Homework Assignment 2
Partial Differential Equations

Problem 1

Let $U \subseteq \mathbb{R}^n$ be an open and bounded set, and let $p \in [1, \infty)$. Suppose that $(u_k)_k \subseteq W^{1,p}(U)$ is a sequence and $u \in W^{1,p}(U)$ such that $u_k \to u$ strongly in $W^{1,p}(U)$. Prove that

- $u_k \rightharpoonup u$ weakly in $L^p(U)$,
- $Du_k \rightharpoonup Du$ weakly in $L^p(U; \mathbb{R}^n)$.

Problem 2

Solve the following two Sturm-Liouville problems:

1. $-u'' = \lambda u, \quad u(0) = u(1) = 0,$
2. $-u'' = \lambda u, \quad u(0) = u'(1) = 0.$

Expand a chosen function into a series of eigenfunctions for one of the problems above such that the series converges in $H_0^1(U)$ and the other one such that the convergence is only in $L^2(U)$ and not in $H_0^1(U)$.

Problem 3

Let $(u_k)_k \subseteq X$ be a sequence in a Banach space X with the property that every subsequence has a further subsequence that converges to a given $u \in X$. Prove that the entire sequence $(u_k)_k$ then converges to u in X.

Problem 4

In the proof of Theorem 3 (Strong Maximum Principle), prove that there exists a $y \in V$ such that:

$$d(y, C) < d(y, \partial U).$$

Problem 5

Let $(w_k)_k$ be a basis of eigenvectors of the operator $-\Delta$ and let $(d^k)_k \subseteq C([0, T])$. Prove that set of all functions of the form:

$$\sum_{k=1}^{n} d^k(t) w_k$$

is dense in $L^2(0, T; H^1(U))$.

Problem 6

Show that:

$$A[u_m(0), u_m(0)] \leq C \|g\|_{H^1}^2$$

where the constant C is independent of m.