MATEMATIKA 1

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1. *(3)* Odredite prirodnu domenu funkcije

$$f(x) := \frac{\arccos\left(\frac{x}{\pi}\right) \cdot \log_\pi(-x)}{\sqrt[4]{1 - \operatorname{ctg} x}}.$$

Rješenje. Realan broj x nalazi se u $\mathcal{D}_f$ ako i samo ako zadovoljava sljedeće uvjete:

- $\dfrac{x}{\pi} \in \mathcal{D}_{\text{arccos}} = [-1, 1] \quad \Leftrightarrow \quad -1 \leq \dfrac{x}{\pi} \leq 1 \quad \Leftrightarrow \quad -\pi \leq x \leq \pi$
- $-x \in \mathcal{D}_{\log_\pi} = \langle 0, +\infty \rangle \quad \Leftrightarrow \quad -x > 0 \quad \Leftrightarrow \quad x < 0$
- $x \in \mathcal{D}_{\text{ctg}} = \mathbb{R} \setminus \{k\pi : k \in \mathbb{Z}\}$
- $1 - \operatorname{ctg} x > 0 \quad \Leftrightarrow \quad \operatorname{ctg} x < 1 \quad \Leftrightarrow \quad x \in \bigcup_{k \in \mathbb{Z}} \left\langle \frac{\pi}{4} + k\pi, \pi + k\pi \right\rangle.$

Prema tome,

$$\mathcal{D}_f = \left\langle -\frac{3\pi}{4}, 0 \right\rangle.$$

2. *(2)* Derivirajte funkciju

$$f(x) := 2^{\operatorname{arctg} x} \cdot \frac{x^2 \cdot \log_2\left(x^2+x+1\right)}{\operatorname{tg}\left(\operatorname{arctg} x\right)}.$$

Rješenje. 1. način. Imamo

$$\begin{aligned}
f'(x) &= \left(2^{\operatorname{arctg} x}\right)' \cdot \frac{x^2 \cdot \log_2\left(x^2+x+1\right)}{\operatorname{tg}\left(\operatorname{arctg} x\right)} + 2^{\operatorname{arctg} x} \cdot \left(\frac{x^2 \cdot \log_2\left(x^2+x+1\right)}{\operatorname{tg}\left(\operatorname{arctg} x\right)}\right)' \\
&= 2^{\operatorname{arctg} x} \ln 2 \cdot \frac{1}{1+x^2} \cdot \frac{x^2 \cdot \log_2\left(x^2+x+1\right)}{\operatorname{tg}\left(\operatorname{arctg} x\right)} \\
&\quad + 2^{\operatorname{arctg} x} \cdot \left(\frac{\left(2x \cdot \log_2\left(x^2+x+1\right) + x^2 \cdot \frac{1}{(x^2+x+1)\ln 2} \cdot (2x+1)\right) \cdot \operatorname{tg}(\operatorname{arctg} x)}{(\operatorname{tg}(\operatorname{arctg} x))^2}\right. \\
&\qquad \left. - \frac{x^2 \cdot \log_2\left(x^2+x+1\right) \cdot \frac{1}{\cos^2(\operatorname{arctg} x)} \cdot \frac{1}{1+x^2}}{(\operatorname{tg}(\operatorname{arctg} x))^2}\right).
\end{aligned}$$

2. način. Kako je $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ dana pravilom

$$f(x) = 2^{\operatorname{arctg} x} \cdot \frac{x^2 \cdot \log_2\left(x^2+x+1\right)}{x} = 2^{\operatorname{arctg} x} \cdot x \cdot \log_2\left(x^2+x+1\right),$$

$f' : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ je dana pravilom

$$\begin{aligned}
f'(x) &= \left(2^{\operatorname{arctg} x} \cdot x \cdot \log_2\left(x^2+x+1\right)\right)' \\
&= \left(2^{\operatorname{arctg} x}\right)' \cdot x \cdot \log_2\left(x^2+x+1\right) + 2^{\operatorname{arctg} x} \cdot x' \cdot \log_2\left(x^2+x+1\right) \\
&\quad + 2^{\operatorname{arctg} x} \cdot x \cdot \left(\log_2\left(x^2+x+1\right)\right)' \\
&= 2^{\operatorname{arctg} x} \ln 2 \cdot \frac{1}{1+x^2} \cdot x \cdot \log_2\left(x^2+x+1\right) + 2^{\operatorname{arctg} x} \cdot 1 \cdot \log_2\left(x^2+x+1\right) \\
&\quad + 2^{\operatorname{arctg} x} \cdot x \cdot \frac{1}{(x^2+x+1)\ln 2} \cdot (2x+1),
\end{aligned}$$

pri čemu smo u drugoj jednakosti iskoristili činjenicu da za sve derivabilne funkcije $f_1, f_2, f_3 : D \subseteq \mathbb{R} \to \mathbb{R}$ vrijedi

$$\begin{aligned}
(f_1 f_2 f_3)' &= ((f_1 f_2) f_3)' = (f_1 f_2)' f_3 + f_1 f_2 f_3' = (f_1' f_2 + f_1 f_2') f_3 + f_1 f_2 f_3' \\
&= f_1' f_2 f_3 + f_1 f_2' f_3 + f_1 f_2 f_3'.
\end{aligned}$$

3. *(2)* Zadane su funkcije

$$f(x) := \left(e^{2x+1} + 1\right)^2 \qquad \text{i} \qquad g(x) := 6e^{2x+1} - 4x.$$

Odredite sve $x_0 \in \mathbb{R}$ sa svojstvom da su tangenta na Γ_f u točki $(x_0, f(x_0))$ i tangenta na Γ_g u točki $(x_0, g(x_0))$ međusobno paralelne. Za svaki određeni x_0 napišite jednadžbe tih dviju tangenti.

Rješenje. Primijetimo da je

$$f(x) = e^{4x+2} + 2e^{2x+1} + 1$$

pa je

$$f'(x) = 4e^{4x+2} + 4e^{2x+1}, \tag{1}$$

dok je

$$g'(x) = 12e^{2x+1} - 4. \tag{2}$$

Tangenta na Γ_f u točki $(x_0, f(x_0))$ i tangenta na Γ_g u točki $(x_0, g(x_0))$ međusobno su paralelne ako i samo ako su njihovi koeficijenti smjera $f'(x_0)$ i $g'(x_0)$ isti, tj. ako i samo ako vrijedi

$$f'(x_0) = g'(x_0),$$

tj., koristeći (1) i (2),

$$4e^{4x_0+2} + 4e^{2x_0+1} = 12e^{2x_0+1} - 4,$$

tj., dijeljenjem s 4,

$$\left(e^{2x_0+1}\right)^2 - 2e^{2x_0+1} + 1 = 0,$$

tj.

$$\left(e^{2x_0+1} - 1\right)^2 = 0,$$

tj.

$$e^{2x_0+1} = 1,$$

tj.

$$2x_0 + 1 = 0,$$

tj.

$$x_0 = -\frac{1}{2}.$$

Preostaje odrediti jednadžbe tangenti na Γ_f odnosno Γ_g u točki s apscisom $-\frac{1}{2}$. Tangenta t_1 na Γ_f u točki $\left(-\frac{1}{2}, f\left(-\frac{1}{2}\right)\right) = \left(-\frac{1}{2}, 4\right)$ dana je jednadžbom

$$t_1 \ldots\ y - 4 = f'\left(-\frac{1}{2}\right) \cdot \left(x - \left(-\frac{1}{2}\right)\right),$$

tj., koristeći (1), jednadžbom

$$t_1 \ldots\ y - 4 = 8\left(x + \frac{1}{2}\right),$$

dakle

$$t_1 \ldots\ y = 8x + 8,$$

dok je tangenta t_2 na Γ_g u točki $\left(-\frac{1}{2}, g\left(-\frac{1}{2}\right)\right) = \left(-\frac{1}{2}, 8\right)$ dana jednadžbom

$$t_2 \ldots\ y - 8 = g'\left(-\frac{1}{2}\right) \cdot \left(x - \left(-\frac{1}{2}\right)\right),$$

tj.

$$t_2 \ldots\ y - 8 = 8\left(x + \frac{1}{2}\right),$$

dakle

$$t_2 \ldots\ y = 8x + 12.$$

4. *(3)* Ispitajte tok i skicirajte graf funkcije

$$f(x) := \frac{x^2+3x+2}{x^2+3x}.$$

Rješenje. Da bismo si olakšali računanje, najprije primijetimo da je formulom

$$f(x) = 1 + \frac{2}{3}\cdot\frac{1}{x} - \frac{2}{3}\cdot\frac{1}{x+3}$$

dan rastav funkcije f na parcijalne razlomke.

1 Domena funkcije f je $\mathcal{D}_f = \mathbb{R}\setminus\{0,-3\} = \langle -\infty,-3\rangle \cup \langle -3,0\rangle \cup \langle 0,+\infty\rangle$.
$\rightsquigarrow$ Rubovi domene: $-\infty, -3, 0, +\infty$.

2 f nije ni parna ni neparna ni periodična.

3 Nultočke: $-2, -1$.

4 $f'(x) = -\frac{2}{3}\cdot\frac{1}{x^2} + \frac{2}{3}\cdot\frac{1}{(x+3)^2} = -2\cdot\frac{2x+3}{x^2(x+3)^2}$.
$\rightsquigarrow$ Stacionarne točke: $-\frac{3}{2}$.

5 $f''(x) = \frac{2}{3}\cdot\frac{2}{x^3} - \frac{2}{3}\cdot\frac{2}{(x+3)^3} = 12\cdot\frac{x^2+3x+3}{x^3(x+3)^3}$.
Kako polinom x^2+3x+3 nema realnih nultočaka, $f''(x)\neq 0$ za sve $x\in\mathcal{D}_f$.

6 Kako je

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty}\left(1+\frac{2}{3}\cdot\frac{1}{x}-\frac{2}{3}\cdot\frac{1}{x+3}\right) = 1,$$

pravac $y=1$ je horizontalna asimptota funkcije f. Nadalje, s obzirom da vrijedi

$$\lim_{x\to 0\pm} f(x) = \lim_{x\to 0\pm}\left(1+\frac{2}{3}\cdot\frac{1}{x}-\frac{2}{3}\cdot\frac{1}{x+3}\right) = \left(1+\frac{2}{3}\cdot\frac{1}{0\pm}-\frac{2}{9}\right) = \pm\infty$$

i

$$\lim_{x\to 3\pm} f(x) = \lim_{x\to 3\pm}\left(1+\frac{2}{3}\cdot\frac{1}{x}-\frac{2}{3}\cdot\frac{1}{x+3}\right) = \left(1-\frac{2}{9}-\frac{2}{3}\cdot\frac{1}{0\pm}\right) = \mp\infty,$$

pravci $x=0$ i $x=3$ su vertikalne asimptote funkcije f.

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	$-\infty$		-3		-2		lok. max $-\frac{3}{2}$		-1		0		$+\infty$
f	$\times$	$\nearrow$ $+$ $\cup$	$\times$	$\nearrow$ $-$ $\cap$	0	$\nearrow$ $+$ $\cap$		$\searrow$ $+$ $\cap$	0	$\searrow$ $-$ $\cap$	$\times$	$\searrow$ $+$ $\cup$	$\times$
f'	$\times$	$+$	$\times$	$+$		$+$	0	$-$		$-$	$\times$	$-$	$\times$
f''	$\times$	$+$	$\times$	$-$		$-$		$-$		$-$	$\times$	$+$	$\times$
$f(x)$		$1 \leftarrow$ $\rightarrow +\infty$		$-\infty \leftarrow$	0		$\frac{1}{9}$		0	$\rightarrow -\infty$		$+\infty \leftarrow$ $\rightarrow 1$	

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