On the asymptotic analysis of elastic rods

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Abstract

Asymptotic expansion method is widely used method in derivation and mathematical justification of the lower dimensional models in continuum mechanics. Important part of the method is *a priori* scaling of the unknowns. Following an idea of Miara in this work we justify the scaling directly from the 3-D elasticity and the *Ansatz* of the asymptotic expansion method.

1 Introduction

Inspired by the pioneering papers of Ciarlet & Destunder [1979a, 1979b] on elastic plates, Bermudez & Viaño [1984] and Aganović & Tutek [1981, 1986] performed an asymptotic analysis of elastic rods, in the framework of linear 3-D elasticity theory. With the help of *a priori* scaling of displacement, stress and external data and by the use of general method of Lions [1973] the leading components of the asymptotic expansion of the displacement and stress due to a small parameter (being a diameter of the cross section of the rod) were identified and a convergence of the scaled displacement and stress was proved. It turned out that the descaled leading displacement is a Bernoulli–Navier field, satisfying the classical equations of rods. The obtained result was considered as a justification of the *a priori* Bernoulli–Navier hypothesis about rigid displacement of the cross sections. With the same technique Cimetière, Geymonat, Le Dret, Raoult & Tutek [1986, 1988] investigated nonlinear rods and suggested how to calculate higher order terms of the asymptotic expansion. For the linear case Trabucho & Viaño [1987, 1989] found out the first three terms for the displacement and first two terms for the stress; in this way some engineering models of rods were justified. The asymptotic method was applied to a number of rod problems; for an extensive list of references see Trabucho & Viaño [1996] and Ciarlet [1997].

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In the mentioned works the justification of the classical models of rods was formal in the sense that instead of a physical hypothesis of Bernoulli–Navier type some *a priori* scaling of displacement, stress and external data was taken. Indeed, Miara [1992a, 1992b, 1994a, 1994b] proved that the scaling taken in the case of plates (Ciarlet & Destuynder [1979a, 1979b]) is a consequence of 3-D elasticity theory and of the *Ansatz* of formal expansion method (see Ciarlet [1997], pp. 89-95, 269). This result provides a complete justification of the scaling for the plates.

In this paper we prove an analogous fact for linear rods. In Section 2 we formulate basic assumptions and Theorem 2.1, containing the main result. The proof of Theorem 2.1 is given in Section 3. In Section 4 we consider an "overlooked" model (see Ciarlet [1997], p. 90-91).

2 Statement of the problem and the main result

In that what follows we assume that Latin indices $i, j, \dots$ take their values from the set $\{1,2,3\}$ and that Greek indices $\alpha, \beta, \dots$ take their values from the set $\{1,2\}$. We assume also a summation convention over repeating indices.

Let $\omega \subset \mathbb{R}^2_{x_1x_2}$ be a bounded, sufficiently regular domain with the properties

$$\int_\omega x_\alpha d\omega = 0, \qquad \int_\omega x_1 x_2 d\omega = 0. \tag{2.1}$$

Let be $\gamma = \partial\omega, \Omega = \omega \times \langle 0,1\rangle$, $\Gamma = \gamma \times \langle 0,1\rangle$, $\Gamma_0 = \omega \times \{0\}$ and $\Gamma_1 = \omega \times \{1\}$. For sufficiently small $\varepsilon > 0$ we define $\omega^\varepsilon = \varepsilon\omega$, $\gamma^\varepsilon = \partial\omega^\varepsilon$, $\Omega^\varepsilon = \omega^\varepsilon \times \langle 0,1\rangle$, $\Gamma^\varepsilon = \gamma^\varepsilon \times \langle 0,1\rangle$, $\Gamma_0^\varepsilon = \omega^\varepsilon \times \{0\}$ and $\Gamma_1^\varepsilon = \omega^\varepsilon \times \{1\}$. Let x^ε denotes a generic point of the domain Ω^ε; then the transformation

$$x^\varepsilon = (x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon) = (\varepsilon x_1, \varepsilon x_2, x_3), \qquad x \in \Omega \tag{2.2}$$

is a bijection $\Omega \to \Omega^\varepsilon$, $\Gamma \to \Gamma^\varepsilon$ etc. We consider the domain Ω^ε as a natural configuration of a linear, homogeneous and isotropic elastic rod with the Lamé coefficients λ^ε and μ^ε. The displacement field we denote by $\boldsymbol{u}^\varepsilon$ and the corresponding strain tensor by $\mathbf{e}(\boldsymbol{u}^\varepsilon) = \operatorname{Sym}\nabla\boldsymbol{u}^\varepsilon$. We shall assume that the rod is clamped on Γ_0^ε and Γ_1^ε. The appropriate function space is then $(V^\varepsilon)^3$, where

$$V^\varepsilon = \left\{v^\varepsilon \in H^1(\Omega^\varepsilon) : v^\varepsilon = 0 \text{ on } \Gamma_0^\varepsilon \text{ and } \Gamma_1^\varepsilon\right\}. \tag{2.3}$$

Let $\boldsymbol{f}^\varepsilon \in L^2(\Omega^\varepsilon)^3$ and $\boldsymbol{g}^\varepsilon \in L^2(\Gamma^\varepsilon)^3$ denote respectively the density of the external body and contact force. Then the equilibrium displacement $\boldsymbol{u}^\varepsilon \in (V^\varepsilon)^3$ is a unique solution to the variational problem

$$\int_{\Omega^\varepsilon} (\lambda^\varepsilon \operatorname{tr}\mathbf{e}(\boldsymbol{u}^\varepsilon)\operatorname{tr}\mathbf{e}(\boldsymbol{v}) + 2\mu^\varepsilon \mathbf{e}(\boldsymbol{u}^\varepsilon)\cdot\mathbf{e}(\boldsymbol{v}))\, dx^\varepsilon =$$
$$= \int_{\Omega^\varepsilon} \boldsymbol{f}^\varepsilon \cdot \boldsymbol{v} dx^\varepsilon + \int_{\Gamma^\varepsilon} \boldsymbol{g}^\varepsilon \cdot \boldsymbol{v} d\Gamma^\varepsilon, \qquad \boldsymbol{v} \in (V^\varepsilon)^3. \tag{2.4}$$

Let

$$V = \left\{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_0 \text{ and } \Gamma_1\right\}. \tag{2.5}$$

For $\boldsymbol{v}^\varepsilon : \Omega^\varepsilon \to \mathbb{R}^N$ ($\boldsymbol{v}^\varepsilon : \Gamma^\varepsilon \to \mathbb{R}^N$), $N = 1,2,3$, and $x \in \Omega$ ($x \in \Gamma$) let be

$$\boldsymbol{v}(\varepsilon)(x) = \boldsymbol{v}^\varepsilon(\varepsilon x_1, \varepsilon x_2, x_3). \tag{2.6}$$

Then, if $\boldsymbol{v}^\varepsilon$ belongs respectively to the space $L^2(\Omega^\varepsilon)^N$ ($L^2(\Gamma^\varepsilon)^N$) and $(V^\varepsilon)^N$, the function $\boldsymbol{v}(\varepsilon)$ belongs to the space $L^2(\Omega)^N$ ($L^2(\Gamma)^N$) and V^N. The following fact is an immediate consequence of (2.4) and the change of variables (2.6).

Lemma 2.1 *The problem (2.4) is equivalent to the problem: find* $\boldsymbol{u}(\varepsilon) \in V^3$ *such that*

$$\begin{aligned} &\varepsilon^{-1} B_{-1}(\boldsymbol{u}(\varepsilon), \boldsymbol{v}) + B_0(\boldsymbol{u}(\varepsilon), \boldsymbol{v}) + \varepsilon B_1(\boldsymbol{u}(\varepsilon), \boldsymbol{v}) = \\ &= \varepsilon \int_\Omega \boldsymbol{f}(\varepsilon) \cdot \boldsymbol{v} dx + \int_\Gamma \boldsymbol{g}(\varepsilon) \cdot \boldsymbol{v} d\Gamma, \qquad \boldsymbol{v} \in V^3, \end{aligned} \tag{2.7}$$

where the forms $B_{-1}, B_0, B_1 : V^3 \times V^3 \to \mathbb{R}$ *are defined as follows:*

$$B_{-1}(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega (\lambda^\varepsilon e_{\alpha\alpha}(\boldsymbol{u}) e_{\beta\beta}(\boldsymbol{v}) + 2\mu^\varepsilon e_{\alpha\beta}(\boldsymbol{u}) e_{\alpha\beta}(\boldsymbol{v}) + \mu^\varepsilon \partial_\alpha u_3 \partial_\alpha v_3)\, dx, \tag{2.8}$$

$$B_0(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega (\lambda^\varepsilon (e_{\alpha\alpha}(\boldsymbol{u}) e_{33}(\boldsymbol{v}) + e_{33}(\boldsymbol{u}) e_{\beta\beta}(\boldsymbol{v})) + \mu^\varepsilon (\partial_3 u_\alpha \partial_\alpha v_3 + \partial_\alpha u_3 \partial_3 v_\alpha))\, dx, \tag{2.9}$$

$$B_1(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega ((\lambda^\varepsilon + 2\mu^\varepsilon) e_{33}(\boldsymbol{u}) e_{33}(\boldsymbol{v}) + \mu^\varepsilon \partial_3 u_\alpha \partial_3 v_\alpha)\, dx. \tag{2.10}$$

In the framework of small displacement theory it is obvious that for a small $\varepsilon > 0$ the relation between external forces and Lamé coefficients has to be under control.

Without loss of generality (see Ciarlet [1990], p. 58) assume that

$$\lambda^\varepsilon = \lambda, \quad \mu^\varepsilon = \mu, \tag{2.11}$$

$$f_i(\varepsilon) = \varepsilon^{p_i} f_i^{p_i}, \quad g_i(\varepsilon) = \varepsilon^{p_i+1} g_i^{p_i+1}, \qquad \text{(no summation over } i) \tag{2.12}$$

where $p_i \in \mathbb{Z}$ and numbers λ, μ as well as functions $f_i^{p_i}, g^{p_i+1}$ don't depend on ε. A special form of the problem (2.7), (2.11), (2.12) suggests an asymptotic analysis: we assume that for each pair $(\boldsymbol{f}(\varepsilon), \boldsymbol{g}(\varepsilon))$ there exist a number $p \in \mathbb{Z}$ and functions $\boldsymbol{u}^p, \boldsymbol{u}^{p+1}, \ldots$, not depending on ε, such that

$$\boldsymbol{u}(\varepsilon) = \varepsilon^p \boldsymbol{u}^p + \varepsilon^{p+1} \boldsymbol{u}^{p+1} + \text{ h.o.t.}, \tag{2.13}$$

where the leading term $\boldsymbol{u}^p$ is nontrivial for at least one pair $(\boldsymbol{f}(\varepsilon), \boldsymbol{g}(\varepsilon))$; h.o.t is an abbreviation for "higher order terms". Because of linearity of the problem (2.7) one can assume that $p = 0$, i.e.

$$\boldsymbol{u}(\varepsilon) = \boldsymbol{u}^0 + \varepsilon \boldsymbol{u}^1 + \text{ h.o.t.} \tag{2.14}$$

We say that $u_i^{l_i} (i = 1, 2, 3, l_i \geq p)$ is *the leading i-th component* of the expansion (2.13) if the following conditions are satisfied:

a) for each pair $(\boldsymbol{f}(\varepsilon), \boldsymbol{g}(\varepsilon))$ and each $k \in \{p, p+1, \ldots, l_i - 1\}$ it holds $u_i^k = 0$;

b) for at least one pair $(\boldsymbol{f}(\varepsilon), \boldsymbol{g}(\varepsilon))$ it holds $u_i^{l_i} \neq 0$.

Obviously, $\min\{l_1, l_2, l_3\} = 0$. The *leading displacement* of the expansion (2.14) is the field $(u_1^{l_1}, u_2^{l_2}, u_3^{l_3})$. We accept the following basic assumptions (the *Ansatz* of the asymptotic expansion method):

A1 The successive terms $\boldsymbol{u}^k, k = 0, 1, \ldots$ in (2.14) satisfy the equations obtained (after inserting (2.14) into (2.7)) by the cancellations of the coefficients of $\varepsilon^m, m \in \mathbb{Z}$.

A2 The leading displacement belongs to the space V^3.

An inspection of coefficients of different powers of ε that appear in (2.7) shows that the pair $(\boldsymbol{f}(\varepsilon), \boldsymbol{g}(\varepsilon))$, defined by (2.12), is trivial if $p_i < -2$, $i = 1, 2, 3$, i.e.

$$f_i^{p_i} = 0, \quad g_i^{p_i+1} = 0 \quad \text{if } p_i < -2, i = 1, 2, 3. \tag{2.15}$$

Our purpose is to found out the smallest numbers p_i, $i = 1, 2, 3$, for which the pair $(\boldsymbol{f}(\varepsilon), \boldsymbol{g}(\varepsilon))$ is not necessarily trivial and to identify the correspondent leading displacement. We shall prove the following result.

Theorem 2.1 *Let be $p_i \in \mathbb{Z}$, $p_i \geq -2$. If for each pair $(\boldsymbol{f}(\varepsilon), \boldsymbol{g}(\varepsilon))$, defined by (2.12), there exist terms $\boldsymbol{u}^0, \ldots, \boldsymbol{u}^3$ of the expansion (2.14), then*

$$p_\alpha = 2, \qquad p_3 = 1, \tag{2.16}$$

i.e.

$$f_\alpha(\varepsilon) = \varepsilon^2 f_\alpha^2, \qquad f_3(\varepsilon) = \varepsilon f_3^1, \tag{2.17}$$

$$g_\alpha(\varepsilon) = \varepsilon^3 g_\alpha^3, \qquad g_3(\varepsilon) = \varepsilon^2 g_3^2. \tag{2.18}$$

The leading displacement is the field (u_1^0, u_2^0, u_3^1), being of the Bernoulli–Navier type:

$$u_\alpha^0(x) = z_\alpha^0(x_3), \qquad u_3^1(x) = z_3^1(x_3) - x_\alpha \partial_3 z_\alpha^0(x_3), \tag{2.19}$$

where

$$z_\alpha^0 \in H_0^2(0,1), \qquad z_3^1 \in H_0^1(0,1). \tag{2.20}$$

Under assumption

$$f_3^1(\varepsilon) \in H^1(0,1; L^2(\omega)), \quad g_3^2(\varepsilon) \in H^1(0,1; L^2(\gamma)) \tag{2.21}$$

the field (z_1^0, z_2^0, z_3^1) is a unique solution to the system

$$\begin{aligned} EI_1 \int_0^1 \partial_{33} z_1^0 \partial_{33} \eta dx_3 &= \int_0^1 F_1 \eta dx_3, \qquad \eta \in H_0^2(0,1), \\ EI_2 \int_0^1 \partial_{33} z_2^0 \partial_{33} \eta dx_3 &= \int_0^1 F_2 \eta dx_3, \qquad \eta \in H_0^2(0,1), \\ EA \int_0^1 \partial_3 z_3^1 \partial_3 \eta dx_3 &= \int_0^1 F_3 \eta dx_3, \qquad \eta \in H_0^1(0,1), \end{aligned} \tag{2.22}$$

where

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \quad \textit{(Young's modulus)}, \tag{2.23}$$

$$I_1 = \int_\omega x_1^2 d\omega, \quad I_2 = \int_\omega x_2^2 d\omega, \quad A = |\omega|, \tag{2.24}$$

$$F_\alpha = \int_\omega \left(f_\alpha^2 + x_\alpha \partial_3 f_3^1\right) d\omega + \int_\gamma \left(g_\alpha^3 + x_\alpha \partial_3 g_3^2\right) d\gamma, \tag{2.25}$$

$$F_3 = \int_\omega f_3^1 d\omega + \int_\gamma g_3^2 d\gamma. \tag{2.26}$$

Remark 2.1 Interpretation of the results (2.17) and (2.18) can be described by paraphrasing that one given for the plates (Ciarlet [1997], p. 94): the linear Bernoulli–Navier rod theory cannot be obtained by an asymptotic analysis of the linear 3-D equations unless the data behave as specific powers of ε, in the following sense: the ratio between (some appropriate measure of) the applied transverse body force and the Lamé constants must behave like ε^2, the ratio between (some appropriate measure of) the applied longitudinal body force and the Lamé constants must behave like ε, etc.

3 Proof of Theorem 2.1

The equations for successive terms of expansion (2.13) and the corresponding boundary conditions shall be find, respectively, by the use of assumptions A1 and A2. Following Miara [1994a] we shall break the proof into several steps.

Step 1. Taking into account (2.15) we assume that

$$f_i(\varepsilon) = \varepsilon^{-2} f_i^{-2}, \qquad g_i(\varepsilon) = \varepsilon^{-1} g_i^{-1}. \tag{3.1}$$

By the cancellation of the coefficient of ε^{-1} in (2.7) and setting respectively $v_3 = 0$ and $v_\alpha = 0$ we obtain the system

$$\int_\Omega \left(\lambda e_{\alpha\alpha}(\boldsymbol{u}^0) e_{\beta\beta}(\boldsymbol{v}) + 2\mu e_{\alpha\beta}(\boldsymbol{u}^0) e_{\alpha\beta}(\boldsymbol{v})\right) dx = \int_\Omega f_\alpha^{-2} v_\alpha dx + \int_\Gamma g_\alpha^{-1} v_\alpha d\Gamma, \tag{3.2}$$

$$\boldsymbol{v} = (v_1, v_2) \in V^2,$$

$$\int_\Omega \mu \partial_\alpha u_3^0 \partial_\alpha v_3 dx = \int_\Omega f_3^{-2} v_3 dx + \int_\Gamma g_3^{-1} v_3 d\Gamma, \qquad v_3 \in V. \tag{3.3}$$

Setting in (3.2) $v_\alpha(x) = \eta_\alpha(x_3)$, $\eta_\alpha \in H_0^1(0,1)$, we conclude immediately that the functions f_α^{-2} and g_α^{-1} must satisfy a compatibility condition

$$\int_\omega f_\alpha^{-2} d\omega + \int_\gamma g_\alpha^{-1} d\gamma = 0 \text{ on } \langle 0, 1\rangle. \tag{3.4}$$

That is in contradiction with the assumption that $\boldsymbol{u}^0$ exists for each pair $(\boldsymbol{f}^{-2}, \boldsymbol{g}^{-1})$; therefore

$$f_\alpha^{-2} = 0, \quad g_\alpha^{-1} = 0. \tag{3.5}$$

Setting in (3.2) $v_\alpha(x) = \vartheta_\alpha(x_1, x_2)\eta(x_3)$, $\vartheta_\alpha \in H^1(\omega)$, $\eta \in H_0^1(0,1)$, and taking into account (3.5) we obtain

$$\int_\omega \left(\lambda e_{\alpha\alpha}(\boldsymbol{u}^0) e_{\beta\beta}(\boldsymbol{\vartheta}) + 2\mu e_{\alpha\beta}(\boldsymbol{u}^0) e_{\alpha\beta}(\boldsymbol{\vartheta})\right) d\omega = 0 \text{ on } \langle 0, 1\rangle. \tag{3.6}$$

From that we conclude that for each $x_3 \in \langle 0, 1\rangle$ the field (u_1^0, u_2^0) is a plane infinitezimal rigid displacement, i.e. that there exist functions $z_\alpha^0, z^0 \in H^1(0,1)$ such that

$$u_\alpha^0(x) = z_\alpha^0(x_3) + \delta_\alpha(x_1, x_2) z^0(x_3), \tag{3.7}$$

where

$$\delta_1(x_1,x_2) = -x_2, \qquad \delta_2(x_1,x_2) = x_1. \tag{3.8}$$

From (3.7) it follows that

$$z^0_\alpha, z^0 \in H^1_0(0,1). \tag{3.9}$$

Analogously, setting in (3.3) respectively $v_3(x) = \eta_3(x_3)$ and $v_3(x) = \vartheta(x_1,x_2)\eta_3(x_3)$, $\vartheta \in H^1(\omega)$, $\eta_3 \in H^1_0(0,1)$, we obtain a compatibility condition

$$\int_\omega f_3^{-2} d\omega + \int_\omega g_3^{-1} d\omega = 0 \text{ on } \langle 0,1\rangle \tag{3.10}$$

and therefore

$$f_3^{-2} = 0, \quad g_3^{-1} = 0 \tag{3.11}$$

and

$$u^0_3(x) = z^0_3(x_3), \tag{3.12}$$

where

$$z^0_3 \in H^1_0(0,1). \tag{3.13}$$

Step 2. Taking into account (3.5) and (3.11) we assume that

$$f_i(\varepsilon) = \varepsilon^{-1} f_i^{-1}, \qquad g_i(\varepsilon) = g_i^0. \tag{3.14}$$

By the cancellation of the coefficient of ε^0 in (2.7) and setting respectively $v_3 = 0$ and $v_\alpha = 0$ we obtain the system

$$\int_\Omega \left(\lambda e_{\alpha\alpha}(\boldsymbol{u}^1) e_{\beta\beta}(\boldsymbol{v}) + 2\mu e_{\alpha\beta}(\boldsymbol{u}^1) e_{\alpha\beta}(\boldsymbol{v}) + \lambda e_{33}(\boldsymbol{u}^0) e_{\beta\beta}(\boldsymbol{v}) + \mu \partial_\alpha u^0_3 \partial_3 v_\alpha\right) dx \tag{3.15}$$

$$= \int_\Omega f_\alpha^{-1} v_\alpha dx + \int_\Gamma g^0_\alpha v_\alpha d\Gamma, \quad \boldsymbol{v} = (v_1, v_2) \in V^2,$$

$$\int_\Omega \left(\mu \left(\partial_\alpha u^1_3 + \partial_3 u^0_\alpha\right) \partial_\alpha v_3 + \lambda \partial_\alpha u^0_\alpha \partial_3 v_3\right) dx = \int_\Omega f_3^{-1} v_3 dx + \int_\Gamma g^0_3 v_3 d\Gamma, \quad v_3 \in V. \tag{3.16}$$

Setting in (3.15) $v_\alpha(x) = \eta_\alpha(x_3)$, $\eta_\alpha \in H^1_0(0,1)$, we obtain a compatibility condition

$$\int_\omega f_\alpha^{-1} d\omega + \int_\gamma g^0_\alpha d\gamma = 0 \quad \text{ on } \langle 0,1\rangle \tag{3.17}$$

and therefore

$$f_\alpha^{-1} = 0, \quad g^0_\alpha = 0. \tag{3.18}$$

Setting in (3.15) $v_\alpha(x) = \vartheta_\alpha(x_1,x_2)\eta(x_3)$, $\vartheta_\alpha \in H^1(\omega)$, $\eta \in H^1_0(0,1)$, and taking into account (3.12) and (3.18) we obtain

$$\int_\omega \left(\lambda e_{\alpha\alpha}(\boldsymbol{u}^1) e_{\beta\beta}(\vartheta) + 2\mu e_{\alpha\beta}(\boldsymbol{u}^1) e_{\alpha\beta}(\vartheta)\right) d\omega + \lambda \partial_3 z^0_3 \int_\omega e_{\beta\beta}(\vartheta) d\omega = 0 \text{ on } \langle 0,1\rangle. \tag{3.19}$$

From (3.19) it follows that there exist functions $z^1_\alpha, z^1 \in H^1(0,1)$ such that

$$u^1_\alpha(x) = z^1_\alpha(x_3) + \delta_\alpha(x_1,x_2) z^1(x_3) - \nu x_\alpha \partial_3 z^0_3(x_3), \tag{3.20}$$

where

$$\nu = \frac{\lambda}{2(\lambda+\mu)} \qquad \text{(Poisson ratio)}. \tag{3.21}$$

From (3.20) we conclude that

$$z_3^0 \in H^2(0,1). \tag{3.22}$$

Analogously, setting in (3.16) $v_3(x) = \eta_3(x_3)$, $\eta_3 \in H_0^1(0,1)$, we obtain a compatibility condition

$$\int_\omega f_3^{-1} d\omega + \int_\omega g_3^0 d\gamma = 0 \text{ on } \langle 0,1\rangle \tag{3.23}$$

and therefore

$$f_3^{-1} = 0, \quad g_3^0 = 0; \tag{3.24}$$

setting $v_3(x_3) = \vartheta(x_1,x_2)\eta_3(x_3)$, $\vartheta \in H^1(\omega)$, $\eta_3 \in H_0^1(0,1)$ and taking into account (3.7) and (3.24) we have

$$\int_\omega \partial_\alpha u_3^1 \partial_\alpha \vartheta d\omega + \partial_3 z_\alpha^0(x_3) \int_\omega \partial_\alpha \vartheta d\omega + \partial_3 z^0(x_3) \int_\omega \delta_\alpha \partial_\alpha \vartheta d\omega = 0 \text{ on } \langle 0,1\rangle. \tag{3.25}$$

From (3.25) we conclude that there exists a function $z_3^1 \in H^1(0,1)$ such that

$$u_3^1(x) = z_3^1(x_3) - x_\alpha \partial_3 z_\alpha^0(x_3) + w(x_1,x_2)\partial_3 z^0(x_3), \tag{3.26}$$

where $w \in H^1(\omega) \cap L_0^2(\omega)$ is the warping function of the domain ω, i.e. solution to the problem

$$\int_\omega (\partial_\alpha w + \delta_\alpha)\, \partial_\alpha \vartheta d\omega = 0, \quad \vartheta \in H^1(\omega) \cap L_0^2(\omega); \tag{3.27}$$

here

$$L_0^2(\omega) = \left\{ \vartheta \in L^2(\omega) : \int_\omega \vartheta d\omega = 0 \right\}. \tag{3.28}$$

From (3.26) we conclude that

$$z^0, z_\alpha^0 \in H^2(0,1). \tag{3.29}$$

Step 3. Because of (3.18) and (3.24) we assume

$$f_i(\varepsilon) = f_i^0, \quad g_i(\varepsilon) = \varepsilon g_i^1. \tag{3.30}$$

By the cancellation of the coefficient of ε in (2.7) and setting respectively $v_3 = 0$ and $v_\alpha = 0$ we obtain the system

$$\int_\Omega \big(\lambda e_{\alpha\alpha}(\boldsymbol{u}^2) e_{\beta\beta}(\boldsymbol{v}) + 2\mu e_{\alpha\beta}(\boldsymbol{u}^2) e_{\alpha\beta}(\boldsymbol{v}) + \lambda e_{33}(\boldsymbol{u}^1) e_{\beta\beta}(\boldsymbol{v}) + \mu(\partial_\alpha u_3^1 + \partial_3 u_\alpha^0)\partial_3 v_\alpha\big) dx \tag{3.31}$$
$$= \int_\Omega f_\alpha^0 v_\alpha dx + \int_\Gamma g_\alpha^1 v_\alpha d\Gamma, \quad \boldsymbol{v} = (v_1, v_2) \in V^2,$$

$$\int_\Omega \big(\mu \partial_\alpha u_3^2 \partial_\alpha v_3 + \lambda e_{\alpha\alpha}(\boldsymbol{u}^1)\partial_3 v_3 + \mu \partial_3 u_\alpha^1 \partial_\alpha v_3 + (\lambda+2\mu) e_{33}(\boldsymbol{u}^0)\partial_3 v_3\big) dx \tag{3.32}$$
$$= \int_\Omega f_3^0 v_3 dx + \int_\Gamma g_3^1 v_3 d\Gamma, \quad v_3 \in V.$$

Setting in (3.31) $v_\alpha(x) = \eta_\alpha(x_3)$, $\eta_\alpha \in H_0^1(0,1)$, we have

$$\mu \int_\Omega (\partial_\alpha u_3^1 + \partial_3 u_\alpha^0)\partial_3 \eta_\alpha dx = \int_\Omega f_\alpha^0 \eta_\alpha dx + \int_\Gamma g_\alpha^1 \eta_\alpha d\Gamma. \tag{3.33}$$

Inserting (3.7) and (3.26) into (3.33) and taking into account that

$$\int_\omega \partial_\alpha w d\omega = 0, \quad \int_\omega \delta_\alpha d\omega = 0 \tag{3.34}$$

we obtain a compatibility condition

$$\int_\omega f_\alpha^0 d\omega + \int_\gamma g_\alpha^1 d\gamma = 0 \quad \text{on } \langle 0,1\rangle \tag{3.35}$$

and therefore

$$f_\alpha^0 = 0, \quad g_\alpha^1 = 0. \tag{3.36}$$

Setting in (3.31) $v_\alpha(x) = \vartheta_\alpha(x_1,x_2)\eta(x_3)$, $\vartheta_\alpha \in H^1(\omega)$, $\eta \in H_0^1(0,1)$, and taking into account (3.7), (3.26), (3.27) and (3.36), after integration by parts with respect to the variable x_3 we obtain

$$\int_\omega \left(\lambda e_{\alpha\alpha}(\boldsymbol{u}^2)e_{\beta\beta}(\vartheta) + 2\mu e_{\alpha\beta}(\boldsymbol{u}^2)e_{\alpha\beta}(\vartheta)\right) d\omega + \lambda\partial_3 z_3^1 \int_\omega \partial_\beta\vartheta_\beta d\omega - \lambda\partial_{33}z_\alpha^0 \int_\omega x_\alpha\partial_\beta\vartheta_\beta d\omega \tag{3.37}$$
$$+\lambda\partial_{33}z^0 \int_\omega w\partial_\beta\vartheta_\beta d\omega - \mu\partial_{33}z^0 \int_\omega (\partial_\alpha w + \delta_\alpha)\vartheta_\alpha d\omega = 0, \quad \text{on } \langle 0,1\rangle.$$

Setting here $\vartheta_\alpha = \delta_\alpha$ and taking into account that

$$D = \int_\omega (\partial_\alpha w + \delta_\alpha)\,\delta_\alpha d\omega = \int_\omega (\partial_\alpha w + \delta_\alpha)^2\, d\omega > 0 \tag{3.38}$$

one gets $\partial_{33}z^0 = 0$ or, because of $(3.9)_2$,

$$z^0 = 0. \tag{3.39}$$

The equation (3.37) takes the form

$$\int_\omega \left(\lambda e_{\alpha\alpha}(\boldsymbol{u}^2)e_{\beta\beta}(\boldsymbol{\vartheta}) + 2\mu e_{\alpha\beta}(\boldsymbol{u}^2)e_{\alpha\beta}(\boldsymbol{\vartheta})\right) d\omega \tag{3.40}$$
$$+\lambda\partial_3 z_3^1 \int_\omega \partial_\beta\vartheta_\beta d\omega - \lambda\partial_{33}z_\alpha^0 \int_\omega x_\alpha\partial_\beta\vartheta_\beta d\omega = 0 \text{ on } \langle 0,1\rangle.$$

From (3.40) we conclude that there exist functions $z_\alpha^2, z^1 \in H^1(0,1)$ such that

$$u_\alpha^2(x) = z_\alpha^2(x_3) + \delta_\alpha(x_1,x_2)z^2(x_3) - \nu x_\alpha\partial_3 z_3^1(x_3) + \nu\Phi_{\alpha\beta}(x_1,x_2)\partial_{33}z_\beta^0, \tag{3.41}$$

where

$$(\Phi_{\alpha\beta}) = \begin{pmatrix} \frac{1}{2}(x_1^2 - x_2^2) & x_1x_2 \\ x_1x_2 & \frac{1}{2}(x_2^2 - x_1^2) \end{pmatrix} \tag{3.42}$$

(see Trabucho & Viaño [1996]). Setting in (3.32) $v_3(x) = \eta_3(x_3)$, $\eta_3 \in H_0^1(0,1)$, we get

$$\int_\Omega \left(\lambda e_{\alpha\alpha}(\boldsymbol{u}^1) + (\lambda + 2\mu) e_{33}(\boldsymbol{u}^0)\right) \partial_3 \eta_3 dx = \int_\Omega f_3^0 \eta_3 dx + \int_\Gamma g_3^1 \eta_3 d\Gamma. \tag{3.43}$$

Inserting here (3.12) and (3.20) and taking into account (3.13) we conclude that $z_3^0 \in H_0^1(0,1)$ is a unique solution to the problem

$$EA \int_0^1 \partial_3 z_3^0 \partial_3 \eta_3 dx_3 = \int_0^1 \left(\int_\omega f_3^0 d\omega + \int_\gamma g_3^1 d\gamma \right) \eta_3 dx_3, \quad \eta_3 \in H_0^1(0,1). \tag{3.44}$$

Because of (3.22) we obtain

$$-EA \partial_{33} z_3^0 = \int_\omega f_3^0 d\omega + \int_\gamma g_3^1 d\gamma. \tag{3.45}$$

Setting in (3.32) $v_3(x_3) = \vartheta(x_1, x_2)\eta_3(x_3)$, $\vartheta \in H^1(\omega)$, $\eta_3 \in H_0^1(0,1)$ and taking into account (3.12) and (3.20), after integration by parts with respect to the variable x_3 we get

$$\begin{aligned} &\mu \int_\omega \partial_\alpha u_3^2 \partial_\alpha \vartheta d\omega + \mu \partial_3 z_\alpha^1 \int_\omega \partial_\alpha \vartheta d\omega + \mu \partial_3 z^1 \int_\omega \delta_\alpha \partial_\alpha \vartheta \\ &-\mu\nu \partial_{33} z_3^0 \int_\omega x_\alpha \partial_\alpha \vartheta d\omega - E \partial_{33} z_3^0 \int_\omega \vartheta d\omega = \int_\omega f_3^0 \vartheta d\omega + \int_\gamma g_3^1 \vartheta d\gamma \text{ on } \langle 0,1 \rangle. \end{aligned} \tag{3.46}$$

Because of (3.45) it is sufficient to consider (3.46) for $\vartheta \in H^1(\omega) \cap L_0^2(\omega)$. According to the assumptions (2.21) there is a unique $U_3^2 \in H^1(\Omega) \cap H^1(0,1; L_0^2(\omega))$ such that

$$\mu \int_\omega \partial_\alpha U_3^2 \partial_\alpha \vartheta d\omega = \int_\omega f_3^0 \vartheta d\omega + \int_\gamma g_3^1 \vartheta d\gamma \text{ on } \langle 0,1 \rangle, \vartheta \in H^1(\omega) \cap L_0^2(\omega). \tag{3.47}$$

From (3.46) and (3.47) it follows that there exists $z_3^2 \in H^1(0,1)$ such that

$$u_3^2(x) = U_3^2(x) + z_3^2(x_3) - x_\alpha \partial_3 z_\alpha^1(x_3) + w(x_1, x_2) \partial_3 z^1(x_3) + \frac{\nu}{2} x_\alpha^2 \partial_{33} z_3^0. \tag{3.48}$$

Step 4. Because of (3.36) we assume

$$f_\alpha(\varepsilon) = \varepsilon f_\alpha^1, \qquad g_\alpha(\varepsilon) = \varepsilon^2 g_\alpha^2. \tag{3.49}$$

Setting in (2.7) $v_3 = 0$ and $v_\alpha(x) = \eta_\alpha(x_3)$, $\eta_\alpha \in H_0^1(0,1)$, by the cancellation of the coefficient of ε^2 we obtain

$$\mu \int_\Omega (\partial_\alpha u_3^2 + \partial_3 u_\alpha^1) \partial_3 \eta_\alpha dx = \int_\Omega f_\alpha^1 \eta_\alpha dx + \int_\Gamma g_\alpha^2 \eta_\alpha d\Gamma. \tag{3.50}$$

Inserting here (3.20) and (3.48) we get

$$\mu \int_0^1 \left(\partial_3 z^1 \int_\omega (\partial_\alpha w + \delta_\alpha) d\omega + \int_\omega \partial_\alpha U_3^2 d\omega \right) \partial_3 \eta_\alpha dx_3 = \int_0^1 \left(\int_\omega f_\alpha^1 d\omega + \int_\gamma g_\alpha^2 d\gamma \right) \eta_\alpha dx_3; \tag{3.51}$$

because of (3.34) and the equality

$$\mu \int_\omega \partial_\alpha U_3^2 d\omega = \int_\omega x_\alpha f_3^0 d\omega + \int_\gamma x_\alpha g_3^1 d\gamma \tag{3.52}$$

we have

$$\int_0^1 \left(\int_\omega x_\alpha f_3^0 d\omega + \int_\gamma x_\alpha g_3^1 d\gamma \right) \partial_3 \eta_\alpha dx_3 = \int_0^1 \left(\int_\omega f_\alpha^1 d\omega + \int_\gamma g_\alpha^2 d\gamma \right) \eta_\alpha dx_3. \tag{3.53}$$

Taking into account assumptions (2.21), after integration by parts with respect to the variable x_3 we obtain a compatibility condition

$$\int_\omega \left(f_\alpha^1 + x_\alpha \partial_3 f_3^0 \right) d\omega + \int_\gamma \left(g_\alpha^2 + x_\alpha \partial_3 g_3^1 \right) d\gamma = 0. \tag{3.54}$$

Therefore

$$f_3^0 = 0, \quad g_3^1 = 0, \tag{3.55}$$

$$f_\alpha^1 = 0, \quad g_\alpha^2 = 0. \tag{3.56}$$

Because of (3.44) it holds

$$z_3^0 = 0 \tag{3.57}$$

and, because of (3.12),

$$u_3^0 = 0. \tag{3.58}$$

Therefore

$$u_3^1 \in V. \tag{3.59}$$

From (3.26) it follows

$$z_\alpha^0 \in H_0^2(0,1), \quad z_3^1 \in H_0^1(0,1). \tag{3.60}$$

Because of (3.55) we assume

$$f_3(\varepsilon) = \varepsilon f_3^1, \quad g_3(\varepsilon) = \varepsilon^2 g_3^2. \tag{3.61}$$

Setting in (2.7) $v_\alpha = 0$ and $v_3(x) = \eta_3(x_3)$, $\eta_3 \in H_0^1(0,1)$, after cancellation of the coefficient of ε^2 we obtain

$$\int_\Omega \left(\lambda \partial_\alpha u_\alpha^2 + (\lambda + 2\mu) \partial_3 u_3^1 \right) \partial_3 \eta_3 dx = \int_\Omega f_3^1 \eta_3 dx + \int_\Gamma g_3^2 \eta_3 d\Gamma. \tag{3.62}$$

Inserting here (3.26) and (3.41) and taking into account $(3.60)_2$ we conclude that z_3^1 is a unique solution to the problem

$$EA \int_0^1 \partial_3 z_3^1 \partial_3 \eta_3 = \int_0^1 \left(\int_\omega f_3^1 d\omega + \int_\gamma g_3^2 d\gamma \right) \eta_3, \quad \eta_3 \in H_0^1(0,1). \tag{3.63}$$

Setting in (2.7) $v_\alpha = 0$ and $v_3(x) = x_\alpha \partial_3 \eta_\alpha(x_3)$, $\eta_\alpha \in H_0^2(0,1)$, after cancellation of the coefficient of ε^2 we obtain

$$\begin{aligned}\mu \int_\Omega \left(\partial_\alpha u_3^3 + \partial_3 u_\alpha^2\right) \partial_3 \eta_\alpha dx &= -\int_\Omega \left(\lambda \partial_\alpha u_\alpha^2 + (\lambda + 2\mu)\partial_3 u_3^1\right) x_\beta \partial_{33} \eta_\beta dx \\ &+ \int_\Omega x_\alpha f_3^1 \partial_3 \eta_\alpha dx + \int_\Gamma x_\alpha g_3^2 \partial_3 \eta_\alpha d\Gamma.\end{aligned} \tag{3.64}$$

By the use of (3.26) and (3.41) on the right hand side we get

$$\begin{aligned}\mu \int_\Omega \left(\partial_\alpha u_3^3 + \partial_3 u_\alpha^2\right) \partial_3 \eta_\alpha dx &= EI_\alpha \int_0^1 \partial_{33} z_\alpha^0 \partial_{33} \eta_\alpha dx_3 \\ &+ \int_0^1 \left(\int_\omega x_\alpha f_3^1 d\omega + \int_\gamma x_\alpha g_3^2 d\gamma\right) \partial_3 \eta_\alpha dx_3.\end{aligned} \tag{3.65}$$

Taking into account the assumptions (2.21), after integration by parts in the second term on the right hand side we obtain

$$\begin{aligned}\mu \int_\Omega \left(\partial_\alpha u_3^3 + \partial_3 u_\alpha^2\right) \partial_3 \eta_\alpha dx &= EI_\alpha \int_0^1 \partial_{33} z_\alpha^0 \partial_{33} \eta_\alpha dx_3 \\ &- \int_0^1 \left(\int_\omega x_\alpha \partial_3 f_3^1 d\omega + \int_\gamma x_\alpha \partial_3 g_3^2 d\gamma\right) \eta_\alpha dx_3.\end{aligned} \tag{3.66}$$

Step 5. Because of (3.56) we assume

$$f_\alpha(\varepsilon) = \varepsilon^2 f_\alpha^2, \qquad g_\alpha(\varepsilon) = \varepsilon^3 g_\alpha^3. \tag{3.67}$$

Setting in (2.7) $v_3 = 0$ and $v_\alpha(x) = \eta_\alpha(x_3)$, $\eta_\alpha \in H_0^2(0,1)$, after cancellation of the coefficient of ε^3 we obtain

$$\mu \int_\Omega \left(\partial_\alpha u_3^3 + \partial_3 u_\alpha^2\right) \partial_3 \eta_\alpha dx = \int_\Omega f_\alpha^2 \eta_\alpha dx + \int_\Gamma g_\alpha^3 \eta_\alpha d\Gamma. \tag{3.68}$$

Taking into account $(3.60)_1$, from (3.66) and (3.68) we conclude that z_α^0 is a unique solution to the problem

$$\begin{aligned}EI_\alpha \int_0^1 \partial_{33} z_\alpha^0 \partial_{33} \eta_\alpha dx_3 &= \int_0^1 \left(\int_\omega f_\alpha^2 \eta_\alpha d\omega + \int_\gamma g_\alpha^3 \eta_\alpha d\gamma\right) \eta_\alpha dx_3 \\ &+ \int_0^1 \left(\int_\omega x_\alpha \partial_3 f_3^1 d\omega + \int_\gamma x_\alpha \partial_3 g_3^2 d\gamma\right) \eta_\alpha dx_3, \quad \eta_\alpha \in H_0^2(0,1).\end{aligned} \tag{3.69}$$

The conclusions of Theorem 2.1 follow now from results of the Steps 1–5.

4 On overlooked models

We shall here discuss the asymptotic analysis for data (2.12) when $p_\alpha \neq 2$ or $p_3 \neq 1$. In such cases analysis leads to some compatibility conditions (see e.g. (3.10)). Therefore the definition of the leading displacement of the expansion (2.14) (see p. 3) has to be changed: under "arbitrary" pair $(\boldsymbol{f}(\varepsilon), \boldsymbol{g}(\varepsilon))$ we have to consider each one that satisfies all necessary compatibility conditions.

If $-2 \le p_i < 0$, it turns out that the compatibility conditions are very complicated and without clear physical interpretation; moreover, the problems for the leading displacements are ill posed.

We shall formulate the result for $p_i = 0$ $(i = 1, 2, 3)$ only. In this case the compatibility conditions are "reasonable", but the leading displacement is not unique. For the sake of simplicity we assume some regularity of the functions $f : \langle 0, 1\rangle \to L^2(\omega)^2$, $g : \langle 0, 1\rangle \to L^2(\gamma)^2$. Repeating the steps of the proof of Theorem 2.1 we obtain the following conclusion.

If there exist the terms $\boldsymbol{u}^0, \dots, \boldsymbol{u}^3$ of the expansion (2.14) with $\boldsymbol{u}^0 \in V^3$, then $\boldsymbol{f}^0$ and $\boldsymbol{g}^1$ satisfy the compatibility conditions

$$\int_\omega f^0_\alpha d\omega + \int_\omega g^1_\alpha d\gamma = 0, \qquad \int_\omega x_\alpha f^0_3 d\omega + \int_\gamma x_\alpha g^1_3 d\gamma = \text{const.} \tag{4.1}$$

The leading displacement is $\boldsymbol{u}^0(x)$, with the components

$$u^0_\alpha(x) = z^0_\alpha(x_3) + \delta_\alpha(x_1, x_2) z^0(x_3), \qquad u^0_3(x) = z^0_3(x_3), \tag{4.2}$$

where

$$z^0_\alpha, z^0, z^0_3 \in H^1_0(0, 1) \cap H^2(0, 1). \tag{4.3}$$

The functions z^0, z^0_3, z^0_α satisfy the equations

$$-\mu D \partial_{33} z^0 = \int_\omega \delta_\alpha f^0_\alpha d\omega + \int_\gamma \delta_\alpha g^1_\alpha d\gamma, \tag{4.4}$$

$$-EA \partial_{33} z^0_3 = \int_\omega f^0_3 d\omega + \int_\gamma g^1_3 d\gamma, \tag{4.5}$$

$$EI_\alpha \int_0^1 \partial_{33} z^0_\alpha \partial_{33} \eta_\alpha dx_3 = \int_0^1 F_\alpha \eta_\alpha, \quad \eta_\alpha \in H^1_0(0, 1) \cap H^2(0, 1), \tag{4.6}$$

where

$$F_\alpha = \lambda \partial_{33} \int_\omega x_\alpha \partial_\beta W_\beta d\omega + (\lambda + 2\mu)(z^0)^{(iv)} \int_\omega x_\alpha w d\omega; \tag{4.7}$$

here $W \in H^2(0, 1; H^1(\omega)^2)$ is a solution to the problem

$$\int_\omega W_\alpha d\omega = 0, \qquad \int_\omega \delta_\alpha W_\alpha d\omega = 0, \tag{4.8}$$

$$\int_\omega \left(\lambda e_{\alpha\alpha}(W) e_{\beta\beta}(\vartheta) + 2\mu e_{\alpha\beta}(W) e_{\alpha\beta}(\vartheta)\right) d\omega \tag{4.9}$$

$$+\partial_{33} z^0 \int_\omega \left(\lambda w \partial_\beta \vartheta_\beta - \mu \left(\partial_\beta w + \delta_\beta\right) \vartheta_\beta\right) d\omega \tag{4.10}$$

$$= \int_\omega f_\alpha \vartheta_\alpha d\omega + \int_\gamma g_\alpha \vartheta_\alpha d\gamma, \tag{4.11}$$

$$\vartheta \in H^1(\omega)^2, \int_\omega \vartheta_\alpha d\omega = 0, \int_\omega \delta_\alpha \vartheta_\alpha d\omega = 0. \tag{4.12}$$

As in the previous sections, w denotes the warping function of the domain ω.

As we see, rotation z^0 and longitudinal displacement z_3^0 are unique, but the transversal displacement z_α^0 is defined up to a polynomial $(a_\alpha x_3 + b_\alpha)x_3(x_3 - 1)$, $a_\alpha, b_\alpha \in \mathbb{R}$.

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