5.6.3. OPĆE NEJEDNAKOSTI

TM 6: NEKA JE $U \subseteq \mathbb{R}^n$ OTVOREN, OGRANIČEN, s C^1 RUBOM ∂U.
NEKA JE $u \in W^{k,p}(U)$.

(i) AKO JE $k < \frac{n}{p}$

TADA JE $u \in L^q(U)$, $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$

VRIJEDI: $\exists C > 0$ (OVISI O k, p, n, U)

$$\|u\|_{L^q(U)} \le C \|u\|_{W^{k,p}(U)}, \quad u \in W^{k,p}(U)$$

(ii) AKO JE $k > \frac{n}{p}$

TADA JE $u \in C^{k - [\frac{n}{p}] - 1, \gamma}(\overline{U})$, GDJE JE

$$\gamma = \begin{cases} [\frac{n}{p}] + 1 - \frac{n}{p}, & \frac{n}{p} \notin \mathbb{N}, \\ \text{BILO KOJI} \in \langle 0, 1 \rangle, & \frac{n}{p} \in \mathbb{N}. \end{cases}$$

VRIJEDI: $\exists C > 0$ (OVISI O k, p, n, γ, U)

$$\|u\|_{C^{k - [\frac{n}{p}] - 1, \gamma}(\overline{U})} \le C \|u\|_{W^{k,p}(U)}.$$

DOK: (i) $u \in W^{k,p}(U) \Rightarrow D^\alpha u \in L^p(U), \quad |\alpha| = k$

$$D^\beta u \in W^{1,p}(U), \quad |\beta| \le k-1$$

S-N-G $\Rightarrow$

$$\|D^\beta u\|_{L^{p^*}(U)} \le C \|D^\beta u\|_{W^{1,p}(U)}, \quad |\beta| \le k-1$$
$$\le C \|u\|_{W^{k,p}(U)}$$

$$\Rightarrow D^\beta u \in L^{p^*}(U), \quad |\beta| \le k-1$$

$$\Rightarrow u \in W^{k-1, p^*}(U) \qquad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$$

ISTI POSTUPAK

$$\Rightarrow u \in W^{k-1, (p^*)^*}(U) \qquad \frac{1}{p^{**}} = \frac{1}{p^*} - \frac{1}{n} = \frac{1}{p} - \frac{2}{n}$$

MOŽEMO NASTAVITI K PUTA

$$\Rightarrow u \in L^q(U) \quad \dots \quad \frac{1}{q} = \frac{1}{p} - \frac{k}{n} > 0$$

(ii) $\underline{k > \frac{n}{p} \notin \mathbb{N}} \Rightarrow \quad \frac{k}{n} > \frac{1}{p} \Rightarrow \quad \frac{1}{p} - \frac{k}{n} < 0$

$\Rightarrow$ NE MOŽEMO NAPRAVITI k ULAGANJA POMOĆU S-N-G.

$\Rightarrow$ MOŽEMO IH NAPRAVITI $l \in \mathbb{N}$:

$$l < \frac{n}{p} < l+1$$

(l JE NAJVEĆI PRIRODNI MANJI OD $\frac{n}{p}$; $l = \left[\frac{n}{p}\right]$)
STROGO!

$$\Rightarrow \quad u \in W^{k - \left[\frac{n}{p}\right], r}(U) \quad , \quad \frac{1}{r} = \frac{1}{p} - \frac{1}{n}\left[\frac{n}{p}\right]$$

$$r = \frac{1}{\frac{1}{p} - \frac{1}{n}\left[\frac{n}{p}\right]} = \frac{pn}{n - p\left[\frac{n}{p}\right]} = \frac{n}{\frac{n}{p} - \left[\frac{n}{p}\right]} > n$$

$\frac{n}{p} - \left[\frac{n}{p}\right] \in \langle 0,1 \rangle$

$$\Rightarrow \quad r > n$$

PRIMIJENIMO MORREYEVU NEJEDNAKOST ~~[illegible]~~

$$\Rightarrow \quad D^\alpha u \in C^{0, 1 - \frac{n}{r}}(\overline{U}) \qquad |\alpha| \leq k - \left[\frac{n}{p}\right] - 1$$

$$1 - \frac{n}{r} = 1 - \frac{n}{p} + \left[\frac{n}{p}\right]$$

$$\Rightarrow \quad u \in C^{k - \left[\frac{n}{p}\right] - 1, \, 1 - \frac{n}{p} + \left[\frac{n}{p}\right]}(\overline{U})$$

$\underline{k > \frac{n}{p} \in \mathbb{N}}$ SLIČNO, ALI NAPRAVIMO S-G-N $\left[\frac{n}{p}\right] - 1$ PUTA
JEDNOM MANJE
DA NAM OSTANE JEDNOM ZA MORREYA!

<u>NAP:</u> NA $\mathbb{R}^n$ NEJEDNAKOSTI SOBOLEVA $\rightsquigarrow$ FOURIEROVOM TRANSFORMACIJOM

5.7. KOMPAKTNA ULAGANJA

G-N-S $\Rightarrow$ $W^{1,p}(U) \hookrightarrow L^{p^*}(U)$, za $p \in [1, n\rangle$

$$p^* = \frac{pn}{n-p}$$

U ograničen $\Rightarrow$ $W^{1,p}(U) \hookrightarrow L^{q}(U)$ $\quad q \in [1, p^*\rangle$

↑
kompaktno!

__DEF:__ X, Y Banachovi p. $X \subset Y$.

$X \subset\subset Y$, X kompaktno uložen u Y

$\overset{c}{\hookrightarrow}$

ako:

(i) $\exists C>0$, $\|x\|_Y \leq C \|x\|_X$, $x \in X$

(ii) $\forall$ ograničen niz u X je predkompaktan u Y (ima kvg podniz u Y)

__TM 1__ (Rellich-Kondrachov teorem kompaktnosti)

Neka je $U \subseteq \mathbb{R}^n$ otvoren i ograničen, ∂U klase C^1, $p \in [1, n\rangle$. Tada je

$$W^{1,p}(U) \subset\subset L^q(U), \quad q \in [1, p^*\rangle.$$

DOK: ~~1~~ KORAK: ZNAMO $W^{1,p}(U) \hookrightarrow L^q(U)$ NEPREKIDNO ULAGANJE

$$\|u\|_{L^q(U)} \leq C \|u\|_{W^{1,p}(U)}$$

OSTAJE POKAZATI: KOMPAKTNOST ULAGANJA:

$(u_m)_m \subseteq W^{1,p}(U)$ OGRANIČEN $\Rightarrow$ IMA KONV. PODNIZ $(u_{m_j})_j$ u $L^q(U)$.

2. KORAK: TEOREM PROŠIRENJA $\Rightarrow$ BSOMP $(u_m) \subseteq W^{1,p}(\mathbb{R}^n)$

$\text{supp}\, u_m \subset\subset V$ KOMPAKTNO SADRŽAN

$$\sup_m \|u_m\|_{W^{1,p}(\mathbb{R}^n)} < \infty$$

3. KORAK: IZGLADIMO NIZ

$$u_m^\varepsilon := \eta_\varepsilon * u_m$$

ZA ε DOVOLJNO MALI $\text{supp}\, u_m^\varepsilon \subset V$, $m \in \mathbb{N}$, ~~ε~~.

4. KORAK: TV: $u_m^\varepsilon \longrightarrow u_m$ u $L^q(V)$, KAD $\varepsilon \to 0$

UNIFORMNO PO m!

DOK TV: NEKA SU u_m GLATKE

$$u_m^\varepsilon(x) - u_m(x) = \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)\, u_m(y)\,dy - \int_{B(x,\varepsilon)} \eta_\varepsilon(y)\, u_m(x)\,dy$$

$$= \int_{B(0,1)} \eta(y) \Big(u_m(x-\varepsilon y) - u_m(x)\Big)\, dy$$

$$\overset{N-L}{=} \int_{B(0,1)} \eta(y) \int_0^1 \frac{d}{dt}\Big(u_m(x-\varepsilon t y)\Big)\,dt\,dy$$

$$= \int_{B(0,1)} \eta(y) \int_0^1 Du_m(x-\varepsilon t y)(-\varepsilon y)\,dt\,dy$$

$$\Rightarrow \quad |u^\varepsilon_m(x) - u_m(x)| \le \varepsilon \int_{B(0,1)} \eta(y) \int_0^1 |Du_m(x-\varepsilon t y)| \underbrace{|y|}_{\le 1} \, dt \, dy$$

$$\Rightarrow \quad \int_V |u^\varepsilon_m(x) - u_m(x)| \, dx \le \varepsilon \int_{B(0,1)} \eta(y) \int_0^1 \underbrace{\int_V |Du_m(x-\varepsilon t y)| \, dx}_{\text{INTEGRAL } Du_m \text{ PO ETTAVOM } V} \, dt \, dy$$

$$\le \varepsilon \int_{B(0,1)} \eta(y) \int_0^1 \left(\int_V |Du_m(z)| \, dz \right) dt \, dy$$

$$= \varepsilon \, \| Du_m \|_{L^1(V)}$$

$$\Rightarrow \quad \| u^\varepsilon_m - u_m \|_{L^1(V)} \le \varepsilon \| Du_m \|_{L^1(V)} \le \varepsilon C \| Du_m \|_{L^p(V)}$$

PROŠIRIMO PO GUSTOĆI NA $u_m \in W^{1,p}(V)$ V OGRANIČEN

OGRANIČENO UNIF. PO m!

$$\Rightarrow \quad \| u^\varepsilon_m - u_m \|_{L^1(V)} \le C \varepsilon$$

NEOVISAN O m

ZA $q \in [1, p^*\rangle$ INTERPOLACIJSKA NEJEDNAKOST §B.2

$$\| u \|_{L^q(V)} \le \| u \|^{\theta}_{L^1(V)} \, \| u \|^{1-\theta}_{L^{p^*}(V)}$$

$$\frac{1}{q} = \frac{\theta}{1} + \frac{1-\theta}{p^*} \qquad \theta \in \langle 0, 1 \rangle$$

$$\Rightarrow \| u^\varepsilon_m - u_m \|_{L^q(V)} \le \| u^\varepsilon_m - u_m \|^{\theta}_{L^1(V)} \underbrace{\| u^\varepsilon_m - u_m \|^{1-\theta}_{L^{p^*}(V)}}_{\le C}$$

UNIF. IZ S-N-G

$$\Rightarrow \quad \| u^\varepsilon_m - u_m \|_{L^q(V)} \le C \varepsilon^{\theta}$$

$\Rightarrow$ UNIF. KVG

5. KORAK: TV: $\forall \varepsilon > 0$ $(u_m^\varepsilon)_m$ JE UNIFORMNO OGRANIČEN I EKVINEPREKIDAN

POK TV:

$$|u_m^\varepsilon(x)| = \left| \int_{B(x,\varepsilon)} \eta_\varepsilon(x-y)\, u_m(y)\, dy \right| \leq \underbrace{\|\eta_\varepsilon\|_{L^\infty(\mathbb{R}^n)}}_{\leq \frac{C}{\varepsilon^n}} \underbrace{\|u_m\|_{L^1(V)}}_{\leq C}$$

$$\leq \frac{C}{\varepsilon^n} < \infty$$

UNIFORMNA OGRANIČENOST

SLIČNO,

$$|Du_m^\varepsilon(x)| \leq \frac{C}{\varepsilon^{n+1}} < \infty$$

EKVINEPREKIDNOST NIZA:

$$\forall \gamma > 0 \quad \exists \delta > 0 \ \forall x,y \ |x-y| < \delta \Rightarrow |u_m^\varepsilon(x) - u_m^\varepsilon(y)| < \gamma, \quad \forall m \in \mathbb{N}$$

RAČUNAM:

$$|u_m^\varepsilon(x) - u_m^\varepsilon(y)| = \left| \int_{B(x,\varepsilon)} \eta_\varepsilon(x-z)\, u_m(z)\, dz - \int_{B(y,\varepsilon)} \eta_\varepsilon(y-z)\, u_m(z)\, dz \right|$$

$$= \left| \int_{B(0,1)} \eta(w) \big(u_m(x-\varepsilon w) - u_m(y-\varepsilon w)\big)\, dw \right|$$

$$= \left| \int_{B(0,1)} \eta(w) \int_0^1 \frac{d}{d\lambda}\big(u_m(\lambda x + (1-\lambda)y - \varepsilon w)\big)\, d\lambda\, dw \right|$$

$$= \left| \int_{B(0,1)} \eta(w) \int_0^1 Du_m(\lambda x + (1-\lambda)y - \varepsilon w)(x-y)\, d\lambda\, dw \right|$$

$$\leq \|\eta\|_{L^\infty(\mathbb{R}^n)} \underbrace{\int_{B(0,1)} \int_0^1 |Du_m(\lambda x + (1-\lambda)y - \varepsilon w)|\, d\lambda\, dw}_{\leq \|Du_m\|_{L^1(V)} < \infty \text{ UNIFORMNO PO } m} |x-y|$$

$$\leq C|x-y|$$

$$\forall \gamma > 0 \quad \exists \delta = \frac{\gamma}{C} \quad \ldots$$

6. KORAK: TV: $\forall \delta > 0$ $\exists$ PODNIZ $(u_{m_j})_j$ T.D.

$$\limsup_{j,k\to\infty} \|u_{m_j} - u_{m_k}\|_{L^2(V)} \leq \delta$$

$$\left(\exists n_0 \ \forall k,j \geq n_0 \quad \|u_{m_j} - u_{m_k}\| \leq 2\delta \right)$$

DOK TV:

ZNAMO: $u_m^{\varepsilon} \xrightarrow[\varepsilon \to 0]{} u_m$ u $L^2(V)$ UNIFORMNO PO m

NEKA JE $\varepsilon > 0$ DOVOLJNO MALI T.D.

$$\|u_m^{\varepsilon} - u_m\|_{L^2(V)} \leq \delta/2, \quad m \in \mathbb{N}$$

(u_m^{ε}) IMAJU NOSAČ U V

5. KORAK $\Rightarrow$ UNIF. OGR & EKVINEPREKIDNE $\}$ ARZELA-ASCOLI TEOREM (u_m^{ε} NEPREKIDNE)

$\Downarrow$

$\exists$ PODNIZ $(u^{\varepsilon}_{m_j})_j$ KOJI KVG UNIF. NA V

$\swarrow$

KVG U L^2 NORMI

$\swarrow$

$$\limsup_{j,k\to+\infty} \|u^{\varepsilon}_{m_j} - u^{\varepsilon}_{m_k}\|_{L^2(V)} = 0$$

$$\Downarrow \left(\exists n_0 \ \forall j,k \geq n_0 \quad \|u^{\varepsilon}_{m_j} - u^{\varepsilon}_{m_k}\|_{L^2(V)} \leq \delta \right)$$

$$\|u_{m_j} - u_{m_k}\|_{L^2(V)} \leq \underbrace{\|u_{m_j} - u^{\varepsilon}_{m_j}\|_{L^2}}_{\leq \frac{\delta}{2}} + \underbrace{\|u^{\varepsilon}_{m_j} - u^{\varepsilon}_{m_k}\|_{L^2}}_{\limsup \to 0 \ (\leq \delta)} + \underbrace{\|u^{\varepsilon}_{m_k} - u_{m_k}\|_{L^2}}_{\leq \frac{\delta}{2}}$$

$$\Rightarrow \limsup_{j,k\to+\infty} \|u_{m_j} - u_{m_k}\|_{L^2(V)} \leq \delta$$

$$\left(\exists n_0 \ \forall k,j \geq n_0 \quad \|u_{m_j} - u_{m_k}\|_{L^2(V)} \leq 2\delta \right)$$

7. KORAK (DIJAGONALNI POSTUPAK)

$$
\begin{array}{lllll}
\delta & & & & \\
1 & u_{m^1_1} & u_{m^1_2} & u_{m^1_3} & u_{m^1_4} \; \cdots \\
\frac{1}{2} & u_{m^2_1} & u_{m^2_2} & u_{m^2_3} & u_{m^2_4} \; \cdots \\
\frac{1}{3} & u_{m^3_1} & u_{m^3_2} & u_{m^3_3} & u_{m^3_4} \; \cdots \\
\frac{1}{4} & u_{m^4_1} & u_{m^4_2} & u_{m^4_3} & u_{m^4_4} \; \cdots \\
\vdots & & & & \ddots
\end{array}
$$

PODNIZ OD (u_m)

PODNIZ OD $(u_{m^1_j})$

PODNIZ OD $(u_{m^2_j})$

PODNIZ OD $(u_{m^3_j})$

DEF: $u_k = u_{m^k_k} \quad k \in \mathbb{N}$

$\Rightarrow u_k$ JE KVG

<u>DOK:</u> $\delta > 0 \Rightarrow \exists\, k_0 \in \mathbb{N}$ T.D. $\frac{1}{k_0} < \delta/2$

$\Rightarrow (u_k)_{k \geq k_0}$ JE PODNIZ OD $(u_{m^{k_0}_j})$

$\Rightarrow \exists\, m_0 \in \mathbb{N}$ T.D. $\forall\, j,k \geq m_0$

$\| u_{m^{k_0}_j} - u_{m^{k_0}_k} \| < \delta$

$\Rightarrow$ ZA DOVOLJNO VELIKI k_0 JE

$\forall k, j \geq k_0 \quad \| u_k - u_j \| < \delta$

<u>NAP:</u>

$$W^{1,p}(U) \subset\subset L^p(U) \quad , \quad p \in [1, +\infty]$$

JER JE $p < p^*$ (TO JE ZA $p < n$).

ZA $p > n$ DIREKTNO IZ MORREYEVE NEJEDNAKOSTI I A-A TM

$W^{1,p}_0(U) \subset\subset L^p(U)$ I ZA ∂U KOJI NIJE C^1!