# The symmetric monoidal category indproVect of filtered-cofiltered vector spaces

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This article is in the process of becoming
a completely written article.

**Abstract**

We construct a symmetric monoidal category $(\text{indproVect}, \tilde{\hat{\otimes}}, k)$ of filtered-cofiltered vector spaces, whose morphisms are linear maps which in a weak sense respect the filtrations and cofiltrations, and whose monoidal product is the usual tensor product of vector spaces formally completed and with a corresponding filtration of cofiltrations. It contains two dual subcategories, the category $(\text{indVect}, \otimes, k)$ of filtered vector spaces and the category $(\text{proVect}, \hat{\otimes}, k)$ of cofiltered vector spaces. The monoidal product in it combines the ordinary tensor product and the completed tensor product. The category indproVect is equivalent to the category of strict ind-pro-objects of at most $\aleph_0$ cofinality in the category of vector spaces. We prove that this category has coequalizers which commute with the tensor product. This makes the definition of the tensor product over a base monoid $A$ in the category indproVect possible and the definition of a monoidal category $({}_A\text{indproVect}_A, \tilde{\hat{\otimes}}_A, A)$ of internal $A$-bimodules in this category. We also give descriptions of coproducts, coequalizers and filtered colimits in the subcategory proVect. We define and use a notion of a formal sum and a formal basis in the category proVect of cofiltered vector spaces.

**Keywords:** cofiltered vector space, formal sum, formal basis, filtered vector space, strict pro-object, strict ind-object, strict ind-pro-object, filtered-cofiltered vector space, completed tensor product, dual vector spaces, algebraic dual, formal completion

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## 2.1 The category indVect of filtered vector spaces

**Definition 2.1.** *A filtration* in the category $\mathcal{V}$ is a directed system $\boldsymbol{V}\colon I \to \mathcal{V}$ whose every connecting morphism is a monomorphism. A filtration $\boldsymbol{V}$ is called an $\aleph_0$-filtration if the indexing category $I$ is of cofinality at most $\aleph_0$. *Morphisms of filtrations* are morphisms of directed systems.

Category of filtrations in the category $\mathcal{V}$ is equivalent to the category of strict ind-objects in $\mathcal{V}$, hence the notation $\mathrm{Ind}^s\mathcal{V}$ for this category.

**Definition 2.2.** *A filtration on the object* $V$ in a category $\mathcal{V}$ is a filtration $\boldsymbol{V}$ in the category $\mathcal{V}$ together with a set of monomorphisms $\iota_i^V\colon \boldsymbol{V}(i) \to V$ such that for each $i \to j \in \mathrm{Mor}(I)$, $\iota_j^V \circ \boldsymbol{V}(i \to j) = \iota_i^V$, or equivalently, with a natural transformation $\iota^V\colon \boldsymbol{V} \to V$ from $\boldsymbol{V}$ to a constant functor.

Given filtrations $\boldsymbol{V}$ and $\boldsymbol{W}$ on objects $V$ and $W$ respectively, we say that a morphism $f\colon V \to W$ *respects filtrations* if for all $i \in I$ there exists $j \in J$ and a morphism $f_{ji}$ for which $f \circ \iota_i^V = \iota_j^W \circ f_{ji}$, where $I$ and $J$ are indexing categories for $\boldsymbol{V}$ and $\boldsymbol{W}$ respectively.

**Definition 2.3.** *A filtered vector space* is a vector space $V$ together with an $\aleph_0$-filtration $\boldsymbol{V}$ in the category of vector spaces such that $V \cong \operatorname{colim} \boldsymbol{V}$ in Vect. *A morphism of filtered vector spaces*, or *a filtered map*, is a linear mapping $f\colon V \to W$ which respects filtrations.

It is easy to check that this indeed is a category. It will be noted by the symbol indVect. Note that the filtration $\boldsymbol{V}$ of a filtered vector space $V \cong \operatorname{colim} \boldsymbol{V}$ is a filtration on the vector space $V$ because all injections from components $V(i)$ to $\operatorname{colim} \boldsymbol{V}$ in Vect are monomorphisms.

**Proposition 2.4.** Every filtration $\boldsymbol{V}$ in the category of vector spaces has a colimit $V \cong \operatorname{colim} \boldsymbol{V}$ in the category of vector spaces and all components of the universal cocone are monomorphisms [**?**].

**Theorem 2.5.** There is a natural bijective correspondence between morphisms $\boldsymbol{V} \to \boldsymbol{W}$ of $\aleph_0$-filtrations in the category Vect and the filtered maps $V \to W$ between filtered vector spaces $V \cong \operatorname{colim} \boldsymbol{V}$, $W \cong \operatorname{colim} \boldsymbol{W}$.

*Proof.* ... PROOF □

**Corollary 2.6.** The category indVect of filtered vector spaces and filtered maps is equivalent to the category $\mathrm{Ind}^s_{\aleph_0}\mathrm{Vect}$ of strict ind-objects of cofinality at most $\aleph_0$ in the category of vector spaces.

## 2.2 The tensor product $\otimes$

**Definition 2.7.** Let $(\mathcal{V}, \otimes, k)$ be a symmetric monoidal category in which the monoidal product of every two monomorphisms is a monomorphism. Define the monoidal product of objects and morphisms of $\mathrm{Ind}^s\mathrm{Vect}$ as follows. For two filtrations $\boldsymbol{V}\colon I \to \mathcal{V}$, $\boldsymbol{W}\colon J \to \mathcal{V}$ define the monoidal product $\boldsymbol{V} \otimes \boldsymbol{W}$ as the composition $I \times J \xrightarrow{(\boldsymbol{V},\boldsymbol{W})} \mathcal{V} \times \mathcal{V} \xrightarrow{\otimes} \mathcal{V}$, and for two morphisms of filtrations, given by premorphisms $f = (\lambda, \{f_i\})$ and $g = (\mu, \{g_j\})$, define the monoidal product by premorphism $f \otimes g := (\lambda \times \mu, \{f_i \otimes g_j\})$.

**Proposition 2.8.** Let $(\mathcal{V}, \otimes, k)$ be a symmetric monoidal category in which the monoidal product of every two monomorphisms is a monomorphism. The category $(\mathrm{Ind}^s\mathcal{V}, \otimes, k)$ is then a symmetric monoidal category.

*Proof.* It is easily seen that the monoidal product of filtrations and of morphisms of filtrations is well defined and that $\otimes\colon \mathrm{Ind}^s\mathcal{V} \times \mathrm{Ind}^s\mathcal{V} \to \mathrm{Ind}^s\mathcal{V}$ is a bifunctor. The associator, unitors and simetrizator of the category $\mathrm{Ind}^s\mathcal{V}$ are built on levels by the same natural transformations in the category $\mathcal{V}$, and all axioms for symmetric monoidal category are satisfied because on each level they are satisfied in $\mathcal{V}$. $\square$

**Definition 2.9.** Define the tensor product of filtered vector spaces $V \cong \operatorname{colim} \boldsymbol{V}$ and $W \cong \operatorname{colim} \boldsymbol{W}$ as the filtered vector space $V \otimes W := \operatorname{colim} \boldsymbol{V} \otimes \boldsymbol{W}$, and the tensor product $f \otimes g$ of filtered maps $f$ and $g$ as the unique filtered map induced by the monoidal product of the morphisms of the corresponding filtrations.

It can be easily proved that $\operatorname{colim} \boldsymbol{V} \otimes \boldsymbol{W}$ is exactly the usual tensor product $V \otimes W$, and that the induced filtered map $f \otimes g$ between tensor products of filtered vector spaces is exactly the usual $f \otimes g$, hence the same symbols as the usual ones. By this definition we get the following equivalence of categories.

**Proposition 2.10.** The category $(\mathrm{indVect}, \otimes, k)$ is a symmetric monoidal category which is strongly monoidally equivalent to the symmetric monoidal category $(\mathrm{Ind}^s_{\aleph_0}\mathrm{Vect}, \otimes, k)$.

## 2.3 Filtered basis and finite sums

**Definition 2.11.** A subset $B \subseteq V$ of a filtered vector space $V \cong \operatorname{colim} \boldsymbol{V}$ is *a filtered basis for* $V$ if there is a cofinal subset $K$ of the indexing set $I$ such that for each $k \in K$ the set $(\iota_k^V)^{-1}(B)$ is a vector space basis of the vector space $\boldsymbol{V}(k)$.

A filtered basis is evidently also a vector space basis for $V$. Given one filtered basis of $V$, each element of the filtered vector space $V$ can be uniquely written as a finite linear combination of the elements of that basis.

**Proposition 2.12.** Every filtered vector space has a filtered basis.

*Proof.* ... Zorn lemma. □

# 3 The category $(\mathrm{proVect}, \hat{\otimes}, k)$

## 3.1 The category proVect of cofiltered vector spaces

**Definition 3.1.** *A cofiltration* in the category $\mathcal{V}$ is an inverse system $\boldsymbol{V}\colon I^{\mathrm{op}} \to \mathcal{V}$ whose every connecting morphism is an epimorphism. A cofiltration $\boldsymbol{V}$ is called an $\aleph_0$-cofiltration if the indexing category $I$ is of cofinality at most $\aleph_0$. *Morphisms of cofiltrations* are morphisms of inverse systems.

Category of cofiltrations in the category $\mathcal{V}$ is equivalent to the category of strict pro-objects in $\mathcal{V}$, hence the notation $\mathrm{Pro}^s\mathcal{V}$ for this category.

**Definition 3.2.** *A cofiltration on the object* $V$ in a category $\mathcal{V}$ is a cofiltration $\boldsymbol{V}$ in the category $\mathcal{V}$ together with a set of epimorphisms $\pi_i^V\colon V \to \boldsymbol{V}(i)$ such that for each $i \to j \in \mathrm{Mor}(I)$, $\boldsymbol{V}(i \to j) \circ \pi_i^V = \pi_j^V$, or equivalently, with a natural transformation $\pi^V\colon V \to \boldsymbol{V}$ from a constant functor to $\boldsymbol{V}$.

Given cofiltrations $\boldsymbol{V}$ and $\boldsymbol{W}$ on objects $V$ and $W$ respectively, we say that the morphism $f\colon V \to W$ *respects cofiltrations* if for all $j \in J$ there exists $i \in I$ and a morphism $f_{ji}$ for which $f_{ji} \circ \pi_i^V = \pi_j^W \circ f$, where $I$ and $J$ are the indexing categories of $\boldsymbol{V}$ and $\boldsymbol{W}$ respectively.

**Definition 3.3.** *A cofiltered vector space* is a vector space $V$ together with an $\aleph_0$-filtration $\boldsymbol{V}$ in the category of vector spaces such that $V \cong \lim \boldsymbol{V}$ in Vect. *A morphism of cofiltered vector spaces*, or *a cofiltered map*, is a linear mapping $f\colon V \to W$ which respects cofiltrations.

It is easy to check that this indeed is a category. It will be noted by the symbol proVect. The cofiltration $\boldsymbol{V}$ of a cofiltered vector space $V \cong \lim \boldsymbol{V}$ is a cofiltration on the vector space $V$ because all projections from $\lim \boldsymbol{V}$ in Vect to components $\boldsymbol{V}(i)$ are epimorphisms when the indexing family is of cofinality at most $\aleph_0$.

**Proposition 3.4.** Every $\aleph_0$-cofiltration $\boldsymbol{V}$ in the category of vector spaces has a limit $V \cong \lim \boldsymbol{V}$ in the category of vector spaces and all components of the universal cone are epimorphisms [?].

**Theorem 3.5.** There is a natural bijective correspondence between morphisms $\boldsymbol{V} \to \boldsymbol{W}$ of $\aleph_0$-cofiltrations in the category Vect and the cofiltered maps $V \to W$ between cofiltered vector spaces $V \cong \lim \boldsymbol{V}$, $W \cong \lim \boldsymbol{W}$.

*Proof.* ... PROOF □

**Corollary 3.6.** The category proVect of cofiltered vector spaces and cofiltered maps is equivalent to the category $\mathrm{Pro}^s_{\aleph_0}\mathrm{Vect}$ of strict pro-objects of cofinality at most $\aleph_0$ in the category of vector spaces.

## 3.2 The completed tensor product $\hat{\otimes}$

**Definition 3.7.** Let $(\mathcal{V}, \otimes, k)$ be a symmetric monoidal category in which the monoidal product of every two epimorphisms is an epimorphism. Define the monoidal product of objects and morphisms of $\mathrm{Pro}^s\mathrm{Vect}$ as follows. For two cofiltrations $\boldsymbol{V} \colon I^{\mathrm{op}} \to \mathcal{V}$, $\boldsymbol{W} \colon J^{\mathrm{op}} \to \mathcal{V}$ define the monoidal product $\boldsymbol{V} \otimes \boldsymbol{W}$ as the composition $I^{\mathrm{op}} \times J^{\mathrm{op}} \xrightarrow{(\boldsymbol{V},\boldsymbol{W})} \mathcal{V} \times \mathcal{V} \xrightarrow{\otimes} \mathcal{V}$, and for two morphisms of filtrations, given by premorphisms $f = (\lambda, \{f_i\})$ and $g = (\mu, \{g_j\})$, define the monoidal product by premorphism $f \otimes g := (\lambda \times \mu, \{f_i \otimes g_j\})$.

**Proposition 3.8.** Let $(\mathcal{V}, \otimes, k)$ be a symmetric monoidal category in which the monoidal product of every two epimorphisms is an epimorphism. The category $(\mathrm{Pro}^s\mathcal{V}, \otimes, k)$ is then a symmetric monoidal category.

*Proof.* It is easily seen that the monoidal product of cofiltrations and of morphisms of cofiltrations is well defined and that $\otimes \colon \mathrm{Pro}^s\mathcal{V} \times \mathrm{Pro}^s\mathcal{V} \to \mathrm{Pro}^s\mathcal{V}$ is a bifunctor. The associator, unitors and simetrizator of the category $\mathrm{Pro}^s\mathcal{V}$ are built on levels by the same natural transformations in the category $\mathcal{V}$, and all axioms for symmetric monoidal category are satisfied because on each level they are satisfied in $\mathcal{V}$. □

**Definition 3.9.** Define the tensor product of cofiltered vector spaces $V \cong \lim \boldsymbol{V}$ and $W \cong \lim \boldsymbol{W}$ as the cofiltered vector space $V \hat{\otimes} W := \lim \boldsymbol{V} \otimes \boldsymbol{W}$, and the tensor product $f \hat{\otimes} g$ of cofiltered maps $f$ and $g$ as the unique cofiltered map induced by the monoidal product of the maps of the corresponding cofiltrations.

By this definition we get the following equivalence of categories.

**Proposition 3.10.** The category $(\mathrm{proVect}, \hat{\otimes}, k)$ is a symmetric monoidal category which is strongly monoidally equivalent to the symmetric monoidal category $(\mathrm{Pro}^s_{\aleph_0}\mathrm{Vect}, \otimes, k)$.

## 3.3 Formal sums and formal basis

**Definition 3.11.** *A formal sum* in a cofiltered vector space $V \cong \lim \boldsymbol{V}$ is an expression of the form $\sum_{\lambda\in\Lambda} v_\lambda$, where $\Lambda$ is an indexing set and each $v_\lambda$ is an element of $V$, for which every projection $\pi_i^V : V \to \boldsymbol{V}(i)$ maps all but finitely many summands $v_\lambda$ to 0. Here, instead of formal sum, we can also say *pro-finite sum.*

It is easy to see that every formal sum $\sum_\lambda v_\lambda$ in a cofiltered vector space $V \cong \lim \boldsymbol{V}$ defines a unique thread of $\lim \boldsymbol{V}$ and hence an element of $V$. That element is called *the value of the formal sum* $\sum_\lambda v_\lambda$.

**Definition 3.12.** Let $V \cong \lim \boldsymbol{V}$ and $W \cong \lim \boldsymbol{W}$ be cofiltered vector spaces. We say that a linear map $A\colon V \to W$ *distributes over formal sums* if for every formal sum $\sum_\lambda v_\lambda$ in $V$ the expresion $\sum_\lambda A(v_\lambda)$ is a formal sum in $W$ and the value of the first formal sum is mapped to the value of the second one:

$$A(\sum_\lambda v_\lambda) = \sum_\lambda A(v_\lambda).$$

**Theorem 3.13.** Let $V \cong \lim \boldsymbol{V}$ and $W \cong \lim \boldsymbol{W}$ be cofiltered vector spaces. A linear map $A\colon V \to W$ distributes over formal sums if and only if it is a cofiltered map.

*Proof.* ... PROOF □

**Definition 3.14.** A subset $B \subset V\backslash\{0\}$ is *a formal basis for* a cofiltered vector space $V \cong \lim \boldsymbol{V}$ if there is a cofinal subset $K$ of the indexing set $I$ such that for every $k \in K$ the projection $\pi_k^V$ maps $B\backslash \operatorname{Ker} \pi_k^V$ bijectively to a vector space basis of the vector space $\boldsymbol{V}(k)$.

**Proposition 3.15.** Every cofiltered vector space has a formal basis.

*Proof.* ... Zorn lemma. □

**Proposition 3.16.** Given a formal basis $B = \{e_\alpha\}_{\alpha\in A}$ of the cofiltered vector space $V \cong \lim \boldsymbol{V}$, every element of $V$ is the value of a unique formal sum $\sum_\alpha a_\alpha e_\alpha$ whose summands are elements of the formal basis $B$ with coefficients in the field $k$.

*Proof.* ... Zorn lemma. □

## 3.4 Completions of vector spaces and linear maps

**Definition 3.17.** Let $V$ be a vector space and $\boldsymbol{V}$ a cofiltration on it. We say that the vector space $V$ is *complete with respect to cofiltration* $\boldsymbol{V}$ if $V \cong \lim \boldsymbol{V}$ via the canonical map to the limit. If the cofiltration $\boldsymbol{V}$ is an $\aleph_0$-cofiltration, the cofiltered vector space $\hat{V} := \lim \boldsymbol{V}$ is called *the completion of* $V$ *with respect to cofiltration* $\boldsymbol{V}$ and the canonical map to the limit $V \to \hat{V}$ is called *the canonical map to completion*.

**Theorem 3.18.** Every element of the completion $\hat{W} \cong \lim \boldsymbol{W}$ of a vector space $W$ with respect to $\aleph_0$-cofiltration $\boldsymbol{W}$ is the value of a formal sum in $\hat{W}$ whose summands are all in the image $\iota(W)$ of the canonical map to completion $\iota\colon W \to \hat{W}$.

*Proof.* ... PROOF ... Uses Zorn lemma. □

**Corollary 3.19.** A vector space $W$ on which an $\aleph_0$-cofiltration $\boldsymbol{W}$ is given is complete with respect to this cofiltration if and only if the canonical map $\iota\colon W \to \hat{W}$ is injective and its image in the completion $\hat{W}$ contains all values of formal sums in $\hat{W}$ which have summands in the image of $W$.

*Proof.* Every element of the cofiltered vector space $\hat{W} \cong \lim \boldsymbol{W}$ is the value of a formal sum of elements of $\iota(W)$ by proposition 3.18. A vector subspace $\iota(W)$ is complete if and only if it contains all values of all formal sums of elements of $\iota(W)$. Indeed, if $\iota(W)$ contains values of all those formal sums, then $\hat{W} \subset \iota(W)$, hence $\hat{W} = \iota(W)$, and if $\iota(W)$ is complete, then of course it contains the values of all the formal sums in $\iota(W)$. □

A linear map which respects cofiltrations can be completed to a cofiltered linear map.

**Proposition 3.20.** Let $V$ and $W$ be vector spaces on which $\aleph_0$-cofiltrations $\boldsymbol{V}$ and $\boldsymbol{W}$ are given, respectively. If a linear mapping $f\colon V \to W$ respects cofiltrations,

$$(\forall j \in J)(\exists i \in I)(\exists f_{ji}\colon \boldsymbol{V}(i) \to \boldsymbol{W}(j))\ f_{ji} \circ \pi_i^V = \pi_j^W \circ f,$$

then it can be completed to a cofiltered map $\hat{f}\colon \hat{V} \to \hat{W}$ in a unique way such that $\hat{f} \circ \iota_V = \iota_W \circ f$. The map $\hat{f}$ is then called *the completion of* the linear mapping $f$.

*Proof.* Uses the same reasoning as in the second part of the proof of the proposition 3.5. □

Notice that the completed tensor product $V \hat{\otimes} W$ is the completion of the usual tensor product $V \otimes W$ with respect to the cofiltration $\boldsymbol{V} \otimes \boldsymbol{W}$.

**Proposition 3.21.** Let $V \cong \lim \boldsymbol{V}$ and $W \cong \lim \boldsymbol{W}$ be cofiltered vector spaces. The cofiltration $\boldsymbol{V} \otimes \boldsymbol{W}$ is a cofiltration on the vector space $V \otimes W$, hence it induces the canonical linear map to the limit $V \otimes W \to V \hat{\otimes} W$. That map is injective,

$$V \otimes W \hookrightarrow V \hat{\otimes} W.$$

The pure tensor $v \otimes w \in V \otimes W$ maps to the same pure tensor $v \otimes w \in V \hat{\otimes} W$.

*Proof.* ... □

## 3.5 Monomorphisms, kernels, subspaces and quotients

**Proposition 3.22.** A cofiltered map $i \colon V \to W$ is a monomorphism in the category proVect if and only if it is a monomorphism in the category Vect.

*Proof.* Assume that a cofiltered map $i \colon V \to W$ is a monomorphism in proVect. We prove that it is injective. Assume the contrary. Then there exist elements $v \neq w$ in $V$ such that $i(v) = i(w)$. Define two linear maps $f, g \colon k \to V$ with $f(1) = v, g(1) = w$. Then trivially $f$ and $g$ are morphisms in proVect, $i \circ f = i \circ g$ but $f \neq g$, which contradicts the first assumption. The other implication is proved easily. □

**Proposition 3.23.** Let $V \cong \lim \boldsymbol{V}$ be a cofiltered vector space and $W \subset V$ a vector subspace of $V$. Then there is a canonical induced cofiltration $\boldsymbol{W} := \boldsymbol{V} \cap W$ on $W$ which we call *the subspace cofiltration*. If $W \cong \lim \boldsymbol{W}$ via the canonical map to the limit, then $W$ is said to be a *complete subspace of* a cofiltered vector space $V$. It is then a cofiltered vector space and the inclusion $W \hookrightarrow V$ is a cofiltered map.

*Proof.* ... Canonical construction of the subspace cofiltration. □

**Proposition 3.24.** A kernel of a cofiltered map $f \colon V \to W$, $\operatorname{Ker} f = \{v \in V \mid f(v) = 0\}$, is a complete subspace of the cofiltered vector space $V$.

*Proof.* By proposition 3.19 and proposition 3.13. □

**Proposition 3.25.** Let $V \cong \lim \boldsymbol{V}$ be a cofiltered vector space and $W \subset V$ a complete subspace of $V$. Then the quotient $V/W$ is a cofiltered vector space with regard to the canonical induced cofiltration $\boldsymbol{V}/\boldsymbol{W}$ which we call *the quotient cofiltration*. The quotient map $V \to V/W$ is a cofiltered map.

*Proof.* ... Canonical construction of the quotient cofiltration and proof that $V/W$ is complete. □

**Proposition 3.26.** Let $V \cong \lim \boldsymbol{V}$ and $W \cong \lim W$ be cofiltered vector spaces and $V' \subset V$ and $W' \subset W$ their complete subspaces (with subspace cofiltrations). Let $f\colon V \to W$ be a cofiltered map such that $f(V') \subset W'$. Then the induced linear map $\bar{f}\colon V/V' \to W/W'$ is also a cofiltered map (with respect to quotient cofiltrations).

*Proof.* ... Checking that the induced linear map respects cofiltrations. □

## 3.6 Coproducts and filtered colimits

**Proposition 3.27.** Coproduct in proVect. ...

*Proof.* ... □

**Theorem 3.28.** Filtered colimit in proVect. ...

*Proof.* ... □

**Proposition 3.29.** Let $\boldsymbol{V}$ be a filtration in the category proVect and let $V \cong \operatorname{colim} \boldsymbol{V}$ in Vect. Then all components $\iota_i^V\colon \boldsymbol{V}(i) \to V$ of the universal cocone are monomorphisms in the category Vect.

*Proof.* By proposition 3.22 and the fact that the same is true in the category of vector spaces by proposition 2.4. □

# 4 The category $(\mathrm{indproVect}, \tilde{\otimes}, k)$

Here we define the category indproVect and prove that it is equivalent to the category of strict ind-pro-objects of cofinality at most $\aleph_0$ in the category of vector spaces. We then define a natural symmetric monoidal product on it. We prove that this monoidal category admits coequalizers and that they commute with the tensor product.

## 4.1 The category indproVect of filtered-cofiltered vector spaces

**Definition 4.1.** *A filtered-cofiltered vector space* is a vector space $V$ together with an $\aleph_0$-filtration $\boldsymbol{V}$ in the category proVect such that $V \cong \operatorname{colim} \boldsymbol{V}$ in Vect. *A morphism of filtered-cofiltered vector spaces*, or *a filtered-cofiltered map*, is a linear mapping $f\colon V \to W$ such that for every $i \in I$ there exists

$j \in J$ and a cofiltered map $f_{ji}$ such that $f \circ \iota_i^V = \iota_j^W \circ f_{ji}$ in Vect, where $I$ and $J$ are the indexing categories of $\boldsymbol{V}$ and $\boldsymbol{W}$ respectively.

Every filtered-cofiltered vector space can also be seen as a colimit in the category proVect, by the proposition 3.28. Hence beside the filtration of cofiltrations there is also the induced cofiltration on it. It can be shown that the filtered-cofiltered morphisms are exactly the cofiltered morphisms which respect filtration. We will not work with such a definition, because of simplicity of the one given above. For example, tensor product $R \tilde{\otimes} H$ of filtered vector space $R \cong \operatorname{colim} \boldsymbol{R}$ and cofiltered vector space $H \cong \lim \boldsymbol{H}$ will have filtering components $R_n \hat{\otimes} H$ each of which will have cofiltering components $R_n \otimes H_k$.

**Proposition 4.2.** There is a natural bijective correspondence between morphisms $\boldsymbol{V} \to \boldsymbol{W}$ of $\aleph_0$-filtrations in the category proVect and the filtered-cofiltered maps $V \to W$ between filtered-cofiltered vector spaces $V \cong \operatorname{colim} \boldsymbol{V}$, $W \cong \operatorname{colim} \boldsymbol{W}$.

*Proof.* By proposition 3.29 ..... The proof is the same as the proof of the theorem 2.5, the only difference is that here the connecting morphisms are cofiltered maps. □

**Corollary 4.3.** The category indproVect of filtered-cofiltered vector spaces and filtered-cofiltered maps is equivalent to the category $\mathrm{Ind}^s_{\aleph_0}\mathrm{Pro}^s_{\aleph_0}\mathrm{Vect}$ of strict ind-pro-objects of cofinality at most $\aleph_0$ in the category of vector spaces.

## 4.2 The semi-completed tensor product $\tilde{\otimes}$

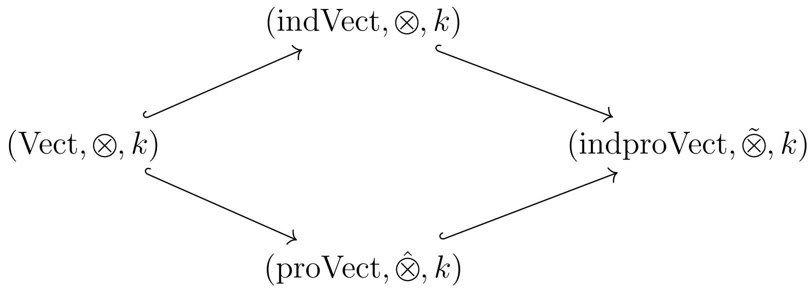
**Proposition 4.4.** The completed tensor product $f \hat{\otimes} g$ of two monomorphisms in the category proVect is a monomorphism.

*Proof.* ... Uses formal basis. □

**Definition 4.5.** Define the tensor product of filtered-cofiltered vector spaces $V \cong \operatorname{colim} \boldsymbol{V}$ and $W \cong \operatorname{colim} \boldsymbol{W}$ as the filtered-cofiltered vector space $V \tilde{\otimes} W := \operatorname{colim} \boldsymbol{V} \hat{\otimes} \boldsymbol{W}$, and the tensor product $f \tilde{\otimes} g$ of two filtered-cofiltered maps $f$ and $g$ as the unique filtered-cofiltered map induced by the monoidal product of the maps of the corresponding filtrations in proVect.

**Proposition 4.6.** The category $(\mathrm{indproVect}, \tilde{\otimes}, k)$ is a symmetric monoidal category. Its full monoidal subcategories are symmetric monoidal categories

$(\mathrm{indVect}, \otimes, k)$ and $(\mathrm{proVect}, \hat{\otimes}, k)$.




*Proof.* ... PROOF □

## 4.3 Duality

**Proposition 4.7.** Categories indVect and proVect are dual categories, as follows. (i) If $V$ is a filtered vector space, with filtration $(\{V_n\}_{n\in I}, \{\phi_{mn}\})$, its algebraic dual $V^* = \mathrm{Vect}(V, k) \cong \mathrm{indVect}(V, k)$ is a cofiltered vector space due to a naturally induced cofiltration $(\{V^*{}_n\}_{n\in I}, \{\phi^*_{nm}\})$, where

$$V^*_n = (V^*)_n := V^*/\,\mathrm{Anih}(V_n) \cong (V_n)^*,$$

$$\phi^*_{nm} = (\phi^*)_{nm} := (\phi_{mn})^*.$$

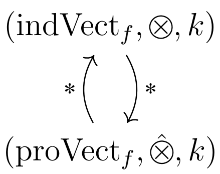
(ii) If $W$ is a cofiltered vector space, with cofiltration $(\{W_n\}_{n\in I}, \{\psi_{nm}\})$, its dual in the category proVect, $W^{*\mathrm{proVect}} = \mathrm{proVect}(W, k) \subset \mathrm{Vect}(W, k)$ is a filtered vector space due to a naturally induced filtration $(\{W^*_n\}_{n\in I}, \{\psi^*_{mn}\})$, where

$$W^*_n = (W^*)_n := W^*/\,\mathrm{Anih}(W_n) \cong (W_n)^*,$$

$$\psi^*_{mn} = (\psi^*)_{mn} := (\psi_{nm})^*.$$

*Proof.* ... PROOF □

**Proposition 4.8.** The subcategories $(\mathrm{indVect}_f, \otimes, k)$ and $(\mathrm{proVect}_f, \hat{\otimes}, k)$ of the monoidal category $(\mathrm{indproVect}, \tilde{\otimes}, k)$ are dual as monoidal categories.




*Proof.* ... □

## 4.4 Semi-completions of linear maps

**Proposition 4.9.** Let $V \cong \operatorname{colim} \boldsymbol{V}$ and $W \cong \operatorname{colim} \boldsymbol{W}$ be filtered-cofiltered vector spaces. The canonical morphism of filtrations $\boldsymbol{V} \otimes \boldsymbol{W} \to \boldsymbol{V} \hat{\otimes} \boldsymbol{W}$ in Vect induces a linear map $V \otimes W \to V \tilde{\otimes} W$ which is an injection and it is a morphism of filtered vector spaces:

$$V \otimes W \hookrightarrow V \tilde{\otimes} W.$$

The pure tensor $v \otimes w \in V \otimes W$ maps to the same pure tensor $v \otimes w \in V \tilde{\otimes} W$.

*Proof.* ... □

**4.10.** Let $V$ and $W$ be vector spaces and $\boldsymbol{V}$ and $\boldsymbol{W}$ filtrations of vector spaces given on them such that on every component of those filtrations a cofiltration is given and such that the connecting morphisms of those filtrations respect those cofiltrations. We say that the linear map $f\colon V \to W$ respects filtrations and cofiltrations if for every ....

**Proposition 4.11.** Let $V \cong \operatorname{colim} \boldsymbol{V}$, $V' \cong \operatorname{colim} \boldsymbol{V'}$, $W \cong \operatorname{colim} \boldsymbol{W}$, $W' \cong \operatorname{colim} \boldsymbol{W'}$ be filtered-cofiltered vector spaces, and let $f\colon V \otimes W \to V' \otimes W'$ be a linear map. If the linear map $f$ respects filtrations and cofiltrations in a way given in 4.10 then it can be completed to a filtered-cofiltered map $\tilde{f}\colon V \tilde{\otimes} W \to V' \tilde{\otimes} W'$,

$$\begin{array}{ccc} V \tilde{\otimes} W & \xrightarrow{\tilde{f}} & V' \tilde{\otimes} W' \\ \uparrow & & \uparrow \\ V \otimes W & \xrightarrow{f} & V' \otimes W' \end{array}$$

The filtered-cofiltered map $\tilde{f}$ is induced by the premorphism whose components are completions of the components of the premorphism of filtrations $\boldsymbol{V} \otimes \boldsymbol{W} \to \boldsymbol{V'} \otimes \boldsymbol{W'}$ induced by the linear map $f$.

*Proof.* ... □

## 4.5 Dual Hopf algebras and the Heisenberg double

**Corollary 4.12.** The algebraic dual $R^*$ of an internal Hopf algebra $R$ in the category $(\mathrm{indVect}_f, \otimes, k)$ is an internal Hopf algebra in the category $(\mathrm{proVect}_f, \hat{\otimes}, k)$. The canonical pairing $R \otimes R^* \to k$ is a morphism in the category indproVect and a Hopf pairing in that category.

*Proof.* Consequence of the propositions 4.8 and 4.10. □

So the algebraic dual $R^*$ of an algebra $R$ filtered by finite-dimensional subspaces is always a coalgebra in this setup, cofiltered by finite-dimensional quotient spaces. We can define a, say right, action of $R^*$ on $R$ as a filtered-cofiltered map and we then have the Heisenberg double $R^*\sharp R$ internal to the category indproVect. We then also have the right action of $R^*\sharp R$ on $R$, which is a map of filtered-cofiltered vector spaces.

## 4.6 Dual bases and canonical elements

**Proposition 4.13.** Let $\{D_\alpha\}_{\alpha\in A}$ be a filtered basis of a filtered vector space $V \cong \operatorname{colim} \boldsymbol{V}$ with finite-dimensional components, an object in the category $\mathrm{indVect}_f$. Then the set of dual functionals $\{e_\alpha\}_{\alpha\in A}$ is a cofiltered basis of the dual vector space $V^* \cong \lim \boldsymbol{V}^*$, an object in the category $\mathrm{proVect}_f$.

*Proof.* ... PROOF □

**Proposition 4.14.** Let $R$ be an internal Hopf algebra $R \cong \operatorname{colim} \boldsymbol{R}$ in the category $(\mathrm{indVect}_f, \otimes, k)$ with bijective antipode and $\{D_\alpha\}_{\alpha\in A}$ its filtered basis. Let $\{e_\alpha\}_{\alpha\in A}$ be the set of the corresponding dual functionals, hence a formal basis of the dual Hopf algebra $R^* \cong \lim \boldsymbol{R}^*$ which is internal to the category $(\mathrm{proVect}_f, \hat{\otimes}, k)$. Let $\phi\colon R \to R$ be a linear mapping. Then the canonical elements $\mathcal{K}(\phi)$ and $\mathcal{L}(\phi)$ defined as

$$\mathcal{K}(\phi) = \sum_{\alpha\in A} e_\alpha \otimes \phi(D_\alpha)$$

$$\mathcal{L}(\phi) = \sum_{\alpha,\beta\in A} e_\alpha S^{-1}(e_\beta) \otimes D_\beta \phi(D_\alpha)$$

are formal sums in the completed tensor product $R^* \hat{\otimes} R$ (which is equal to $R^* \tilde{\otimes} \bar{R}$, where $\bar{R}$ is $R$ with filtration forgotten) and as such define the elements of $R^* \hat{\otimes} R$ as their values.

*Proof.* ... PROOF □

The canonical elements $\mathcal{K}(\phi)$ and $\mathcal{L}(\phi)$ do not depend on the choice of the filtered basis $\{D_\alpha\}_{\alpha\in A}$.

**Theorem 4.15.** Let $A$ be an internal Hopf algebra $A \cong \operatorname{colim} \boldsymbol{A}$ in the category $(\mathrm{indVect}_f, \otimes, k)$ with bijective antipode. The canonical element mappings $\mathcal{K}$ and $\mathcal{L}$ are bijections of $\mathrm{Vect}(A, A)$ and $A^* \hat{\otimes} A$.

*Proof.* ... PROOF □

**Theorem 4.16.** Let $A$ be an internal Hopf algebra in $(\mathrm{indVect}_f, \otimes, k)$ with bijective antipode. For each $Y \in A$, define $\phi_Y \in \mathrm{Vect}(A, A)$ by $\phi_Y : X \mapsto YX$, and define

$$\mathrm{Lu}(Y) := \mathcal{L}(\phi_Y) = \sum_{\beta,\gamma\in B} e_\beta S^{-1}(e_\gamma) \otimes D_\gamma Y D_\beta.$$

Then the image of the mapping $\mathrm{Lu}\colon A \to A^* \hat{\otimes} A$ is inside $A^* \otimes A$ if and only if all adjoint orbits of $A$ are finite-dimensional. Furthermore, in that case the correstriction $\mathrm{Lu}\colon A \to A^* \sharp A$ is a filtered-cofiltered map and an antiisomorphism of algebras.

*Proof.* ... PROOF □

## 4.7 Coequalizers commute with the monoidal product

Theorems that follow use monomorphisms, subspace cofiltrations, completenes of kernel, quotient cofiltrations, quotient cofiltered maps and colimits in proVect.

**Theorem 4.17.** The category indproVect admits coequalizers.

*Proof.* ... PROOF □

**Theorem 4.18.** The coequalizers in the monoidal category $(\mathrm{indproVect}, \tilde{\otimes}, k)$ commute with the monoidal product.

*Proof.* ... PROOF □

Hence for each internal monoid $A$ in the symmetric monoidal category $(\mathrm{indproVect}, \tilde{\otimes}, k)$ the monoidal product $\tilde{\otimes}_A$ over the monoid $A$ can be defined, and the category of internal $A$-bimodules in the category indproVect is a monoidal category, $({}_A\mathrm{indproVect}_A, \tilde{\otimes}_A, A)$.

# 5 Internal algebraic structures

## 5.1 Internal Hopf algebroid

In the article [5], the definition of a Hopf algebroid internal to any symmetric monoidal category with coequalizers that commute with the monoidal product is given and the scalar extension theorem is proven.

**Theorem 5.1.** Let $(\mathcal{V}, \tilde{\otimes}, 1)$ be a symmetric monoidal category with coequalizers that commute with the monoidal product. Let $R$ be a braided-commutative right-left Yetter-Drinfeld module algebra over a Hopf algebra

$H$ with bijective antipode, all internal to category $(\mathcal{V}, \tilde{\otimes}, 1)$. Then the smash product $H\sharp R$ is an internal Hopf algebroid over $R$ in that category. Furthermore, it can be written as a smash product in several different ways:

$$H\sharp R \cong L\sharp H \cong H^{\mathrm{co}}\sharp L \cong R\sharp H^{\mathrm{co}},$$

where $L \cong R^{\mathrm{op}}$.

## 5.2 Heisenberg doubles of countably-dimensional Hopf algebras

In the article [6], these theorems are proven.

**Theorem 5.2.** Let $R$ be an internal Hopf algebra in $(\mathrm{indVect}_f, \otimes, k)$ with bijective antipode and finite-dimesional adjoint orbits. Then $R$ is a braided-commutative Yetter-Drinfeld module algebra over the internal Hopf algebra $R^*$ in $(\mathrm{proVect}, \hat{\otimes}, k)$ internal to the category $(\mathrm{indproVect}, \tilde{\otimes}, k)$ of filtered-cofiltered vector spaces.

**Theorem 5.3.** Let $R$ be an internal Hopf algebra in $(\mathrm{indVect}_f, \otimes, k)$ with bijective antipode and finite-dimesional adjoint orbits. Then the Heisenberg double $R^*\sharp R$ is a filtered-cofiltered vector space and has a structure of a Hopf algebroid over $R$ internal to the category $(\mathrm{indproVect}, \tilde{\otimes}, k)$ of filtered-cofiltered vector spaces.

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