MATEMATIKA 1 – rješenja četvrte zadaće
2020./2021.

1. *(4 = 2 + 2)* Zadani su vektori

$$\vec{a} = [1, 0, -1] \quad \text{i} \quad \vec{b} = [1, 2, 3].$$

Izračunajte:

(a) $\left(\vec{a} \cdot \vec{b}\right)\left(\vec{a} \times \vec{b} - \vec{b} \times \vec{a}\right)$

(b) $\vec{a} \times \left(\left(\vec{a} + \vec{b}\right) \times \vec{b}\right)$.

Rješenje. Imamo

$$\vec{a} \cdot \vec{b} = [1, 0, -1] \cdot [1, 2, 3] = 1 \cdot 1 + 0 \cdot 2 - 1 \cdot 3 = -2 \tag{1}$$

i

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 1 & 2 & 3 \end{vmatrix} = [2, -4, 2]. \tag{2}$$

(a) Imamo

$$\begin{aligned} \left(\vec{a} \cdot \vec{b}\right)\left(\vec{a} \times \vec{b} - \vec{b} \times \vec{a}\right) &= \left(\vec{a} \cdot \vec{b}\right)\left(\vec{a} \times \vec{b} + \vec{a} \times \vec{b}\right) \\ &= 2\left(\vec{a} \cdot \vec{b}\right)\left(\vec{a} \times \vec{b}\right) \\ &\overset{(1),(2)}{=} 2(-2)[2, -4, 2] \\ &= [-8, 16, -8]. \end{aligned}$$

(b) Imamo

$$\begin{aligned} \vec{a} \times \left(\left(\vec{a} + \vec{b}\right) \times \vec{b}\right) &= \vec{a} \times \left(\vec{a} \times \vec{b} + \vec{b} \times \vec{b}\right) \\ &= \vec{a} \times \left(\vec{a} \times \vec{b} + \vec{0}\right) \\ &= \vec{a} \times \left(\vec{a} \times \vec{b}\right) \\ &\overset{(2)}{=} [1, 0, -1] \times [2, -4, 2] \\ &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -1 \\ 2 & -4 & 2 \end{vmatrix} \\ &= [-4, -4, -4]. \end{aligned}$$

2. *(2)* Neka je $\{\vec{a}, \vec{b}, \vec{c}\}$ baza prostora V^3 sa sljedećim svojstvima:

$$|\vec{a}\,| = |\vec{b}\,| = |\vec{c}\,| = 1, \quad \alpha := \measuredangle(\vec{b}, \vec{c}) = \frac{\pi}{2}, \quad \beta := \measuredangle(\vec{c}, \vec{a}) = \frac{\pi}{2}, \quad \gamma := \measuredangle(\vec{a}, \vec{b}) = \frac{\pi}{6}.$$

Izračunajte

$$\measuredangle\left(\vec{a} + \vec{b} + \vec{c},\ \vec{a} + \vec{b} - \vec{c}\right).$$

Rješenje. Imamo

$$\begin{aligned}
\vec{a} \cdot \vec{b} &= |\vec{a}\,| \cdot |\vec{b}\,| \cdot \cos\gamma = 1 \cdot 1 \cdot \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}, \\
\vec{a} \cdot \vec{c} &= |\vec{a}\,| \cdot |\vec{c}\,| \cdot \cos\beta = 1 \cdot 1 \cdot \cos\frac{\pi}{2} = 0, \\
\vec{b} \cdot \vec{c} &= |\vec{b}\,| \cdot |\vec{c}\,| \cdot \cos\alpha = 1 \cdot 1 \cdot \cos\frac{\pi}{2} = 0.
\end{aligned} \tag{3}$$

Dakle,

$$\begin{aligned}
\cos\measuredangle\left(\vec{a} + \vec{b} + \vec{c},\ \vec{a} + \vec{b} - \vec{c}\right) &= \frac{\left(\vec{a} + \vec{b} + \vec{c}\right) \cdot \left(\vec{a} + \vec{b} - \vec{c}\right)}{|\vec{a} + \vec{b} + \vec{c}\,| \cdot |\vec{a} + \vec{b} - \vec{c}\,|} \\
&= \frac{1 + \sqrt{3}}{\sqrt{3 + \sqrt{3}} \cdot \sqrt{3 + \sqrt{3}}} = \frac{1 + \sqrt{3}}{3 + \sqrt{3}} = \frac{1 + \sqrt{3}}{\sqrt{3}(1 + \sqrt{3})} = \frac{1}{\sqrt{3}},
\end{aligned} \tag{4}$$

pri čemu smo u drugoj jednakosti iskoristili da vrijedi:

- $\left(\vec{a} + \vec{b} + \vec{c}\right) \cdot \left(\vec{a} + \vec{b} - \vec{c}\right) = |\vec{a}\,|^2 + |\vec{b}\,|^2 - |\vec{c}\,|^2 + 2\vec{a} \cdot \vec{b} \overset{(3)}{=} 1 + \sqrt{3}$
- $$\begin{aligned}
|\vec{a} + \vec{b} + \vec{c}\,| &= \sqrt{\left(\vec{a} + \vec{b} + \vec{c}\right) \cdot \left(\vec{a} + \vec{b} + \vec{c}\right)} \\
&= \sqrt{|\vec{a}\,|^2 + |\vec{b}\,|^2 + |\vec{c}\,|^2 + 2\left(\vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}\right)} \\
&\overset{(3)}{=} \sqrt{3 + \sqrt{3}}
\end{aligned}$$
- $$\begin{aligned}
|\vec{a} + \vec{b} - \vec{c}\,| &= \sqrt{\left(\vec{a} + \vec{b} - \vec{c}\right) \cdot \left(\vec{a} + \vec{b} - \vec{c}\right)} \\
&= \sqrt{|\vec{a}\,|^2 + |\vec{b}\,|^2 + |\vec{c}\,|^2 + 2\left(\vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{c} - \vec{b} \cdot \vec{c}\right)} \\
&\overset{(3)}{=} \sqrt{3 + \sqrt{3}}.
\end{aligned}$$

Iz (4) vidimo da je

$$\measuredangle\left(\vec{a} + \vec{b} + \vec{c},\ \vec{a} + \vec{b} - \vec{c}\right) = \arccos\frac{1}{\sqrt{3}}.$$

3. *(2)* Odredite kanonski oblik jednadžbe ravnine π koja ne siječe z-os ni x-os i prolazi točkom $T = (e, \pi, e - \pi)$.

Rješenje. Kako π ne siječe z-os ni x-os, paralelna je s njima, pa onda i s cijelom ravninom koju one razapinju, a to je xz-ravnina. Dakle, $\vec{n}_\pi \parallel \vec{n}_{xz} \parallel [0, 1, 0]$ pa možemo staviti $\vec{n}_\pi = [0, 1, 0]$.

Sad uvrštavanjem podataka $[A, B, C] = \vec{n}_\pi = [0, 1, 0]$ i $(x_0, y_0, z_0) = T = (e, \pi, e - \pi)$ u jednadžbu

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

ravnine kroz točku (x_0, y_0, z_0) s vektorom normale $[A, B, C]$ dobivamo da je kanonski oblik jednadžbe ravnine π dan sa

$$\pi \ldots \ y = \pi.$$

4. *(2)* Odredite kanonski oblik jednadžbe pravca p koji je okomit na pravce

$$q_1 \ldots \ \frac{x-1}{1} = \frac{y-1}{2} = \frac{z-3}{0} \qquad \text{i} \qquad q_2 \ldots \ \frac{x-6}{2} = \frac{y-1}{2} = \frac{z-3}{0}$$

i prolazi njihovim sjecištem.

Rješenje. Kako je $p \perp q_1, q_2$, vrijedi $\vec{s}_p \perp \vec{s}_{q_1}, \vec{s}_{q_2}$ pa je

$$\vec{s}_p \parallel \vec{s}_{q_1} \times \vec{s}_{q_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 2 & 2 & 0 \end{vmatrix} = [0, 0, -2],$$

dakle možemo staviti npr. $\vec{s}_p = -\frac{1}{2}[0, 0, -2] = [0, 0, 1]$.

Sjecište pravaca q_1 i q_2 je točka $S = (x, y, z) \in \mathbb{R}^3$ koja zadovoljava (parametarsku) jednadžbu i od q_1 i od q_2, tj. koja zadovoljava sustav

$$\begin{cases} x = t + 1 = 2s + 6 \\ y = 2t + 1 = 2s + 1 \\ z = 3 \end{cases} \quad \text{za neke } s, t \in \mathbb{R}.$$

Jedino je rješenje ovog sustava točka $(x, y, z) = (-4, -9, 3)$ (za $s = t = -5$), dakle $S = (-4, -9, 3)$.

Sad uvrštavanjem podataka $[a, b, c] = \vec{s}_p = [0, 0, 1]$ i $(x_0, y_0, z_0) = S = (-4, -9, 3)$ u formulu

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

za kanonski oblik jednadžbe pravca p kroz točku (x_0, y_0, z_0) s vektorom smjera $[a, b, c]$ dobivamo da je kanonski oblik jednadžbe pravca p dan sa

$$p \ldots \ \frac{x+4}{0} = \frac{y+9}{0} = \frac{z-3}{1}.$$