0$$

(iii) FIKS: $x^0 \in \mathbb{R}^n$, $\varepsilon>0$

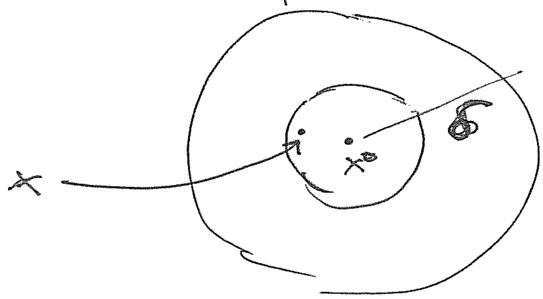
$$|u(x,t) - g(x^0)| = \left| \int_{\mathbb{R}^n} \Phi(x-y,t)\left(g(y) - g(x^0)\right) dy \right|$$

$$\le \underbrace{\int_{B(x^0,\delta)} \Phi(x-y,t)\, |g(y)-g(x^0)|\, dy}_{=I} + \underbrace{\int_{\mathbb{R}^n \setminus B(x^0,\delta)} \Phi(x-y,t)\, |g(y)-g(x^0)|\, dy}_{=J}$$

$\lim_{y\to x^0} g(y) = g(x^0)$: $\exists \delta>0$ t.d. $\quad |y-x^0|<\delta \Rightarrow |g(y)-g(x^0)|<\varepsilon$

ZA ~~x~~ T.D. $|x-x^0| < \frac{\delta}{2}$

$$I \le \varepsilon \int_{B(x^0,\delta)} \Phi(x-y,t)\, dy \le \varepsilon \int_{\mathbb{R}^n} \Phi(x-y,t)\, dy = \varepsilon$$



$$y \in \mathbb{R}^n \setminus B(x^0,\delta) \Rightarrow |y-x^0| \geq \delta$$

$$|y-x^0| \leq |y-x| + |x-x^0| \leq |y-x| + \frac{\delta}{2} \leq |y-x| + \frac{1}{2}|y-x^0|$$

$$\Rightarrow \quad \frac{1}{2}|y-x^0| \leq |y-x|$$

$$J = \int_{\mathbb{R}^n \setminus B(x^0,\delta)} \Phi(x-y,t) \underbrace{|g(y)-g(x^0)|}_{\leq 2\|g\|_{L^\infty(\mathbb{R}^n)}} dy$$

$$\leq \frac{C}{t^{n/2}} \int_{\mathbb{R}^n \setminus B(x^0,\delta)} e^{-\frac{|x-y|^2}{4t}} dy$$

$$-\frac{1}{2}|y-x^0| \geq -|y-x|$$

$$\leq \frac{C}{t^{n/2}} \int_{\mathbb{R}^n \setminus B(x^0,\delta)} e^{-\frac{|y-x^0|^2}{16t}} dy$$

$$= \frac{C}{t^{n/2}} \int_\delta^{+\infty} e^{-\frac{r^2}{16t}} r^{n-1} dr = \left| \begin{array}{l} \frac{r}{4\sqrt{t}} = \varrho \\ \frac{dr}{(4\sqrt{t})} = d\varrho \end{array} \right|$$

$$= \cancel{\frac{C}{t^{n/2}}} \, 4^n \cancel{t^{n/2}} \int_{\frac{\delta}{4\sqrt{t}}}^{+\infty} e^{-\varrho^2} \varrho^{n-1} 4^{n-1} \cancel{t^{\frac{n-1}{2}}} d\varrho$$

$$= C \, 4^n \int_{\frac{\delta}{4\sqrt{t}}}^{+\infty} e^{-\varrho^2} \varrho^{n-1} d\varrho \longrightarrow 0$$

JER INTEGRAL POSTOJI!

$$t \to 0+ \Rightarrow \frac{\delta}{4\sqrt{t}} \to +\infty$$

ZA t DOVOLJNO MALI $< \varepsilon$

NAP: (i) FORMALNO: $\Phi_t - \Delta \Phi = 0$ $\quad \mathbb{R}^n \times \langle 0, \infty \rangle$

$\Phi = \delta_0 \quad \mathbb{R}^n \times \{0\}$

(ii) $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $g \geq 0$, $g \not\equiv 0$. TADA

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y)\,dy > 0$$

$$\forall \ (x,t) \in \mathbb{R}^n \times \langle 0, +\infty \rangle$$

∞ BRZINA ŠIRENJA POREMEĆAJA

POČETNA TEMPERATURA

ZA $t > 0$ TEMPERATURA > 0 SVUGDJE!

NEHOMOGENA C.Z.

$$\begin{aligned} u_t - \Delta u &= f & &\text{u } \mathbb{R}^n \times \langle 0, +\infty \rangle \\ u &= 0 & &\text{na } \mathbb{R}^n \times \{0\} \end{aligned}$$

S > 0 FIX

$$(x,t) \longmapsto \Phi(x-y, t-s) \quad \text{Rj. J. PROVOĐENJA (HOMOGENE)}$$

$$u(x,t;s) := \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s)\, dy$$

PO PRETHODNOM ZADATKU:

$$\begin{aligned} u_t(\cdot;s) - \Delta u(\cdot;s) &= 0 & &\text{u } \mathbb{R}^n \times \langle s, +\infty \rangle \\ u(\cdot;s) &= f(\cdot,s) & &\text{na } \mathbb{R}^n \times \{s\} \end{aligned}$$

DUHAMELOV PRINCIP:

$$\begin{aligned} u(x,t) &= \int_0^t u(x,t;s)\, ds \quad , \quad x \in \mathbb{R}^n, t > 0 \\ &= \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s) f(y,s)\, dy\, ds \end{aligned}$$

DEF:

$$C_1^2(\mathbb{R}^n \times [0,+\infty\rangle) = \{ u : \mathbb{R}^n \times [0,+\infty\rangle \to \mathbb{R} : u, D_x u, D_x^2 u, u_t \in C(\mathbb{R}^n \times [0,+\infty\rangle) \}$$

TM2 (RJEŠENJE NEHOMOGENE J.T.)

NEKA JE $f \in C_1^2(\mathbb{R}^n \times [0,+\infty\rangle)$, S KOMPAKTNIM NOSAČEM I

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s)\, f(y,s)\, dy\, ds$$

TADA:

(i) $u \in C_1^2(\mathbb{R}^n \times \langle 0,+\infty\rangle)$

(ii) $u_t - \Delta u = f$ u $\mathbb{R}^n \times \langle 0,+\infty\rangle$

(iii) $\displaystyle\lim_{\substack{(x,t)\to(x^0,0)\\ \in\, \mathbb{R}^n\times\langle 0,+\infty\rangle}} u(x,t) = 0$, $x^0 \in \mathbb{R}^n$

DOK: (i) IMAMO NEPRAVI INTEGRAL I PO S
MORAMO OPRAVDATI ~~[illegible]~~ DERIVIRANJE

IDEJA: ZAMIJENA VARIJABLI

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)\, f(x-y, t-s)\, dy\, ds$$

UNIF. KVG. $\Rightarrow$

$$u_t(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)\, f_t(x-y,t-s)\, dy\, ds + \int_{\mathbb{R}^n} \Phi(y,t)\, f(x-y,0)\, dy$$

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s) \frac{\partial^2 f}{\partial x_i \partial x_j}(x-y,t-s)\, dy\, ds$$

$\Rightarrow u_t, u, D_x u, D_x^2 u \in C(\mathbb{R}^n \times [0,\infty\rangle)$

$\Rightarrow u \in C_1^2(\mathbb{R}^n \times \langle 0,+\infty\rangle)$

(ii)

$$u_t(x,t) - \Delta_x u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)\,(f_t - \Delta_x f)(x-y,t-s)\,dy\,ds + \int_{\mathbb{R}^n} \Phi(y,t)\, f(x-y,0)\,dy$$

$$= \int_\varepsilon^t \int_{\mathbb{R}^n} \Phi(y,s)\,(-f_s - \Delta_y f)\,(x-y,t-s)\,dy\,ds \quad =: I_\varepsilon \to 0$$

$$+ \int_0^\varepsilon \int_{\mathbb{R}^n} \Phi(y,s)\,(-f_s - \Delta_y f)(x-y,t-s)\,dy\,ds \quad =: J_\varepsilon = * - K$$

$$+ \int_{\mathbb{R}^n} \Phi(y,t)\, f(x-y,0)\,dy \quad =: K$$

$$|J_\varepsilon| \leq \left(\|f_s\|_\infty + \|\Delta_y f\|_\infty\right) \int_0^\varepsilon \underbrace{\int_{\mathbb{R}^n} \Phi(y,s)\,dy}_{=1}\,ds = C\varepsilon$$

$$I_\varepsilon = \int_\varepsilon^t \int_{\mathbb{R}^n} \underbrace{(\Phi_s - \Delta_y \Phi)}_{=0}(y,s)\, f(x-y,t-s)\,dy\,ds \quad = 0$$

$$+ \int_{\mathbb{R}^n} \Phi(y,\varepsilon)\, f(x-y,t-\varepsilon)\,dy \quad = *$$

$$- \int_{\mathbb{R}^n} \Phi(y,t)\, f(x-y,0)\,dy \quad = K \qquad \text{(SKRATI SE}$$

$$\Rightarrow \quad u_t(x,t) - \Delta_x u(x,t) = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} \Phi(y,\varepsilon)\, f(x-y,t-\varepsilon)\,dy = f(x,t)$$

KAO ZA POČ. U. U TM 1!

(iii)

$$|u(x,t)| \leq \int_0^t \int_{\mathbb{R}^n} \Phi(y,s)\,dy\,ds\ \|f\|_{L^\infty(\mathbb{R}^n\times[0,+\infty))} = t\,\|f\|_{L^\infty} \xrightarrow{\text{KAD } t\to 0} 0$$

NAP:

$$u_t - \Delta u = f \quad u \quad \mathbb{R}^n \times \langle 0, +\infty \rangle$$

$$u = g \quad \text{na} \quad \mathbb{R}^n \times \{0\}$$

Rj:

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y) dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s) dy\, ds$$

2.3.2 SVOJSTVO SREDNJE VRIJEDNOSTI

$U \subseteq \mathbb{R}^n$ OTVOREN, $T > 0$

$U_T := U \times \langle 0, T]$ PARABOLIČKI CILINDAR

$\Gamma_T := \overline{U}_T \setminus U_T$ PARABOLIČKA GRANICA CILINDRA