University of Zagreb

Completed Hopf algebroids

Doctoral presentation

Martina Stojić

Faculty of Science
Department of Mathematics

October 20, 2017

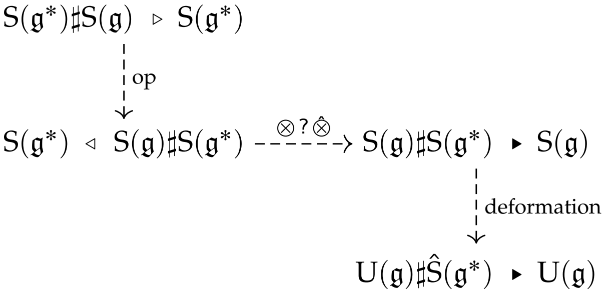
3. INTERNAL HOPF ALGEBROID AND SCALAR EXTENSION

4. HEISENBERG DOUBLES OF FILTERED HOPF ALGEBRAS AND GENERALIZATIONS

5. EXAMPLES

- Heisenberg double $\mathrm{U}(\mathfrak{g})^*\sharp\mathrm{U}(\mathfrak{g})$
- Noncommutative phase space $\mathrm{U}(\mathfrak{g})\sharp\hat{\mathrm{S}}(\mathfrak{g}^*)$
- Minimal scalar extension $\mathrm{U}(\mathfrak{g})^{min}\sharp\mathrm{U}(\mathfrak{g})$
- Reduced Heisenberg double $\mathrm{U}(\mathfrak{g})^\circ\sharp\mathrm{U}(\mathfrak{g})$
- Minimal algebra $\mathcal{O}^{min}(G)\sharp\mathrm{U}(\mathfrak{g})$ of differential operators
- Algebra $\mathcal{O}(\mathrm{Aut}(\mathfrak{g}))\sharp\mathrm{U}(\mathfrak{g})$
- Heisenberg double $U_q(\mathfrak{sl}_2)^*\sharp U_q(\mathfrak{sl}_2)$ when q is a root of unity

INTRODUCTION



1. INTRODUCTION

- Weyl algebra $S(\mathfrak{g}^*)\sharp S(\mathfrak{g})$
- Deformation of Weyl algebra
- Problems with Weyl algebra deformations
- Yetter-Drinfeld module algebra and Hopf algebroid
- Idea for the solution – the thesis

Weyl algebra $\mathrm{S}(\mathfrak{g}^*)\sharp\mathrm{S}(\mathfrak{g})$

Weyl algebra

$\cong$ $\langle x_1,\ldots,x_n,\partial_1,\ldots,\partial_n\rangle/I$, where the ideal I is generated by $\partial_\alpha x_\beta - x_\beta\partial_\alpha - \delta_{\alpha\beta},\ \alpha,\beta\in\{1,\ldots,n\}$

$\cong$ ring $\mathrm{Diff}(\mathbb{R}^n)\cong\left\{\sum_{I=0}^{K} p_I(x)\partial_I \mid K\in\mathbb{N}_0^n,\ p_I \text{ polynomials}\right\}$

$\cong$ smash product $\mathrm{S}(\mathfrak{g}^*)\sharp\mathrm{S}(\mathfrak{g})$, for $\mathfrak{g}=T_0V\cong V$ vector space $\mathrm{S}(\mathfrak{g}^*)\cong k[V^*]\cong k[x_1,\ldots,x_n]$, $\mathrm{S}(\mathfrak{g})=\mathrm{U}(\mathfrak{g})\cong k[\partial_1,\ldots,\partial_n]$

Smash product $\mathrm{S}(\mathfrak{g}^*)\sharp\mathrm{S}(\mathfrak{g})$ is $\mathrm{S}(\mathfrak{g}^*)\otimes\mathrm{S}(\mathfrak{g})$ with multiplication

- $f\sharp D\cdot g\sharp E=\sum f(D_{(1)}\triangleright g)\sharp D_{(2)}E$, where $D\triangleright f=Df$
- coproduct $\Delta(D)=\sum D_{(1)}\otimes D_{(2)}$, $D\in\mathrm{U}(\mathfrak{g})$, is defined with $\Delta(D)(f\otimes g)=D(f\cdot g)=\sum D_{(1)}f\cdot D_{(2)}g$ (Leibniz rule)

Dual Hopf algebras $\mathrm{S}(\mathfrak{g}^*)\cong k[V^*]$ and $\mathrm{S}(\mathfrak{g})=\mathrm{U}(\mathfrak{g})$

- product dual to coproduct, unit to counit, etc.

Deformation of Weyl algebra

Deformation $\rightsquigarrow$ noncommutative coordinates

- first $\mathrm{Diff}(\mathbb{R}^n)^{\mathrm{op}} \cong (\mathrm{S}(\mathfrak{g}^*)\sharp\mathrm{S}(\mathfrak{g}))^{\mathrm{op}} \cong \mathrm{S}(\mathfrak{g})\sharp\mathrm{S}(\mathfrak{g}^*)$ (geometry: algebra of diff. operators that act to the left $\triangleleft$)
- now $\mathrm{S}(\mathfrak{g}) \cong k[\hat{x}_1, \ldots, \hat{x}_n]$, $\mathrm{S}(\mathfrak{g}^*) \cong k[\hat{\partial}_1, \ldots, \hat{\partial}_n]$
- deformation: $\mathfrak{g}$ becomes a noncommutative Lie algebra, generated by $\hat{x}_1, \ldots, \hat{x}_n$, mod the ideal J generated by $[\hat{x}_\alpha, \hat{x}_\beta] - \sum_\sigma C^\sigma_{\alpha\beta}\hat{x}_\sigma$, $\alpha, \beta \in \{1, \ldots, n\}$
- $\mathrm{S}(\mathfrak{g})$ deforms as algebra to $\mathrm{U}(\mathfrak{g})$, $\mathrm{S}(\mathfrak{g}^*)$ deforms as coalgebra to a Hopf algebra dual to $\mathrm{U}(\mathfrak{g})$

Meljanac, Škoda, Stojić, *Lie algebra type noncommutative phase spaces are Hopf algebroids*, Lett. Math. Phys. 107:3, 475–503 (2017)

- $\mathrm{U}(\mathfrak{g})\sharp\hat{\mathrm{S}}(\mathfrak{g}^*)$ Why is it OK? Comparison with t-deformations.

Problems with Weyl algebra deformations

Problem with infinite dimensionality of $\mathrm{S}(\mathfrak{g})$ and $\mathrm{U}(\mathfrak{g})$:

- coproduct $\Delta\colon \mathrm{S}(\mathfrak{g}^*) \to \mathrm{S}(\mathfrak{g}^*) \otimes \mathrm{S}(\mathfrak{g}^*)$ deforms to coproduct $\mathrm{S}(\mathfrak{g}^*) \to \mathrm{S}(\mathfrak{g}^*) \hat{\otimes} \mathrm{S}(\mathfrak{g}^*)$ with completion

One possible solution:

- coproduct $\hat{\Delta}\colon \hat{\mathrm{S}}(\mathfrak{g}^*) \to \hat{\mathrm{S}}(\mathfrak{g}^*) \hat{\otimes} \hat{\mathrm{S}}(\mathfrak{g}^*)$ and smash product $\mathrm{U}(\mathfrak{g})\sharp\hat{\mathrm{S}}(\mathfrak{g}^*)$ defined out of the action $\hat{\mathrm{S}}(\mathfrak{g}^*) \triangleleft \mathrm{U}(\mathfrak{g})$

Problem: combining $\otimes$ and $\hat{\otimes}$

- we need to work with the 'action' of the deformed differential operators $\hat{\mathrm{S}}(\mathfrak{g}^*) \blacktriangleright \mathrm{U}(\mathfrak{g})$ there are no axioms for
- there is no definition of a 'completed' Hopf algebroid

Solution *ad hoc* in the LMP article:

- 'action' without axioms... infinite sums... coordinates...
- unstable definition of a 'completed' Hopf algebroid...

Yetter-Drinfeld module algebra and Hopf algebroid

Algebra of formal diff. operators around the unit of a Lie group:

$$\mathrm{Diff}^{\omega}(G,e) \cong J^{\infty}(G,e)\sharp \mathrm{U}(\mathfrak{g}^L) \cong \mathrm{U}(\mathfrak{g}^L)^*\sharp \mathrm{U}(\mathfrak{g}^L)$$
$$\mathrm{Diff}^{\omega}(G,e) \cong J^{\infty}(G,e)^{\mathrm{co}}\sharp \mathrm{U}(\mathfrak{g}^R) \cong \mathrm{U}(\mathfrak{g}^R)^*\sharp \mathrm{U}(\mathfrak{g}^R)$$

Noncommutative phase space is the opposite algebra:

$$\mathrm{U}(\mathfrak{g}^L)\sharp \hat{\mathrm{S}}(\mathfrak{g}^*) \cong (\hat{\mathrm{S}}(\mathfrak{g}^*)^{\mathrm{co}}\sharp \mathrm{U}(\mathfrak{g}^R))^{\mathrm{op}} \cong \mathrm{Diff}^{\omega}(G,e)^{\mathrm{op}}$$
$$\hat{\mathrm{S}}(\mathfrak{g}^*) \cong J^{\infty}(G,e) \cong \mathrm{U}(\mathfrak{g}^L)^*$$

'Completed' Heisenberg double $\mathrm{U}(\mathfrak{g})^*\sharp \mathrm{U}(\mathfrak{g})$?

Corrolary. (Lu)

- If A is a finite-dimensional Hopf algebra, then the Heisenberg double $A^*\sharp A$ is a Hopf algebroid over A.

Theorem. (Brzeziński, Militaru) Scalar extension.

- If A is a braided-commutative Yetter-Drinfeld module algebra over H, then the smash product $H\sharp A$ is a Hopf algebroid over A.

Idea for the solution – the thesis

1. Category
   - new category which has vector spaces with filtrations and vector spaces with cofiltrations, and $\mathrm{U}(\mathfrak{g})^*\sharp\mathrm{U}(\mathfrak{g})$
   - it has to have a monoidal product $\tilde{\otimes}$ which is equal to $\otimes$ when vector spaces are filtered and $\hat{\otimes}$ when cofiltered
   - it has to admit coequalizers and they have to commute with the monoidal product for the definition of $\tilde{\otimes}_A$ to be possible
2. Definition of an internal Hopf algebroid
   - based on the definition of internal bialgebroid of Gabi Böhm
3. The scalar extension theorem
   - simetrical definition, antipod antiisomorphism, geometry
4. Proof that $\mathrm{U}(\mathfrak{g})$ is an internal braided-commutative YD-module algebra over $\mathrm{U}(\mathfrak{g})^*$
5. What can be more generally known about $A^*\sharp A$ and $H\sharp A$ for dual infinite-dimensional H and A?

THE CATEGORY indproVect

$$(\mathrm{Vect}, \otimes, k) \hookrightarrow (\mathrm{indVect}, \otimes, k) \hookrightarrow (\mathrm{indproVect}, \tilde{\otimes}, k)$$

$$(\mathrm{Vect}, \otimes, k) \hookrightarrow (\mathrm{proVect}, \hat{\otimes}, k) \hookrightarrow (\mathrm{indproVect}, \tilde{\otimes}, k)$$

2. THE CATEGORY indproVect

- Requirements, intuition and strategy
- Categories indVect and proVect
- Dual subcategories of Grothendieck's categories
- The category indproVect
- Tensor products, formal sums and formal basis
- Commutation of the tensor product and coequalizers

Requirements, intuition and strategy

Vector spaces with structure and tensor products

1. A, B 'filtered' vector spaces $\Rightarrow A \otimes B = \operatorname{colim}_{n,m} A_n \otimes B_m$
   H, K 'cofiltered' vector spaces $\Rightarrow H \hat{\otimes} K = \lim_{k,l} H_k \otimes K_l$
2. Filtering components $A_n \hookrightarrow A$ are subspaces, duality $\Rightarrow$ cofiltering components $H \twoheadrightarrow H_k$ are quotients
3. A fin-dim-filtered, H fin-dim-cofiltered $\Rightarrow A \tilde{\otimes} H = A \otimes H$
   $\Rightarrow$ let's try with $A \tilde{\otimes} H = \operatorname{colim}_n \lim_k A_n \otimes H_k = \operatorname{colim}_n A_n \hat{\otimes} H$
   $\Rightarrow$ 'filtered-cofiltered' vector space $V = \operatorname{colim}_n \lim_k V_n^k$
4. Hopefully this is a symmetric monoidal category.
5. Hopefully it admits coequalizers and the monoidal product $\tilde{\otimes}$ commutes with them.

Let's name these categories: indVect, proVect and indproVect.

Requirements, intuition and strategy

Morphisms that respect this structure

6. Multiplication $A \otimes A \to A$, comultiplication $H \to H \hat{\otimes} H$ and action $\blacktriangleright: H \otimes A \to A$ should be morphisms in this category.
7. Axiom of action:
$$(H \hat{\otimes} H) \otimes A \to H \otimes A \to A \text{ i } H \hat{\otimes} (H \otimes A) \to H \otimes A \to A$$
$$\text{become } H \tilde{\otimes} H \tilde{\otimes} A \to H \tilde{\otimes} A \to A.$$
8. When cofiltered algebra H 'acts on' filtered vector space A,
$$\left(\sum_{\alpha \in \mathbb{N}_0^n} a_\alpha \hat{\partial}_1^{\alpha_1} \hat{\partial}_2^{\alpha_2} \cdots \hat{\partial}_n^{\alpha_n} \right) \blacktriangleright \hat{x}_1^{\beta_1} \hat{x}_2^{\beta_2} \cdots \hat{x}_n^{\beta_n}$$
the result is always a finite sum
$$\bar{\sum}_{\alpha \in \mathbb{N}_0^n} a_\alpha \left(\hat{\partial}_1^{\alpha_1} \hat{\partial}_2^{\alpha_2} \cdots \hat{\partial}_n^{\alpha_n} \blacktriangleright \hat{x}_1^{\beta_1} \hat{x}_2^{\beta_2} \cdots \hat{x}_n^{\beta_n} \right)$$
even though each summand of the infinite sum acts.
$\Rightarrow$ Infinitness is controlled by interaction of filtrations and cofiltrations. Formalization: morphisms in indproVect.

Objects of categories $\mathrm{indVect}$ and $\mathrm{proVect}$

Definition. A functor $\boldsymbol{V}\colon I \to \mathcal{V}$ is an $\aleph_0$-*filtration* if I is a small directed category of cofinality of at most $\aleph_0$ and all connecting morphisms are monomorphisms.

Definition. A functor $\boldsymbol{V}\colon I \overset{\mathrm{op}}{\to} \mathcal{V}$ is an $\aleph_0$-*cofiltration* if I is a small directed category of cofinality of at most $\aleph_0$ and all connecting morphisms are epimorphisms.

- natural generalizations of standard notions of filtration and decreasing filtration (equivalently, cofiltration)
- objects of categories $\mathrm{Ind}^{s}_{\aleph_0}\mathcal{V}$ and $\mathrm{Pro}^{s}_{\aleph_0}\mathcal{V}$ respectively
- Why $\aleph_0$? Why monomorphisms and epimorphisms?

Definition. *Filtered* (resp. *cofiltered*) *vector space* is a vector space V together with an $\aleph_0$-filtration (resp. $\aleph_0$-cofiltration) $\boldsymbol{V}$ in Vect such that $V \cong \operatorname{colim} \boldsymbol{V}$ (resp. $V \cong \lim \boldsymbol{V}$) in Vect.

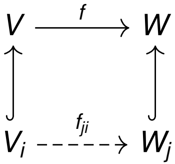
Morphisms of categories indVect and proVect

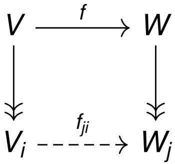
Definition. *Morphism of filtered vector spaces*, or a *filtered map*, from $V \cong \operatorname{colim} \boldsymbol{V}$ to $W \cong \operatorname{colim} \boldsymbol{W}$ is a linear map $f\colon V \to W$ such that

$$(\forall i \in I)(\exists j \in J)(\exists f_{ji}\colon V_i \to W_j)(f \circ \iota_i^V = \iota_j^W \circ f_{ji}).$$

Definition. *Morphism of cofiltered vector spaces*, or a *cofiltered map*, from $V \cong \lim \boldsymbol{V}$ to $W \cong \lim \boldsymbol{W}$ is a linear map $f\colon V \to W$ such that

$$(\forall j \in J)(\exists i \in I)(\exists f_{ji}\colon V_i \to W_j)(f_{ji} \circ \pi_i^V = \pi_j^W \circ f).$$





Dual subcategories of Grothendieck categories

Theorems.

- The category $\mathrm{indVect}$ is equivalent to the category of strict ind-objects of cofinality of at most $\aleph_0$ in the category Vect,

$$\mathrm{indVect} \cong \mathrm{Ind}^{s}_{\aleph_0}\mathrm{Vect}.$$

- The category $\mathrm{proVect}$ is equivalent to the category of strict pro-objects of cofinality of at most $\aleph_0$ in the category Vect,

$$\mathrm{proVect} \cong \mathrm{Pro}^{s}_{\aleph_0}\mathrm{Vect}.$$

- The categories $\mathrm{indVect}$ and $\mathrm{proVect}$ are dual to each other.

$\leftsquigarrow$ These theorems are not theorems if:

  - we don't have monomorphisms $A_n \hookrightarrow A$ in filtrations and epimorphisms $H \twoheadrightarrow H_k$ in cofiltrations.
  - the cofinality is not at most $\aleph_0$. But maybe it still works without $\aleph_0$-assumption if we take $\mathrm{VectFin}$ instead of Vect.

The category indproVect

Definition. *Filtered-cofiltered vector space* is a vector space V together with an $\aleph_0$-filtration $\boldsymbol{V}$ in proVect such that $V \cong \operatorname{colim} \boldsymbol{V}$ in Vect.

⇜ Well def.: **Proposition.** mono in proVect ⇒ mono in Vect.

Definition. *Morphism of filtered-cofiltered vector spaces*, or a *filtered-cofiltered map*, is a linear map $f\colon V \to W$ such that

$$(\forall i \in I)(\exists j \in J)(\exists f_{ji}\colon V_i \to W_j \text{ in proVect})(f \circ \iota_i^V = \iota_j^W \circ f_{ji}).$$

Theorem. (Subcategory of Grothendieck's category.)

- The category indproVect is equivalent to the category of strict ind-pro-objects of cofinality of at most $\aleph_0$ in the category Vect,

$$\mathrm{indproVect} \cong \mathrm{Ind}^s_{\aleph_0}\mathrm{Pro}^s_{\aleph_0}\mathrm{Vect}.$$

⇜ Later, deeper reasons: **Theorem.** proVect has colimits and **Theorem.** Filtered colimit in proVect is colimit in Vect.

Tensor products, formal sums and formal basis

Tensor product is easily defined by lifting it

- to $\mathrm{indVect}$ from filtrations: $V \otimes W = \mathrm{colim}\, \boldsymbol{V} \otimes \boldsymbol{W}$
- to $\mathrm{proVect}$ from cofiltrations: $V \hat{\otimes} W = \lim \boldsymbol{V} \otimes \boldsymbol{W}$
- to $\mathrm{indproVect}$ from filtrations of cofiltrations: $V \tilde{\otimes} W = \mathrm{colim}\, \boldsymbol{V} \hat{\otimes} \boldsymbol{W}$.

Advantage over abstract ind-pro-objects: concrete categories.
♪ Formal sums in $\mathrm{proVect}$. ♫ Formal basis in $\mathrm{proVect}$.

Propositions.

- Categories $(\mathrm{indVectFin}, \otimes, k)$ and $(\mathrm{proVectFin}, \hat{\otimes}, k)$ are dual.
- Morphisms in $\mathrm{proVect}$ = ones that distribute over formal sums.
- If $\{D_\alpha\}_\alpha$ is a filtered basis of V in $\mathrm{indVectFin} \rightsquigarrow$ dual functionals $\{e_\alpha\}_\alpha$ comprise a formal basis of V^* in $\mathrm{proVectFin}$.

Examples. $\mathrm{U}(\mathfrak{g})^* \sharp \mathrm{U}(\mathfrak{g})$, Heisenberg doubles of Hopf algebras filtered by finite-dimensional components, ...

Commutation of the tensor product and coequalizers

Proposition 1. (Coequalizers in proVect.)

- The category proVect admits coequalizers. The coequalizers in $(\mathrm{proVect}, \hat{\otimes}, k)$ commute with the monoidal product.

Proposition 2. (Complete subspaces and quotients in proVect.)

- Vector subspace is complete if and only if it contains values of all formal sums. Quotient by a complete subspace is a cofiltered vector space and the quotient map is a cofiltered map.

Proposition 3. (Coproduct in proVect.)

- The category proVect has coproducts. Description of coproduct in proVect.

Sketch of the proof. Existence: use equivalence with the category $\mathrm{Pro}^{s}_{\aleph_0}\mathrm{Vect}$. Description: use the notion of formal sum, Proposition 2. about formal sums and completions, and Proposition 3. for description of cofiltration on quotients.

Theorem 4. (Filtered colimit in proVect.)

▸ The category proVect has colimits. Description of filtered colimit in proVect.

Sketch of the proof. Complex proof. ...

Theorem 5. (Existence of coequalizers in indproVect.)

▸ The category indproVect admits coequalizers.

Sketch of the proof. Use: existence of colimit in proVect, quotient maps by complete subspaces in proVect, completeness of kernels of maps in proVect, ...

Theorem 6. (Coequalizers commute with $\tilde{\otimes}$ in indproVect.)

▸ Coequalizers in $(\mathrm{indproVect}, \tilde{\otimes}, k)$ commute with the monoidal product.

Sketch of the proof. Complex proof. ... Uses: properties of formal sums, description of filtered colimit in proVect, ...

This makes the definition of $\tilde{\otimes}_A$, for internal monoid A, possible.

Internal Hopf algebroid and scalar extension

$$\mathcal{G} \times_M \mathcal{G} \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; \overset{\circlearrowright}{\mathcal{G}} \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; M$$

$$\mathrm{Fun}(\mathcal{G} \times_M \mathcal{G}) \cong \mathrm{Fun}(\mathcal{G}) \otimes_{\mathrm{Fun}(M)} \mathrm{Fun}(\mathcal{G}) = \mathcal{H} \otimes_A \mathcal{H}$$

$$\mathcal{H} \otimes_A \mathcal{H} \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \overset{\circlearrowright}{\mathcal{H}} \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; A$$

Hopf algebroids, motivation and definition

Functions on a group G = (commutative) Hopf algebra H, functions on a groupoid $\mathcal{G}$ = (commutative) Hopf algebroid $\mathcal{H}$.

General Hopf algebras and Hopf algebroids = functions on spaces with noncommutative coordinates (with the structure of a 'group' or a 'groupoid') = quantum group, quantum groupoid.

It is more complicated:

- Coproduct $\Delta\colon H \to H \otimes H$ becomes coproduct $\Delta\colon \mathcal{H} \to \mathcal{H} \otimes_A \mathcal{H}$. If A is noncommutative, $\mathcal{H} \otimes_A \mathcal{H}$ is not an algebra – hence there is a problem with the definition of multiplicativity of coproduct... Takeuchi product.
- Unit $\eta\colon k \to H$ becomes left unit $\alpha\colon A \to \mathcal{H}$ and right unit $\beta\colon A \to \mathcal{H}$.
- Much more complexity in axioms: left bialgebroid, right bialgebroid and antipode. Lu, Day & Street, ... , Böhm.

Internal bialgebroid of Gabriella Böhm

Modern definition of a Hopf algebroid: Gabriella Böhm, Handbook of Algebra.

bialgebra H over k	(left) bialgebroid $\mathcal{H}$ over A
$\mu\colon H \otimes H \to H$	$\mu\colon \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$
$\eta\colon k \to H$	$\alpha\colon A \to \mathcal{H},\ \beta\colon A^{\mathrm{op}} \to \mathcal{H}$
$\Delta\colon H \to H \otimes H$	$\Delta\colon \mathcal{H} \to \mathcal{H} \otimes_A \mathcal{H}$
$\epsilon\colon H \to k$	$\epsilon\colon \mathcal{H} \to A$
Hopf algebra over k	Hopf algebroid $\mathcal{H}$ over A
$(H, \mu, \eta, \Delta, \epsilon)$	$\mathcal{H}_L = (\mathcal{H}, \mu, \alpha_L, \beta_L, \Delta_L, \epsilon_L)$
together with	$\mathcal{H}_R = (\mathcal{H}, \mu, \alpha_R, \beta_R, \Delta_R, \epsilon_R)$
$S\colon H^{\mathrm{op}} \to H$	$\mathcal{S}\colon \mathcal{H}^{\mathrm{op}} \to \mathcal{H}$

In a symmetric monoidal categoy with coequalizers that commute with the monoidal product Gabriella Böhm defines an internal bialgebroid. Complication: Takeuchi product – she replaces it with actions ρ and λ. No elements – only diagrams.

Definition of internal Hopf algebroid

First work out Gabi's definition of internal left bialgebroid and internal right bialgebroid, with actions ρ and λ.

$$(H \otimes_L H) \otimes (H \otimes H) \overset{\rho}{\dashrightarrow} (H \otimes_L H)$$

$$\begin{array}{ccc} H \otimes (H \otimes H) & \xrightarrow{\mathrm{id} \otimes \pi} & H \otimes (H \otimes_L H) \\ \Big\downarrow{\scriptstyle \Delta \otimes \mathrm{id}} & & \Big\downarrow{\scriptstyle \lambda} \\ (H \otimes_L H) \otimes (H \otimes H) & \xrightarrow{\rho} & (H \otimes_L H) \end{array}$$

Definition. Internal Hopf algebroid $(\mathcal{H}_L, \mathcal{H}_R, \mathcal{S})$ in a symmetric monoidal category which admits coequalizers that commute with the monoidal product.

These properties of coproducts Δ_L and Δ_R were needed for it to be well defined.

Proposition. Δ_L is an R-bimodule map, with ${}_R\mathcal{H}_R$. Δ_R too, ${}_L\mathcal{H}_L$.

Scalar extensions of Lu, Brzeziński and Militaru

Theorem. (Lu) Quantum transformation groupoid.

- If A is a braided-commutative module algebra over Drinfeld double $\mathcal{D}(H)$ of a finite-dimensional Hopf algebra H, then $H\sharp A$ is a Hopf algebroid over A.

About the proof. Lu's definition of Hopf algebroid. Finite dimensionality. Uses canonical elements: $\{a_s\}$ basis of A, $\{x_s\}$ dual basis of A^*, $\beta(a) = \sum_t x_t S^{-1}(x_s) \otimes a_s a a_t$.

Theorem. (Brzeziński, Militaru) Scalar extension.

- If A is a braided-commutative YD-module algebra over Hopf algebra H, then $H\sharp A$ is a Hopf algebroid over A.

About the proof. Lu's definition of Hopf algebroid. Omission in the proof: it is not proved antipode is an antihomomorphism. It is not clear to me whether the proof can be completed without the additional assumption of bijectivity of antipode $S\colon A^{op} \to A$.

Internal scalar extension theorem

Theorem. (Internal scalar extension in indproVect.)

- If A is a braided-commutative YD-module algebra over Hopf algebra H with bijective antipode, then $H\sharp A$ is a Hopf algebroid over A.

Sketch of the proof. Geometry: for $\mathrm{U}(\mathfrak{g}^R)^* = J^\infty(G,e)^{\mathrm{co}} =: H$,

$$H\sharp\mathrm{U}(\mathfrak{g}^R) \cong \mathrm{U}(\mathfrak{g}^L)\sharp H \cong \mathrm{Diff}^\omega(G,e).$$

- $\mathcal{H}_L = L\sharp H$ is a pretty left bialgebroid over L, $\mathrm{U}(\mathfrak{g}^L)\sharp H$, $\mathcal{H}_R = H\sharp R$ is a pretty right bialgebroid over R, $H\sharp\mathrm{U}(\mathfrak{g}^R)$,
- isomorphism of algebras $\Phi\colon \mathcal{H}_L \to \mathcal{H}_R$, formula is extracted from the geometrical example,
- antipode $S\colon \mathcal{H} \to \mathcal{H}$ is an antihomomorphism (complete the proof of B-M: hard, use isomorphism Φ: easy).

Abstract Sweedler notation. The proof works for any symmetric monoidal category with the needed coequalizers property.

HEISENBERG DOUBLES OF FILTERED HOPF ALGEBRAS AND GENERALIZATIONS

Here we study pairings $A\tilde{\otimes}H \to k$ which are non-degenerate in variable in H, hence by which $H \hookrightarrow A^*$. The question is:

When is A over H a braided-commutative YD-module algebra in $\mathrm{indproVect}$, and hence $H\sharp A$ a Hopf algebroid over A?

Is $\mathrm{U}(\mathfrak{g})$ over $\mathrm{U}(\mathfrak{g})^*$ a braided-commutative YD-module algebra in $\mathrm{indproVect}$?

4. HEISENBERG DOUBLES OF FILTERED HOPF ALGEBRAS AND GENERALIZATIONS

- Canonical elements and representations
- Theorem about Yetter-Drinfeld module algebra
- Theorem with canonical elements for A in $\mathrm{indVectFin}$
- Theorem with anihilators for A in $\mathrm{indVect}$ and H in $\mathrm{proVect}$

Canonical elements and representations

♪ Action of $H\tilde{\otimes} A$ on A (from the right).

$$\mathcal{S}_1 : H\tilde{\otimes} A \to \mathrm{Hom}(A, A)$$

$$\mathcal{S}_1(\sum_\lambda h_\lambda \otimes a_\lambda) : b \mapsto \sum_\lambda \langle b, h_\lambda \rangle a_\lambda$$

Canonical element $\mathcal{K}$ is such that $\hat{\mathcal{S}}_1 \circ \mathcal{K} = \mathrm{id}_{\mathrm{Hom}(A,A)}$.

$$\mathcal{K} : \mathrm{Hom}(A, A) \to A^* \hat{\otimes} A$$

$$\mathcal{K}(\phi) = \sum_\alpha e_\alpha \sharp \phi(x_\alpha)$$

♫ Action $H\sharp A$ on A (from the right).

$$\mathcal{T}_1 : H\tilde{\otimes} A \to \mathrm{Hom}(A, A)$$

$$\mathcal{T}_1(\sum_\lambda h_\lambda \otimes a_\lambda) : b \mapsto \sum_\lambda (b \blacktriangleleft h_\lambda) a_\lambda$$

Canonical element $\mathcal{L}$ is such that $\hat{\mathcal{T}}_1 \circ \mathcal{L} = \mathrm{id}_{\mathrm{Hom}(A,A)}$.

$$\mathcal{L} : \mathrm{Hom}(A, A) \to A^* \hat{\otimes} A$$

$$\mathcal{L}(\phi) = \sum_{\alpha,\beta} e_\beta S^{-1}(e_\alpha) \sharp x_\alpha \phi(x_\beta)$$

For $\mathcal{L}(\phi)$, Hopf algebra A has to have a bijective antipode S.

Theorem about Yetter-Drinfeld module algebra

Theorem 1. (About Yetter-Drinfeld module algebra.)

- Let A and H be in Hopf pairing in indproVect which is non-degenerated in variable in H. Assume $\mathcal{T}_2$ is injective,

$$\mathcal{T}_2\colon H\tilde{\otimes} H\tilde{\otimes} A\to \mathrm{Hom}(A\tilde{\otimes} A, A)$$

$$\mathcal{T}_2(\textstyle\sum_\lambda h_\lambda\otimes h'_\lambda\otimes a_\lambda)\colon b\otimes b'\mapsto \sum_\lambda (b\blacktriangleleft h_\lambda)(b'\blacktriangleleft h'_\lambda)a_\lambda$$

Then A is over H a braided-commutative YD-module algebra (with action defined from pairing) if and only if there exists a morphism $\rho\colon A\to H\tilde{\otimes} A$ such that $x\blacktriangleleft \rho(a) = ax$.

Sketch of the proof. Then $\mathcal{T}_1$ is also injective. By acting with the left and the right side of axiom equation on an element of $A\tilde{\otimes} A$, or A, we prove the equation. To prove that ρ is a coaction, we use $\mathcal{T}_2$, and to prove the YD-condition and that A is an algebra, we use $\mathcal{T}_1$. For example, YD-condition is in $H\tilde{\otimes} A$,

$$\sum h_{(2)}(a\blacktriangleleft h_{(1)})_{[-1]}\otimes (a\blacktriangleleft h_{(1)})_{[0]} = \sum a_{[-1]}h_{(1)}\otimes (a_{[0]}\blacktriangleleft h_{(2)}).$$

Theorem with canonical elements for A in $\mathrm{indVectFin}$

$$\mathcal{K}(\phi) = \sum_\alpha e_\alpha \otimes \phi(x_\alpha)$$
$$\mathcal{L}(\phi) = \sum_{\alpha,\beta} e_\beta S^{-1}(e_\alpha) \otimes x_\alpha \phi(x_\beta)$$

For $\phi_a \colon x \mapsto ax$, we know $x \,\hat{\blacktriangleleft}\, \mathcal{L}(\phi_a) = ax$. Put $\mathrm{Lu}(a) := \mathcal{L}(\phi_a)$.

Is $\mathcal{T}_2$ injective and when is $\mathrm{Lu}(A) \subseteq A^* \sharp A$?

Theorem 2. (Heisenberg double of A from $\mathrm{indVectFin}$.)
- Let Hopf algebra A in $\mathrm{indVectFin}$ have a bijective antipode.

Then A is over A^* a braided-commutative YD-module algebra (with action defined from pairing) if and only if the adjoint orbits of A are finite-dimensional.

Sketch of the proof. $\hat{\mathcal{S}}_1 \circ \mathcal{K} = \mathrm{id}$ and $\hat{\mathcal{T}}_1 \circ \mathcal{L} = \mathrm{id}$. Propositions.
- $\mathcal{S}_0$ injective $\Rightarrow \mathcal{S}_1, \mathcal{S}_2$ injective. Similarly, $\hat{\mathcal{S}}_1, \hat{\mathcal{S}}_2$ are injective.
- $\hat{\mathcal{S}}_1 \circ \mathcal{L} \colon \phi_a \mapsto \mathrm{ad}_{S^{-1}(a)}$, for $\phi_a \colon x \mapsto ax$. It follows: $\hat{\mathcal{S}}_1$ bijective.

$\mathcal{L}, \mathcal{M}$ bijective, hence $\mathcal{T}_1, \mathcal{T}_2$ injective. From ad, adjoint orbits.

Theorem with canonical elements for A in $\mathrm{indVectFin}$

Theorem 3. (For A in $\mathrm{indVectFin}$ and H in $\mathrm{proVect}$.)

- Let A with a bijective antipode in $\mathrm{indVectFin}$ and H in $\mathrm{proVect}$ be in Hopf pairing in $\mathrm{indproVect}$ which is non-degenerate in variable in H. Then A is over H a braided-commutative YD-module algebra (with action defined from pairing) if and only if $\mathrm{Lu}(A) \subseteq H\sharp A$.

Consequence. If $\mathrm{Lu}(A) \subseteq A^*\sharp A$, then there exists the smallest Hopf subalgebra $A^{min} \subseteq A^*$ which has all functionals needed for $\mathrm{Lu}(A) \subseteq A^{min}\sharp A$. The theorem is then true for all H such that $A^{min} \subseteq H$.

- For $\mathrm{U}(\mathfrak{g})$, we found this minimal subalgebra explicitly. Examples $\mathrm{U}(\mathfrak{g})^{min}\sharp\mathrm{U}(\mathfrak{g})$, $\mathrm{U}(\mathfrak{g})^{\circ}\sharp\mathrm{U}(\mathfrak{g})$, $\mathrm{U}(\mathfrak{g})^*\sharp\mathrm{U}(\mathfrak{g})$.
- For $U_q(\mathfrak{sl}_2)$, we didn't, but maybe it is possible with better knowledge of q-binomial coefficients. $U_q(\mathfrak{sl}_2)^*\sharp U_q(\mathfrak{sl}_2)$

Theorem with anihilators for A in indVect, H in proVect

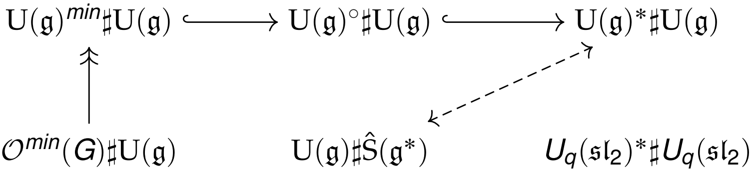
Theorem 4. (For A in indVect and H in proVect.)

▸ Let A in indVect and H in proVect be in Hopf pairing in indproVect which is non-degenerate in variable in H. Assume that Δ_A satisfies $\Delta_A(a) - a \otimes 1 \in A_{n-1} \otimes A$ for $a \in A_n$ and $A_0 \cong k$. Then A is over H a braided-commutative YD-module algebra (with action induced by pairing) if and only if there exists a morphism $\rho\colon A \to H\sharp A$ such that $x \blacktriangleleft \rho(a) = ax$.

Sketch of the proof. Prove $\mathcal{T}_2$ is injective, but without canonical elements. For $t \in H \hat{\otimes} H \hat{\otimes} A_n$, $t \neq 0$, let (k, l) be minimal such that there exists $d \in A_k \hat{\otimes} A_l$ for which $\mathcal{S}_2(t)(d) \neq 0$.

$$\begin{array}{ccc}
A_k \hat{\otimes} A_l \hat{\otimes} H \hat{\otimes} H \hat{\otimes} A_n & \xrightarrow{(\mathcal{T}_2)} & A_m \\
\big\downarrow & & \big\| \\
A_k \hat{\otimes} A_l \hat{\otimes} H/\operatorname{Anih}(A_k) \hat{\otimes} H/\operatorname{Anih}(A_l) \hat{\otimes} A_n & \xrightarrow{(\mathcal{T}_2 \sim \mathcal{S}_2)} & A_m
\end{array}$$

Examples



5. Examples

- Heisenberg double $\mathrm{U}(\mathfrak{g})^{*}\sharp\mathrm{U}(\mathfrak{g})$
- Noncommutative phase space $\mathrm{U}(\mathfrak{g})\sharp\hat{\mathrm{S}}(\mathfrak{g}^*)$
- Minimal scalar extension $\mathrm{U}(\mathfrak{g})^{min}\sharp\mathrm{U}(\mathfrak{g})$
- Reduced Heisenberg double $\mathrm{U}(\mathfrak{g})^{\circ}\sharp\mathrm{U}(\mathfrak{g})$
- Minimal algebra $\mathcal{O}^{min}(G)\sharp\mathrm{U}(\mathfrak{g})$ of differential operators
- Algebra $\mathcal{O}(\mathrm{Aut}(\mathfrak{g}))\sharp\mathrm{U}(\mathfrak{g})$
- Heisenberg double $U_q(\mathfrak{sl}_2)^*\sharp U_q(\mathfrak{sl}_2)$ when q is a root of unity

Heisenberg double $\mathrm{U}(\mathfrak{g})^*\sharp\mathrm{U}(\mathfrak{g})$

Proposition. Adjoint orbits are finite-dimensional.

By Theorem 2 & The Internal Scalar Extension Theorem: $\mathrm{U}(\mathfrak{g})^*\sharp\mathrm{U}(\mathfrak{g})$ is a Hopf algebroid over $\mathrm{U}(\mathfrak{g})$ in $\mathrm{indproVect}$.

It is the algebra of formal differential operators around the unit of a Lie group G:

$$\mathrm{U}(\mathfrak{g}^R)^*\sharp\mathrm{U}(\mathfrak{g}^R) \cong J^\infty(G,e)^{\mathrm{co}}\sharp\mathrm{U}(\mathfrak{g}^R) \cong \mathrm{Diff}^\omega(G,e)$$

$$H\sharp\mathrm{U}(\mathfrak{g}^R) \cong \mathrm{U}(\mathfrak{g}^L)\sharp H \cong \mathrm{Diff}^\omega(G,e)$$

Let $X_1,\ldots,X_n$ be a basis of $\mathfrak{g}^L$, let $Y_1,\ldots,Y_n$ in $\mathfrak{g}^R$ be such that $(Y_\alpha)_e = (X_\alpha)_e$. Then, for $L = \mathrm{U}(\mathfrak{g}^L)$ and $R = \mathrm{U}(\mathfrak{g}^R)$, we have

$$\alpha_L(X_\alpha) = X_\alpha \qquad \alpha_R(Y_\alpha) = Y_\alpha$$

$$\beta_L(X_\alpha) = Y_\alpha + \textstyle\sum_\beta C^\beta_{\alpha\beta} \qquad \beta_R(Y_\alpha) = X_\alpha = \textstyle\sum_\beta \mathcal{O}^\beta_\alpha\sharp Y_\beta$$

$$\mathcal{S}(X_\alpha) = Y_\alpha,\ \ \mathcal{S}(f) = Sf$$

Noncommutative phase space $\mathrm{U}(\mathfrak{g})\sharp\hat{\mathrm{S}}(\mathfrak{g}^*)$

It is the algebra opposite to algebra $\mathrm{Diff}^\omega(G,e)$:

$$\mathrm{U}(\mathfrak{g}^L)\sharp\hat{\mathrm{S}}(\mathfrak{g}^*) \cong (\hat{\mathrm{S}}(\mathfrak{g}^*)^{\mathrm{co}}\sharp\mathrm{U}(\mathfrak{g}^R))^{\mathrm{op}} \cong \mathrm{Diff}^\omega(G,e)^{\mathrm{op}}$$

$$\mathrm{U}(\mathfrak{g}^L)\sharp\hat{\mathrm{S}}(\mathfrak{g}^*) \cong \hat{\mathrm{S}}(\mathfrak{g}^*)\sharp\mathrm{U}(\mathfrak{g}^R)$$

Let $\hat{x}_1,\ldots,\hat{x}_n$ be a basis of $\mathfrak{g}^L$, let $\hat{y}_1,\ldots,\hat{y}_n$ be in $\mathfrak{g}^R$ such that $(\hat{y}_\alpha)_e = (\hat{x}_\alpha)_e$. These are noncommutative coordinates, and

$$\hat{\mathrm{S}}(\mathfrak{g}^*) \cong k[[\hat{\partial}_1,\ldots,\hat{\partial}_n]]$$

has a coproduct dual to product on $\mathrm{U}(\mathfrak{g}^L) = L$. With $\mathrm{U}(\mathfrak{g}^R) = R$,

$$\begin{aligned} \alpha_L(\hat{x}_\alpha) &= \hat{x}_\alpha & \alpha_R(\hat{y}_\alpha) &= \hat{y}_\alpha \\ \beta_L(\hat{x}_\alpha) = \hat{y}_\alpha &= \textstyle\sum_\beta \hat{x}_\beta\sharp\mathcal{O}^\beta_\alpha & \beta_R(\hat{y}_\alpha) &= \hat{x}_\alpha - \textstyle\sum_\beta C^\beta_{\alpha\beta} \\ \mathcal{S}(\hat{y}_\alpha) &= \hat{x}_\alpha & \mathcal{S}(\hat{\partial}_\alpha) &= -\hat{\partial}_\alpha \\ \hat{\partial}_\alpha \blacktriangleright \hat{x}_\beta = \delta_{\alpha\beta},\ \ \hat{y}_\alpha \blacktriangleright \hat{x}_\beta &= \hat{x}_\beta\hat{x}_\alpha & \hat{\partial}_{\alpha_1\cdots\alpha_s} &= \hat{\partial}_{\alpha_1}\cdots\hat{\partial}_{\alpha_s} + def. \end{aligned}$$

Matrix $\mathcal{O} = \exp\mathcal{C}$, where $\mathcal{C}^\alpha_\beta = \sum_\sigma C^\alpha_{\beta\sigma}\hat{\partial}^\sigma$.

Examples $\mathrm{U}(\mathfrak{g})^{min}\sharp\mathrm{U}(\mathfrak{g})$ and $\mathrm{U}(\mathfrak{g})^{\circ}\sharp\mathrm{U}(\mathfrak{g})$

Proposition. Minimal Hopf subalgebra $\mathrm{U}(\mathfrak{g})^{min}\subseteq\mathrm{U}(\mathfrak{g})^*$ for which $\mathrm{Lu}(\mathrm{U}(\mathfrak{g}))\subseteq\mathrm{U}(\mathfrak{g})^{min}\sharp\mathrm{U}(\mathfrak{g})$ is generated with $\bar{\mathcal{U}}^\alpha_\beta,\mathcal{U}^\alpha_\beta$, $\alpha,\beta\in\{1,\ldots,n\}$, which satisfy:

$$\sum_\sigma \mathcal{U}^\alpha_\sigma\bar{\mathcal{U}}^\sigma_\beta=\delta^\alpha_\beta=\sum_\sigma\bar{\mathcal{U}}^\alpha_\sigma\mathcal{U}^\sigma_\beta$$

$$\Delta(\mathcal{U}^\alpha_\beta)=\sum_\sigma\mathcal{U}^\alpha_\sigma\otimes\mathcal{U}^\sigma_\beta,\quad \Delta(\bar{\mathcal{U}}^\alpha_\beta)=\sum_k\bar{\mathcal{U}}^\sigma_\beta\otimes\bar{\mathcal{U}}^\alpha_\sigma$$

$$\epsilon(\mathcal{U}^\alpha_\beta)=\delta^\alpha_\beta=\epsilon(\bar{\mathcal{U}}^\alpha_\beta)$$

$$S(\mathcal{U}^\alpha_\beta)=\bar{\mathcal{U}}^\alpha_\beta,\quad S(\bar{\mathcal{U}}^\alpha_\beta)=\mathcal{U}^\alpha_\beta$$

$$\langle X_\gamma,\mathcal{U}^\alpha_\beta\rangle=C^\alpha_{\gamma\beta}$$

$$\sum_{\sigma,\tau}C^\alpha_{\sigma\tau}\mathcal{U}^\sigma_\beta\mathcal{U}^\tau_\gamma=\sum_\rho\mathcal{U}^\alpha_\rho C^\rho_{\beta\gamma},\quad \sum_{\sigma,\tau}C^\alpha_{\sigma\tau}\bar{\mathcal{U}}^\sigma_\beta\bar{\mathcal{U}}^\tau_\gamma=\sum_\rho\bar{\mathcal{U}}^\alpha_\rho C^\rho_{\beta\gamma}$$

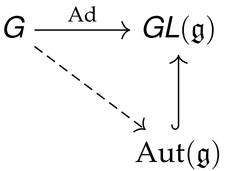
Generators were found by computing $\mathrm{Lu}(X_1),\ldots,\mathrm{Lu}(X_n)$:

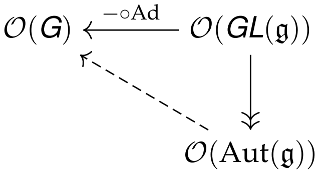
$$\mathrm{Lu}(X_\alpha)=\sum_\beta\bar{\mathcal{U}}^\beta_\alpha\sharp X_\beta\quad\ldots\quad\bar{\mathcal{U}}=\exp\tilde{\mathcal{C}}_n\cdots\exp\tilde{\mathcal{C}}_1,\ (\tilde{\mathcal{C}}_\sigma)^\alpha_\beta=C^\alpha_{\beta\sigma}e_{X_\sigma}.$$

It follows: $\mathrm{U}(\mathfrak{g})^{min}\sharp\mathrm{U}(\mathfrak{g})\hookrightarrow\mathrm{U}(\mathfrak{g})^\circ\sharp\mathrm{U}(\mathfrak{g})\hookrightarrow\mathrm{U}(\mathfrak{g})^*\sharp\mathrm{U}(\mathfrak{g})$ are HA.

Algebras $\mathcal{O}^{min}(G)\sharp\mathrm{U}(\mathfrak{g})$ and $\mathcal{O}(\mathrm{Aut}(\mathfrak{g}))\sharp\mathrm{U}(\mathfrak{g})$

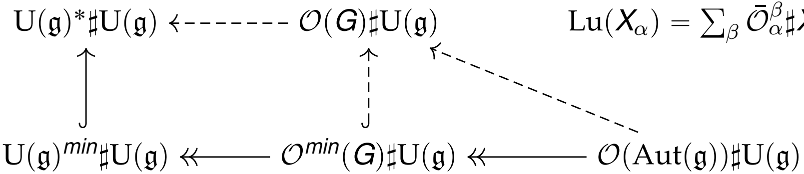
All formulas for components of matrices $\mathcal{U}, \bar{\mathcal{U}}$ are true for components of matrices $\mathcal{O} = \mathrm{Ad}$, $\bar{\mathcal{O}} = \mathcal{O}^{-1}$.





Proposition. Algebras $\mathcal{O}^{min}(G)\sharp\mathrm{U}(\mathfrak{g})$ and $\mathcal{O}(\mathrm{Aut}(\mathfrak{g}))\sharp\mathrm{U}(\mathfrak{g})$ are Hopf algebroids over $\mathrm{U}(\mathfrak{g})$.

Sketch of the proof. Algebraicly using generators and relations.



$$\mathrm{Lu}(X_\alpha) = \sum_\beta \bar{\mathcal{O}}_\alpha^\beta \sharp X_\beta$$

Example $U_q(\mathfrak{sl}_2)^*\sharp U_q(\mathfrak{sl}_2)$ when q is a root of unity

Proposition. Adjoint orbits are finite-dimensional.

Sketch of the proof. We compute

$$\begin{aligned}
\mathrm{ad}'_K(E^nF^mK^r) &= q^{-2n+2m}E^nF^mK^r \\
\mathrm{ad}'_E(E^nF^mK^r) &= q^{-2-2n+2m}(q^{2r}-1)E^{n+1}F^mK^{r-1}+ \\
&\quad +\tfrac{q^{-1-2n+2m+2r}}{(q-q^{-1})^2}(q^{-2m}-1)E^nF^{m-1}K^r+ \\
&\quad +\tfrac{q^{-1-2n+2m+2r}}{(q-q^{-1})^2}(q^{2m}-1)E^nF^{m-1}K^{r-2} \\
\mathrm{ad}'_E(E^nK^r) &= q^{-2-2n}(q^{2r}-1)E^{n+1}K^{r-1} \\
&\;\;\vdots
\end{aligned}$$

By playing combinatorially with exponents, and using $q^d=1$, we get $\mathrm{ad}'_{E^NF^MK^R}(E^nF^mK^r)=0$ when $M>(n+1)e$ or $N>me+(n+1)e^2$. Here e is minimal such that $q^{2e}=1$.

Remark. If q is not a root of unity, this is not true. Maybe it is true more generally, for $U_q(\mathfrak{sl}_n)$.

Thank you!