MATEMATIKA 1

Rješenja treće zadaće 2020./21.

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1. *(3 = 1 + 1 + 1)* Izračunajte limese:

(a) $\lim\limits_{x\to+\infty}\left(3^{x+3}-\sqrt{9^{x+3}-3}\right)$

(b) $\lim\limits_{x\to 0}\dfrac{1+\cos\left(x^3+\pi\right)}{\sin\left(x^6\right)}$

(c) $\lim\limits_{x\to\pi}(1+\sin x)^{\frac{1}{\sin(2x)}}$.

Rješenje. (a) Imamo

$$\begin{aligned}
\lim_{x\to+\infty}\left(3^{x+3}-\sqrt{9^{x+3}-3}\right) &= \lim_{x\to+\infty}\left(3^{x+3}-\sqrt{9^{x+3}-3}\right)\cdot\frac{3^{x+3}+\sqrt{9^{x+3}-3}}{3^{x+3}+\sqrt{9^{x+3}-3}}\\
&= \lim_{x\to+\infty}\frac{\left(3^{x+3}\right)^2-\left(\sqrt{9^{x+3}-3}\right)^2}{3^{x+3}+\sqrt{9^{x+3}-3}}\\
&= \lim_{x\to+\infty}\frac{9^{x+3}-\left(9^{x+3}-3\right)}{3^{x+3}+\sqrt{9^{x+3}-3}}\\
&= \lim_{x\to+\infty}\frac{3}{3^{x+3}+\sqrt{9^{x+3}-3}}\\
&= \left(\frac{3}{+\infty}\right)\\
&= 0.
\end{aligned}$$

(b) *1. način.* Imamo

$$\begin{aligned}
\lim_{x\to 0}\frac{1+\cos\left(x^3+\pi\right)}{\sin\left(x^6\right)} &= \lim_{x\to 0}\frac{1-\cos\left(x^3\right)}{\sin\left(x^6\right)} = \lim_{x\to 0}\frac{\frac{1-\cos\left(x^3\right)}{x^6}}{\frac{\sin\left(x^6\right)}{x^6}} = \frac{\lim_{x\to 0}\frac{1-\cos\left(x^3\right)}{x^6}}{\lim_{x\to 0}\frac{\sin\left(x^6\right)}{x^6}}\\
&= \begin{bmatrix} t=x^3 & x\to 0\ \Rightarrow\ t\to 0\\ s=x^6 & x\to 0\ \Rightarrow\ s\to 0\end{bmatrix} = \frac{\lim_{t\to 0}\frac{1-\cos t}{t^2}}{\lim_{s\to 0}\frac{\sin s}{s}} = \frac{\frac{1}{2}}{1} = \frac{1}{2}.
\end{aligned}$$

2. način. Imamo

$$\begin{aligned}
\lim_{x\to 0}\frac{1+\cos\left(x^3+\pi\right)}{\sin\left(x^6\right)} &= \left(\frac{0}{0}\right) \overset{\text{L'H}}{=} \lim_{x\to 0}\frac{\left(1+\cos\left(x^3+\pi\right)\right)'}{\left(\sin\left(x^6\right)\right)'}\\
&= \lim_{x\to 0}\frac{-\sin\left(x^3+\pi\right)\cdot 3x^2}{\cos\left(x^6\right)\cdot 6x^5}\\
&= \lim_{x\to 0}\frac{-\sin\left(x^3+\pi\right)}{\cos\left(x^6\right)\cdot 2x^3}\\
&= \left(\frac{0}{0}\right) \overset{\text{L'H}}{=} \lim_{x\to 0}\frac{\left(-\sin\left(x^3+\pi\right)\right)'}{\left(\cos\left(x^6\right)\cdot 2x^3\right)'}\\
&= \lim_{x\to 0}\frac{-\cos\left(x^3+\pi\right)\cdot 3x^2}{-\sin\left(x^6\right)\cdot 2x^3+\cos\left(x^6\right)\cdot 6x^2}\\
&= \lim_{x\to 0}\frac{-3\cos\left(x^3+\pi\right)}{-\sin\left(x^6\right)\cdot 2x+6\cos\left(x^6\right)}\\
&= \frac{-3\cos\left(0^3+\pi\right)}{-\sin\left(0^6\right)\cdot 2\cdot 0+6\cos\left(0^6\right)}\\
&= \frac{1}{2}.
\end{aligned}$$

(c) *1. način.* Imamo

$$\lim_{x\to\pi}(1+\sin x)^{\frac{1}{\sin(2x)}} = \lim_{x\to\pi}(1+\sin x)^{\frac{1}{2\sin x\cdot\cos x}} = \lim_{x\to\pi}\left((1+\sin x)^{\frac{1}{\sin x}}\right)^{\frac{1}{2\cos x}} = e^{-\frac{1}{2}},$$

pri čemu zadnja jednakost vrijedi jer je

$$\lim_{x\to\pi}(1+\sin x)^{\frac{1}{\sin x}} = \begin{bmatrix} t=\sin x\\ x\to\pi \;\Rightarrow\; t\to 0\end{bmatrix} = \lim_{t\to 0}(1+t)^{\frac{1}{t}} = e,$$

dok je

$$\lim_{x\to\pi}\frac{1}{2\cos x} = \frac{1}{2\cos\pi} = -\frac{1}{2}.$$

2. način. Imamo

$$\lim_{x\to\pi}(1+\sin x)^{\frac{1}{\sin(2x)}} = \lim_{x\to\pi}\left(e^{\ln(1+\sin x)}\right)^{\frac{1}{\sin(2x)}} = \lim_{x\to\pi} e^{\frac{\ln(1+\sin x)}{\sin(2x)}} = e^{\lim_{x\to\pi}\frac{\ln(1+\sin x)}{\sin(2x)}} = e^{-\frac{1}{2}},$$

pri čemu zadnja jednakost vrijedi jer je

$$\lim_{x\to\pi}\frac{\ln\left(1+\sin x\right)}{\sin(2x)} = \left(\frac{0}{0}\right) \overset{\text{L'H}}{=} \lim_{x\to\pi}\frac{\left(\ln\left(1+\sin x\right)\right)'}{\left(\sin(2x)\right)'} = \lim_{x\to\pi}\frac{\frac{\cos x}{1+\sin x}}{2\cos(2x)} = \frac{\frac{\cos\pi}{1+\sin\pi}}{2\cos(2\pi)} = -\frac{1}{2}.$$

2. *(3 = 1 + 1 + 1)* Izračunajte integrale:

(a) $\displaystyle\int_{-\frac{\pi}{3}}^{-\frac{\pi}{6}} \cos^2(3x)\cdot\sin(3x)\,\mathrm{d}x$

(b) $\displaystyle\int \frac{\ln\left(1+\frac{2}{x}\right)}{x^2}\,\mathrm{d}x$

(c) $\displaystyle\int_{-\infty}^{-2} \frac{\mathrm{d}x}{x^2+4x+8}.$

Rješenje. (a) Imamo

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{6}} \cos^2(3x)\cdot\sin(3x)\,\mathrm{d}x = \begin{bmatrix} t=\cos(3x) & -\frac{\pi}{3}\mapsto -1 \\ \mathrm{d}t = -3\sin(3x)\,\mathrm{d}x & \frac{\pi}{6}\mapsto 0 \end{bmatrix} = \int_{-1}^{0} t^2\,\frac{\mathrm{d}t}{-3} = -\frac{1}{9}t^3\Big|_{-1}^{0} = -\frac{1}{9}.$$

(b) Imamo

$$\int \frac{\ln\left(1+\frac{2}{x}\right)}{x^2}\,\mathrm{d}x = \begin{bmatrix} t = 1+\frac{2}{x} \\ \mathrm{d}t = -\frac{2}{x^2}\,\mathrm{d}x \end{bmatrix} = \int \ln t\,\frac{\mathrm{d}t}{-2} = -\frac{1}{2}t(\ln t - 1) + C$$
$$= -\frac{1}{2}\left(1+\frac{2}{x}\right)\left(\ln\left(1+\frac{2}{x}\right)-1\right)+C,$$

pri čemu predzadnja jednakost vrijedi jer je

$$\int \ln t\,\mathrm{d}t = \begin{bmatrix} u=\ln t & \mathrm{d}u = \frac{\mathrm{d}t}{t} \\ \mathrm{d}v = \mathrm{d}t & v = t \end{bmatrix} = t\,\ln t - \int \mathrm{d}t = t\,\ln t - t + C = t(\ln t - 1) + C.$$

(c) Imamo

$$\int_{-\infty}^{-2} \frac{\mathrm{d}x}{x^2+4x+8} = \lim_{R\to-\infty}\int_{R}^{-2} \frac{\mathrm{d}x}{x^2+4x+8} = \lim_{R\to-\infty}\int_{R}^{-2} \frac{\mathrm{d}x}{(x+2)^2+4}$$
$$= \begin{bmatrix} t = x+2 & R\mapsto R+2 \\ \mathrm{d}t = \mathrm{d}x & -2\mapsto 0 \end{bmatrix} = \lim_{R\to-\infty}\int_{R+2}^{0} \frac{\mathrm{d}t}{t^2+2^2} = \lim_{R\to-\infty}\frac{1}{2}\operatorname{arctg}\frac{t}{2}\Big|_{R+2}^{0}$$
$$= \lim_{R\to-\infty}\left(-\frac{1}{2}\operatorname{arctg}\frac{R+2}{2}\right) = -\frac{1}{2}\left(-\frac{\pi}{2}\right) = \frac{\pi}{4}.$$

3. *(4 = 2 + 2)*

(a) Skicirajte dio ravnine omeđen grafovima funkcija

$$f(x) := \sin x \qquad \text{i} \qquad g(x) := x^2 - \pi x$$

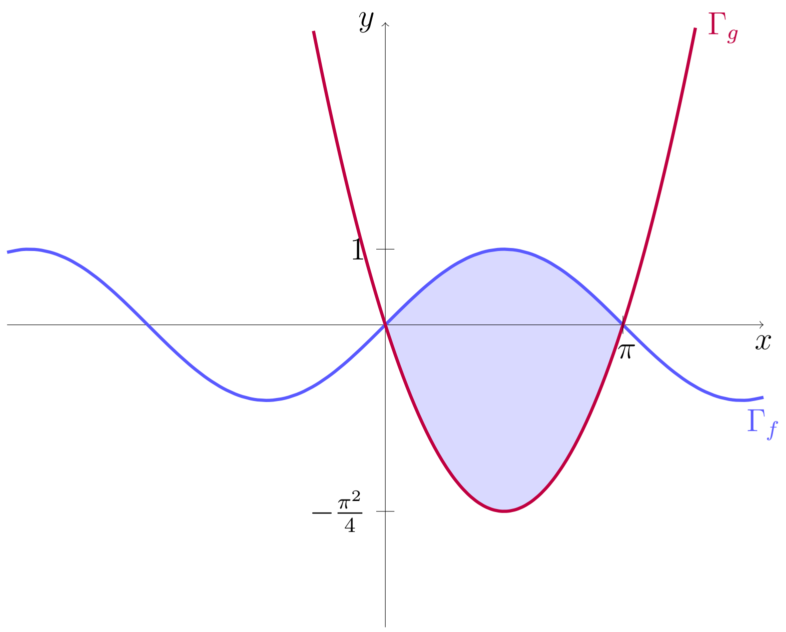
i izračunajte njegovu površinu.

(b) Skicirajte dio ravnine omeđen grafovima funkcija

$$f(x) := e^x + 1, \qquad g(x) := e^{-x} + 1, \qquad h(x) := e^x - 1 \qquad \text{i} \qquad i(x) := e^{-x} - 1$$

i izračunajte njegovu površinu.

Rješenje. (a) Zadani je dio ravnine skiciran na Slici 1.



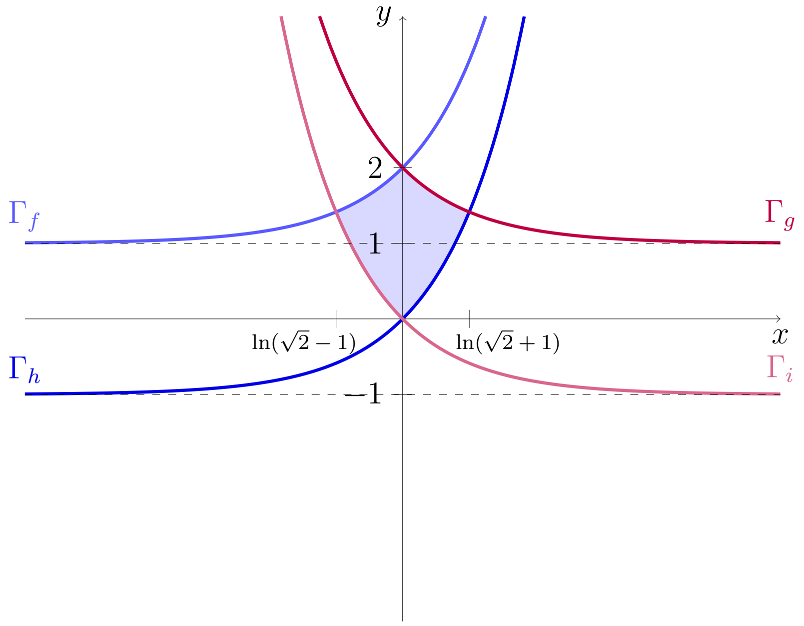
Slika 1: Zadatak 3(a)

Njegova je površina

$$\int_0^\pi (f(x) - g(x))\,\mathrm{d}x = \int_0^\pi \left(\sin x - x^2 + \pi x\right)\,\mathrm{d}x = \left(-\cos x - \frac{x^3}{3} + \frac{\pi}{2}x^2\right)\Bigg|_0^\pi$$
$$= 1 - \frac{\pi^3}{3} + \frac{\pi^3}{2} + 1 = 2 + \frac{\pi^3}{6}$$

kvadratnih jedinica.

(b) Zadani je dio ravnine skiciran na Slici 2.



Slika 2: Zadatak 3(b)

Kako je zrcalno simetričan s obzirom na y-os, njegova je površina

$$
\begin{aligned}
2\int_0^{\ln\left(\sqrt{2}+1\right)}(g(x)-h(x))\,\mathrm{d}x &= 2\int_0^{\ln\left(\sqrt{2}+1\right)}\left(e^{-x}+1-(e^x-1)\right)\,\mathrm{d}x \\
&= 2\int_0^{\ln\left(\sqrt{2}+1\right)}\left(e^{-x}-e^x+2\right)\,\mathrm{d}x \\
&= 2\left(-e^{-x}-e^x+2x\right)\Big|_0^{\ln\left(\sqrt{2}+1\right)} \\
&= 2\left(-\frac{1}{\sqrt{2}+1}-\left(\sqrt{2}+1\right)+2\ln\left(\sqrt{2}+1\right)+2\right) \\
&= 2\left(-\left(\sqrt{2}-1\right)-\left(\sqrt{2}+1\right)+2\ln\left(\sqrt{2}+1\right)+2\right) \\
&= 4\ln\left(\sqrt{2}+1\right)+4-4\sqrt{2}
\end{aligned}
$$

kvadratnih jedinica.