The geometrical setting of gauge theories of the Yang-Mills type

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The appropriate language for describing the pure Yang-Mills theories is introduced. An elementary but precise presentation of the mathematical tools which are necessary for a geometrical description of gauge fields is given. After recalling basic notions of differential geometry, it is shown in what sense a gauge potential is a connection in some fiber bundle, and the corresponding gauge field the associated curvature. It is also shown how the global aspects of the theory (e.g., boundary conditions) are coded into the structure of the bundle. Gauge transformations and equations of motion, as well as the self-duality equations, acquire then a global character, once they are defined in terms of operations in the bundle space. Finally the orbit space, that is to say, the set of gauge inequivalent potentials, is defined, and its is shown why there is no continuous gauge fixing in the non-Abelian case.

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INTRODUCTION

Why fiber bundles?

The pure Yang-Mills theory defined in the four-dimensional Euclidean space has a rich and interesting structure even at the classical level. The discovery of regular solutions to the Yang-Mills field equations, which correspond to absolute minima of the action (Belavin *et al.*, 1975), has led to an intensive study of such a classical theory. One hopes that a deep understanding of the classical theory will be invaluable when one tries to quantize such a theory.

All the finite action solutions at present known are characterized by a topological quantity called the instanton number, or Pontryagin index, or second Chern class. This topological quantity is a manifestation of nontrivial boundary conditions that one imposes on the gauge potentials through the requirement of the finiteness of the action. We are therefore led to a global topological problem of considerable complexity.

Now, there exists in mathematics a theory sufficiently general to deal with such a problem. This is the theory of fiber bundles, which was originally introduced to formulate and solve global topological problems. The notion of a fiber bundle is very appropriate also for local problems of differential geometry and field theory (gauge field theory in particular). So we are in a happy situation where a preexisting theory in mathematics could be used as an operational tool in our study of the topological aspects of a pure Yang-Mills theory.

In this review, we shall adopt at the very beginning the framework of fiber-bundle language. As a bonus we shall get the geometrization of the gauge potential: the gauge potentials will become the coordinates of a connection form in a principal fiber bundle. It is clear from the well-known rule for gauge transformations,

$$A_\mu(x) \to {}^gA_\mu(x) = g^{-1}(x)A_\mu(x)g(x) + g^{-1}(x)\partial_\mu g(x),$$

that, due to the inhomogeneous term $g^{-1}\partial_\mu g$, A_μ is not of a tensor type. The above transformation law is characteristic of another geometric type, the connection type.

The real power of the language, however, is revealed by the following circumstance: the finiteness of the action (an integrability requirement and hence an analytic boundary condition) dictates (under reasonable physical assumptions) the asymptotic behavior of the Euclidean potentials. This asymptotic behavior in turn goes into the actual construction of a principal fiber bundle over some appropriate compactification of the four-dimensional Euclidean space. In this way the global boundary conditions are automatically taken into account.

The theory of fiber bundles is also indispensable in the study of the group of all gauge transformations, $\mathcal{G}$. In R^4, $\mathcal{G}$ can be described in a fairly straightforward manner. However, when we compactify R^4, $\mathcal{G}$ acquires a nontrivial topology. Consequently, the study of the orbit space, that is, the space of all gauge inequivalent potentials, over which we do our functional integrals, requires great care. Here it seems that the language of fiber bundles is indispensable.

There exists a clear and short review (Stora, 1977) of applications of topology and differential geometry to the study of instantons. For a more extensive review on the fiber bundle approach to gauge theories, we refer to Mayer (1977). Some of the early papers in the physics literature where fiber bundles are used are Kerner (1968, 1970), Trautmann (1970), Cho (1975), and Ezawa *et al.* (1976).

I. MARRYING SPACE-TIME AND ISOSPIN SYMMETRY: PRINCIPAL FIBER BUNDLE (GEOMETRY WITHOUT MATTER); ASSOCIATED FIBER BUNDLE (GEOMETRY WITH MATTER)

A. The Cartesian product of space-time and isospace and its generalization: The fiber bundle

We consider space-time to be a four-dimensional Euclidean space R^4. We are interested in gauge theories of the Yang-Mills (YM) type defined in R^4.

The basic idea of a gauge theory is that an isospin rotation at any point of space-time, affecting gauge potentials and fields, leads to a different description of the same physical reality.

Any such change is represented by a smooth assignment of an element of the gauge group to any point of space-time, that is, a map: $R^4 \to G$. The graphs of these maps live in a space $P = R^4 \times G$ (Fig. 1).

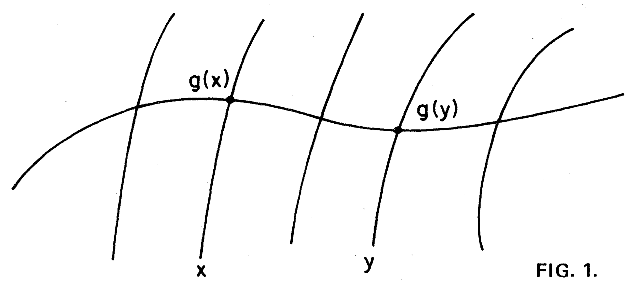

FIG. 1.

Consider now the product space P. The group G has a natural action on P.

If $p \in P$, $p = (x, g)$, $x \in R^4$, $g \in G$.

For $a \in G$, we define $R_a: P \to P$ by $R_a(p) = pa = (x, ga)$.

This action of G on P is *free*; that is, if there is any $p \in G$ such that $R_a(p) = p$, then $a = e$ (e is the identity element of G).

This action determines an equivalence relation between points of P: $p \sim p' \Leftrightarrow$ there exists $a \in G$ such that $p' = pa$.

We readily see that the equivalence classes can be labeled by the points of R^4. In other words, the quotient of P by the equivalence relation is just R^4.

We note also that the equivalence relation gives rise to a canonical projection $\pi: P \to R^4$, defined by $\pi(x, g) = x$. That is, two equivalent points, (x, g) and (x, ga), project to the same point $x \in R^4$.

$\pi^{-1}(x)$ is called a fiber over x and is isomorphic to G.

The triplet (P, G, π) is an example of a trivial principal fiber bundle over R^4 with structure group G and projection π.

The above structure is natural because the fiber $\pi^{-1}(x)$ reproduces the local gauge freedom at x by erecting a gauge group at that point. In fact, each fiber is the same as G except that one forgets which element is the identity (Nelson, 1967). But this is precisely in accordance with the physical desiderata. To see this, imagine, at each point in space-time, a vector space V_x, that is, a representation space for G. Such a vector space is available in the presence of matter fields. We can choose a reference frame in V_x. Once this choice is made, there is a one-to-one correspondence between the set of frames and the group: the group elements take the original reference frame into any other one. This is also the situation with the fiber $\pi^{-1}(x)$ in a principal fiber bundle: once we have chosen a point on the fiber, we can get any other point on the (same) fiber by the action of a (unique) group element.

Before we proceed to the definition of a principal fiber bundle in its full generality, we would like to remark that a fiber bundle (over R^4) is a generalization of a product space which allows for a possible twisting in the large space and, therefore, gives rise to a nontrivial fusion of space-time with isospace.

In order to see this, we can give the following simple example (Kirilov, 1976).

Suppose we have two strips of paper, (1) and (2) (see Fig. 2). We can glue (1) and (2) together in such a way

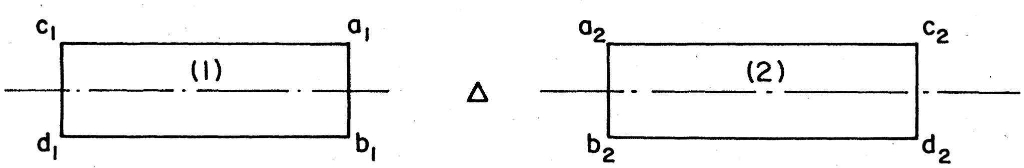

FIG. 2.

that a_1 coincides with a_2, b_1 with b_2, c_1 with c_2, and d_1 with d_2. We then get a cylinder (Fig. 3):

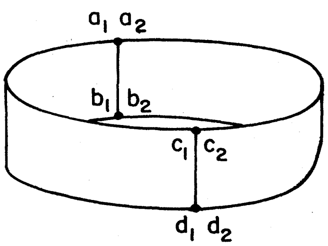

FIG. 3.

But we can glue strips (1) and (2) together with a twist, such that a_1 will still coincide with a_2 and b_1 with b_2, but that c_1 will coincide with d_2 and d_1 with c_2. We then get the Möbius strip (Fig. 4):

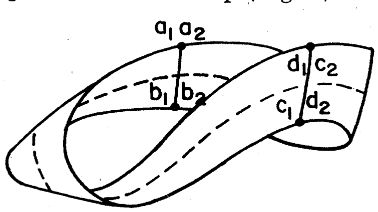

FIG. 4.

Consider now the edges B and B' of Figs. 3 and 4, respectively. B is made out of *two* closed curves, although B' is made out of *one* closed curve.

We can define an action of the group $G=\{+1,-1\}$ (group with two elements) on the points of B (respectively, on the points of B') by looking at the original strips (see Fig. 2).

$+1=$ identity map,

$-1=$ symmetry with respect to the axis Δ,

B is the Cartesian product of a closed loop C with G,

B' is not such a Cartesian product.

Nevertheless, if we identify the points of B (respectively, of B') that are related by a group operation, we get the same closed curve (see Figs. 5 and 6):

We will say that B and B' are bundles with G as a structure group and the closed curve C as a base space. B is a trivial bundle; it is a *product* bundle. B' is not.

We take now the general definition of a principal fiber bundle directly from Kobayashi and Nomizu, (1963).

Definition: Let P be a manifold and G a Lie group.

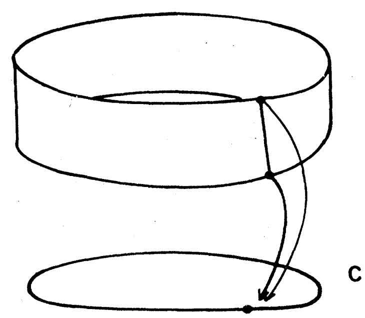

FIG. 5.

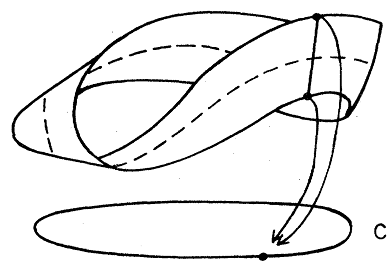

FIG. 6.

A differentiable principal fiber bundle over M with group G consists of a manifold P and an action of G on P satisfying the following conditions:

(1) G acts freely on P on the right: $P\times G\to P$ is denoted by $(u,a)\in P\times G\to ua\in P$.

(2) M is the quotient space of P by the equivalence relation induced by G, $M=P/G$, and the canonical projection $\pi:P\to M$ is differentiable.

(3) P is *locally trivial*, that is, every point $x\in M$ has a neighborhood U such that $\pi^{-1}(U)$ is isomorphic with $U\times G$ in the following sense: there exists a diffeomorphism $\psi:\pi^{-1}(U)\to U\times G$ such that $\psi(u)=(\pi(u),\varphi(u))$, where φ is a mapping of $\pi^{-1}(U)$ into G satisfying $\varphi(ua)=\varphi(u)\cdot a$ for all $u\in\pi^{-1}(U)$ and $a\in G$.

A principal fiber bundle will be denoted by $P(M,G)$ or P. Sometimes we will refer to $P(M,G)$ as a G bundle over M. P is called the total space or bundle space, M is the base space, G the structure group, and π the projection.

For each $x\in M$, $\pi^{-1}(x)$ is a closed submanifold of P, called the fiber over x. Clearly, if u is a point of $\pi^{-1}(x)$, then $\pi^{-1}(x)$ is the set of points ua, $a\in G$ and is called the fiber through u. Every fiber is diffeomorphic to G.

B. Examples of principal fiber bundles

1. The case discussed at the beginning of this section, where the bundle space P is simply the product space $R^4 \times G$, is an example of a trivial G bundle over R^4. B is also a trivial bundle (see Fig. 5).

2. Consider a circle S^1 and the group $G=(1,-1)=Z_2$ (i.e., group with two elements) with the following action on S^1: $+1=$ identity map, $-1=$ antipodal map. The action of G is free. $M=S^1/G=P^1=$ one-dimensional projective space (i.e., space of diameters). π (point of S) $=$ diameter through that point. We have to check the local triviality. Given an element in P^1, that is, a direction in R^2 through the origin 0, there always exists an open cone U (in R^2) containing that direction, and $\pi^{-1}(U)=U\times\{+1,-1\}$ (see Fig. 7).

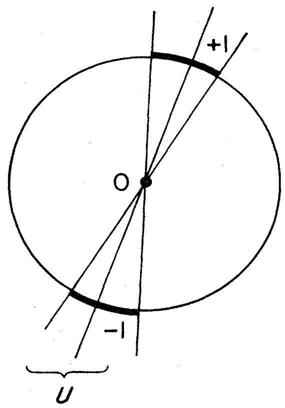

FIG. 7.

Hence S^1 is the total space of a principal fiber bundle with group G and base space P^1. We shall denote this bundle by $S^1(P^1, Z_2)$.

Actually, we can identify this circle (together with the action of Z_2) with B' (with the action defined above), since they are *homotopically equivalent*.

C. More definitions

1. Cross section of a principal bundle

Definition: A (global) *cross section* of a principal bundle is a map σ from the base space to the bundle space P such that $\pi(\sigma(x))=x$ for all $x\in M$ [a local section over $U_\alpha \subset M$ is a map $\sigma_\alpha : U_\alpha \to P$ such that $\pi(\sigma_\alpha(x))=x \ \ \forall x\in U_\alpha$]. Example: Fig. 8 shows a section of B.

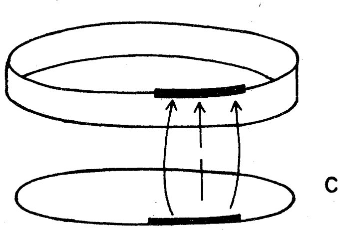

FIG. 8.

Figure 9 shows an example of a local section for the bundle $S^1(P^1, Z_2)$ over the cone U_α.

It is noteworthy that the local triviality condition for a principal fiber bundle [see condition (3) of the definition of the principal fiber bundle] provides us with a preferred set of local sections.

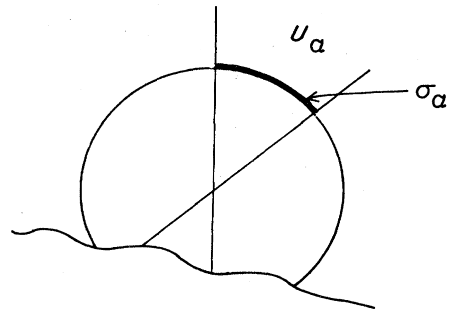

FIG. 9.

Indeed, given a point x on M, there always exists U_α such that $x\in U_\alpha$. Choose a point u in $\pi^{-1}(x)$ and define

$$\sigma_\alpha(x)=u\cdot\varphi_\alpha^{-1}(u)\,.$$

This definition makes sense since $u\cdot\varphi_\alpha^{-1}(u)$ is independent of the choice of the point u in the fiber $\pi^{-1}(x)$. To see this, we need only use the property of φ_α:

$$\varphi_\alpha(ua)=\varphi_\alpha(u)\cdot a \quad \forall a\in G\,.$$

Any point in $\pi^{-1}(x)$ can be written $v=ua$. Then

$$v\cdot\varphi_\alpha^{-1}(v)=u\cdot\varphi_\alpha^{-1}(u)\,.$$

Moreover, $\pi(\sigma_\alpha(x))=x$.

Note that $\psi_\alpha(\sigma_\alpha(x))=(x,e)$, i.e., the local cross section σ_α corresponds, under the diffeomorphism $\psi_\alpha : \pi^{-1}(U_\alpha)\to U_\alpha\times G$, to $U_\alpha\times\{e\}$.

2. Transition functions

We assume that M admits a covering as above. Let $\{U_\alpha\}$ $\alpha\in A$ be such an open covering of M. We shall use the preferred set of local sections $\sigma_\alpha(x)=u\cdot\varphi_\alpha^{-1}(u)$.

Suppose $x\in U_\alpha\cap U_\beta$. Then

$$\sigma_\beta(x)=\sigma_\alpha(x)\varphi_\alpha(u)\circ\varphi_\beta^{-1}(u)\,.$$

Since the action of G on P is free, and since σ_α and σ_β depend only on x, we can define:

$$\psi_{\alpha\beta}(x)=\varphi_\alpha(u)\cdot\varphi_\beta^{-1}(u)\,.$$

The maps $\psi_{\alpha\beta} : U_\alpha\cap U_\beta \to G$ are called transition functions.

They verify the so-called co-cycle condition:

$$\psi_{\alpha\beta}(x)=\psi_{\alpha\gamma}(x)\cdot\psi_{\gamma\beta}(x) \quad \forall x\in U_\alpha\cap U_\beta\cap U_\gamma\,.$$

The transition functions are differentiable.

An example of a transition function for the bundle $S^1(P^1, Z_2)$ is seen in Fig. 10.

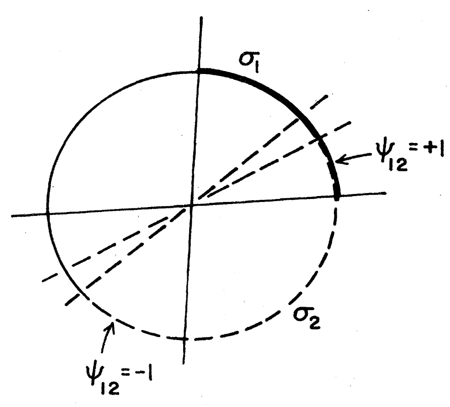

FIG. 10.

3. Principal coordinate bundle

Let $P(M,G)$ be a principal fiber bundle and let $\{U_\alpha\}$ be a covering of M (as above). Consider the transition functions $\psi_{\alpha\beta}$ corresponding to the covering $\{U_\alpha\}$ and constructed from the preferred set of local sections σ_α. We will say that $(M, G, \{U_\alpha\}, \psi_{\alpha\beta})$ constitutes a *principal coordinate bundle* in the sense of Steenrod (1951). We stress that the existence of this coordinate bundle is part of the definition of the principal fiber bundle; it is in fact the expression of the local triviality condition. However, the existence of a group action on the bundle space P gives rise to other possible coordinate bundles.

Suppose we construct, using this group action, a different set σ'_α $(\alpha \in A)$ of local cross sections. Of course, $\sigma'_\alpha(x) = \sigma_\alpha(x) \circ g_\alpha(x)$, where $g_\alpha(x) \in G$. Indeed, $\sigma_\alpha(x)$ and $\sigma'_\alpha(x)$ have the same projection x, and hence belong to the same fiber.

From the new local sections we could construct another set of transition functions $\psi'_{\alpha\beta}$. These transition functions are related to the old ones by

$$\psi'_{\alpha\beta}(x) = g_\alpha^{-1}(x)\psi_{\alpha\beta}(x)g_\beta(x).$$

We will say that the principal coordinate bundles, $(M, G, \{U_\alpha\}, \psi_{\alpha\beta})$ and $(M, G, \{U_\alpha\}, \psi'_{\alpha\beta})$, are *equivalent* (this is an equivalence relation). Actually, the original principal fiber bundle is the equivalence class of these coordinate bundles.

It is remarkable that the open covering of the base space, together with the transition functions, verifying the co-cycle condition, completely determines the principal fiber bundle.

There exists a reconstruction theorem such that any set of functions $\psi_{\alpha\beta}$ (with values in G) defined for a covering $\{U_\alpha\}$ of M and satisfying the condition

$$\psi_{\alpha\beta}(x) = \psi_{\alpha\gamma}(x) \cdot \psi_{\gamma\beta}(x) \quad \forall x \in U_\alpha \cap U_\beta \cap U_\gamma (\forall\alpha, \forall\beta, \forall\gamma)$$

uniquely determines a principal fiber bundle $P(M,G)$ whose transition functions relative to the covering $\{U_\alpha\}$ are the $\psi_{\alpha\beta}$'s.

Example: Wu and Yang in their study of the Dirac static monopole (Wu and Yang, 1975) have used the principal coordinate bundle given by $(M = R^3 - \{O\}$, $G = U(1)$, $\{U_1, U_2\}$, $\psi_{12} = e^{in\alpha})$.

U_1, (respectively, U_2) are the points of R^3–{origin} that are not contained in the half-cone C_1 (respectively, C_2) (see Fig. 11).

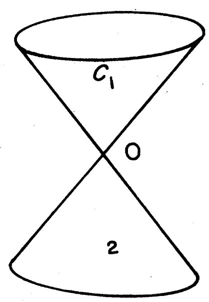

FIG. 11.

n is an integer.

The relevant principal bundle in this case is $P(R^3 - \{O\}, U(1))$. Its base space is the whole of space minus the position of the monopole (taken to be the origin of R^3). Its structure group $U(1)$ is the Abelian group of gauge transformations (phase transformations of matter fields).

Their terminology of "global gauge" corresponds to the concept of a principal fiber bundle. In their formalism the Dirac quantization condition (namely, $2eg = n$, where e is the electric charge and g is the magnetic charge) results from the requirement of single-valuedness of the transition function ψ_{12}. It is related to the classification of $U(1)$ bundles over S^2, the retract of $R^3 - \{O\}$ (see sec. III).

D. Trivial principal fiber bundle

A principal fiber bundle P is *trivial* if we can construct from it (in the manner described above) a principal coordinate bundle such that all transition functions are equal to unity $[\psi_{\alpha\beta}(x) = e]$.

The bundle space P is then diffeomorphic to $M \times G$ and admits a global section (Steenrod, 1951). The trivial bundle is also called the *product bundle*.

As a consequence, the bundle $S^1(P^1, Z_2)$ is not trivial, since it admits no continuous global section: to get such a cross section, we would have to open the cone C (of angle θ) of Fig. 12 until θ reached the value π.

The section σ cannot be continuous, since it does not close when $\theta = \pi$.

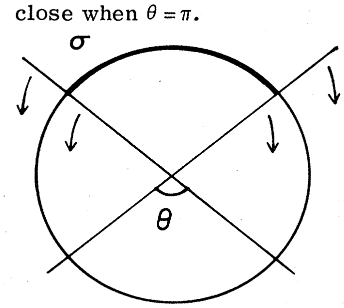

FIG. 12.

The topology of the bundle spaces we introduce will turn out to be relevant to the study of the topologically stable gauge field configurations of the theories under consideration (see Sec. III). However, the bundle space is not given *a priori*. What is usually provided is space-time and the gauge symmetry group. These entities are blended together by certain transition functions.

We will see how these transition functions are related to the boundary conditions imposed on the gauge potentials.

This blending leads first to a coordinate principal bundle and then, in a natural way, to a principal bundle (via the reconstruction theorem). The topology of the bundle space will come from the topology of the base space, the topology of the group, and the transition functions.

In fact, there is a homotopic classification of all principal bundles having a given base space and a given structure group. It is related both to the possible nontrivial homotopy of the transition functions and to the possible obstructions that one meets in trying to extend a local cross section to a global one.

Now, in order to accomodate matter fields within the framework of bundle theory, we would need the concept of fiber bundle associated to a given principal fiber bundle. Although in this review we intend to study the pure Yang-Mills theory (with no matter fields), the notion of an associated fiber bundle turns out to be useful in the study of the group of all gauge transformations and in the geometrical interpretation of the equation of motion (Secs. IV and V).

E. Associated fiber bundle

Suppose the group G acts on a space F. This F can be, for example, a linear representation space for the group G like the vector spaces used in the description of matter fields.

It is possible to use any principal bundle $P(M,G)$ in order to erect at any point of M a copy of F, but in a way which preserves the possibly nontrivial topology of P (or at least part of it).

This procedure completes the generalization of the Cartesian product which the principal bundles achieve for groups. The bundle space E will be a generalized $M \times F$. E will be called a *fiber bundle associated to* P, *with standard fiber* F.

Before giving the definition [again taken from Kobayashi and Nomizu (1963)] of an associated fiber bundle, let us give the picture of two different bundles over a circle, with standard fiber R (the real line).

We can use the trivial principal bundle B of Fig. 5. The associated bundle $E(B,R)$ is obtained by gluing a real line to all pairs of related points of B, as in Fig. 13.

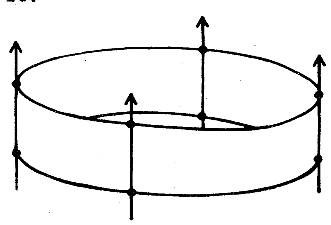

FIG. 13.

If we use the principal fiber bundle B' (see Fig. 6), we get the form shown in Fig. 14:

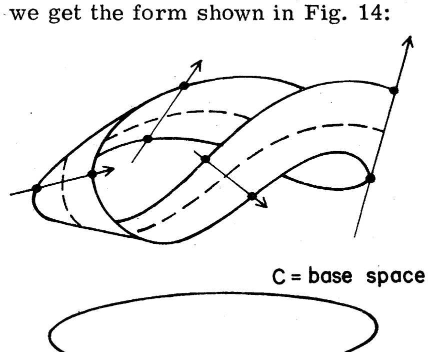

FIG. 14.

The bundle E is the Cartesian product $R \times \{$closed curve $C\}$. The bundle E' is not such a Cartesian product.

The nontriviality of E' shows in the fact that we cannot choose continuously around C an orientation of the real line. However, there exist sections of E' as well as sections of E.

Definition: Let $P(M,G)$ be a principal fiber bundle with projection $\pi: P \to M$, and let F be a manifold on which G acts to the left (we denote by $a\xi$, the result of the action of the group element a on ξ, $\xi \in F$.).

Define the action of $a \in G$ on $P \times F$ by

$$(p, \xi) \to (pa, a^{-1}\xi).$$

This action determines an equivalence relation between points of $P \times F$.

By definition the fiber bundle associated to P, with standard fiber F, is the space $E(M,G,F,P) = P \times F/G$, equipped with the following differentiable structure:

If we define the projection $\pi_E: E \to M$ by

$$\pi_E \text{ (equivalence class of } (u, \xi)) = \pi(u),$$

we have for E again a local triviality property:

$$\forall x \in M, \ \exists \text{ open } U_x \subset M, \ x \in U_x \text{ such that } \pi_E^{-1}(U_x) \approx U_x \times F.$$

The differentiable structure on E is introduced by the requirement that $\pi_E^{-1}(U_x)$ is an open submanifold of E.

The projection π_E is then a differentiable mapping of E onto M.

A cross section of a bundle $E(M,G,F,P)$ is again a mapping $\sigma: M \to E$ such that $\pi_E \circ \sigma$ is the identity map of M.

Figures 13 and 14 show two bundles associated to B (respectively, to B') with standard fiber R.

A field in the trivial case (Fig. 13) would be a function from M to F. It is a cross section of E. The generalization to the nontrivial case is then that a field over M is a cross section of some bundle $E(M,G,F,P)$ (rather than an F-valued function on M).

There will thus exist twisted fields (Avis and Isham, 1978), but we shall not get into the study of those objects here.

Note, however, that these cross sections are *locally* mappings from M to F. Globally they are not. The nontriviality arises from the actual pasting of the copies of F at each point of M (to erect the bundle space E).

In order to complete the description of the associated bundles, let us give a technical lemma we will use later (Secs. IV and V):

Lemma: Let P be a principal fiber bundle with base space M and structure group G, and let F be a manifold on which G acts. There will be a one-to-one correspondence between cross sections of $E(M,G,F,P)$ and functions φ from P to F which have the property $\varphi(ua) = a^{-1} \cdot \varphi(u)$ $\forall u \in P$, $\forall a \in G$.

Proof: Suppose φ is such a function. $\forall x \in M$, we can choose in P a point u such that $\pi(u) = x$. The couple $(u, \varphi(u))$ determines a point of E, namely, the class of points of $P \times F$ of the form $(ua, a^{-1}\varphi(u))$. Clearly, this class does not depend on the choice of u in the fiber $\pi^{-1}(x)$. In other words, we have constructed a section of E. Conversely, given a section of E, we can determine a mapping $\varphi: P \to F$ such that $\varphi(ua) = a^{-1}\varphi(u)$ $\forall u \in P$, $\forall a \in G$.

II. CONNECTION FORM: A GEOMETRIZATION OF THE GAUGE POTENTIAL

A. Rapid course in differential geometry

1. Vectors

We will assume that the notion of an n-dimensional differentiable manifold M is understood. It will be a topological space with local coordinates. By the existence of local coordinates we mean the existence of an open covering of the space such that each one of its constituents (coordinate neighborhoods) is homeomorphic to an open subset of R^n.

Consider a point p on M, and all smooth curves through p. To any such curve we can associate the operation of taking directional derivatives at p of any smooth real-valued function f on M. This defines the vector X_p tangent to the curve at p:

$X_p f$ is the directional derivative of f at p.

The tangent vectors at a given point form an n-dimensional real vector space denoted by $T_p(M)$.

Let $u^1, \ldots, u^n$ be a local coordinate system in a neighborhood U of p. Then $\partial/\partial u^1, \ldots, \partial/\partial u^n$ form a basis of $T_p(M)$.

A vector field X on a smooth manifold M is an assignment of a vector $X_p \in T_p(M)$ to each point p of M. Locally, that is to say, in a given coordinate neighborhood, X may be expressed by

$$X = \sum_{j=1}^{n} \xi^j \cdot \frac{\partial}{\partial u^j},$$

where ξ^j are functions defined in that neighborhood.

X acts on smooth real-valued functions on M. We will denote by $X \cdot f$ the result of this action.

Vector fields form an infinite-dimensional module[1] over the ring of smooth real-valued functions on M. Moreover, in addition to the usual operations defined in a module, another important operation is defined on this set, the Lie product:

Given any two vector fields X, Y on M, we can define a new vector field, their Lie "bracket" $[X, Y]$ by

$$[X, Y] \cdot f = X \cdot (Y \cdot f) - Y \cdot (X \cdot f).$$

2. Forms

Consider now the n-dimensional space $T_p^*(M)$ dual to $T_p(M)$. This is the space of covectors at p. $du^1, \ldots, du^n$ is a basis for $T_p^*(M)$ dual to $\partial/\partial u^1, \ldots, \partial/\partial u^n$:

$$du^i\left(\frac{\partial}{\partial u^j}\right) = \delta^i_j.$$

A 1 form on M is an assignment of a covector at each point p of M. Locally, a 1 form can be written as

$$\omega = \sum_{j=1}^{n} f_j \, du^j.$$

From $T_p^*(M)$ we can construct the exterior algebra $\wedge T_p^*(M)$ (which is skew symmetric).

An r form ω is an assignment of an element of degree r in $\wedge T_p^*(M)$ to each point of M. Locally,

$$\omega = \sum_{i_1 < \cdots < i_r} f_{i_1 \cdots i_r} du^{i_1} \wedge \cdots \wedge du^{i_r}.$$

Of course, $f_{i_1 \cdots i_r}$ is antisymmetric with respect to an interchange of any two of its indices.

3. d operation

To any differentiable real-valued function f on M, we can associate its total derivative df, which is a 1 form defined by $df(X) = X \circ f$, for any vector field X.

An important operation on forms is the exterior differentiation. For a beautiful introduction to forms and the intrinsic geometrical nature of the exterior differentiation we refer to Misner *et al.* (1973). See also Flanders (1963).

The operation is expressed by a linear operator d, called the *exterior derivative*, acting on forms (the o forms are the functions): (i) It takes r forms to $(r+1)$ forms. (ii) It takes functions into their total derivative. (iii) If α is an r form, $d(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^r \alpha\wedge d\beta$. (iv) $d(d\omega) = d^2\omega = 0$ for all ω. The above properties define completely the d operation in a coordinate independent manner. Locally, if ω is an r form,

$$d\omega = \sum_{i_1 < \cdots < i_r} \frac{\partial f_{i_1 \cdots i_r}}{\partial u^{i_0}} du^{i_0} \wedge du^{i_1} \wedge \cdots \wedge du^{i_r}.$$

That is, the components of $d\omega$ are

$$(d\omega)_{i_1 \cdots i_{r+1}} = \sum_{s=1}^{r+1} (-1)^{s+1} \frac{\partial}{\partial u^s} f_{i_1 \cdots \hat{i}_s \cdots i_{r+1}}.$$

4. * operation (duality of forms)

Another important operation can be defined on forms if M is a Riemannian manifold endowed with a metric: the operation of taking the adjoint of a form. Let g_{ij} be the metric on M. We follow de Rham (1960) and define:

$$e_{1\cdots n} = \begin{vmatrix} g_{11} \cdots g_{1n} \\ g_{n1} \cdots g_{nn} \end{vmatrix}^{1/2} \qquad e_{i_1 \cdots i_n} = \delta^{1 \cdots n}_{i_1 \cdots i_n} e_{1 \cdots n},$$

where the Krönecker symbol $\delta^{j_1 \cdots j_p}_{i_1 \cdots i_p}$ $(1 \le p \le n)$ is defined by the following properties:

(i) $\delta^{j_1 \cdots j_p}_{i_1 \cdots i_p}$ is antisymmetric in the set of indices i_κ and in the set of indices j_κ.

(ii) For $i_1 < \cdots < i_p$, $j_1 < \cdots < j_p$

$$\delta^{j_1 \cdots j_p}_{i_1 \cdots i_p} = \begin{cases} 1 \text{ for } i_\kappa = j_\kappa, & \kappa = 1, 2, \ldots, p \\ 0 \text{ for } i_\kappa \ne j_\kappa. \end{cases}$$

The volume element of M is given in local coordinates by:

$$dM = e_{1\cdots n} \, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n.$$

Consider now a p form α given locally by

$$\alpha = \sum_{i_1 < \cdots < i_p} \alpha_{i_1 \cdots i_p} du^{i_1} \wedge \cdots \wedge du^{i_p}.$$

The form *adjoint to* α is $*\alpha$, an $(n-r)$ form, defined locally by:

$$*\alpha = \sum_{j_1 < \cdots < j_{n-p}} (*\alpha)_{j_1 \cdots j_{n-p}} du^{j_1} \wedge \cdots \wedge du^{j_{n-p}}, \tag{2.1}$$

with

$$(*\alpha)_{j_1 \cdots j_{n-p}} = \sum_{i_1 < \cdots < i_p} \delta^{1 \cdots n}_{i_1 \cdots i_p j_1 \cdots j_{n-p}} \cdot e_{1 \cdots n} \cdot \alpha^{i_1 \cdots i_p} \tag{2.2}$$

[1]A module is a vector space with more structure. Let $\mathfrak{X}(M)$ be the set of all differentiable vector fields on M. It is a real vector space. Moreover, if f is a function and X is a vector field on M, the $fX \in \mathfrak{X}(M)$ and is defined by $(fX)_p = f(p)X_p$.

and

$$\alpha^{i_1 \cdots i_p} = g^{i_1 k_1} \cdots g^{i_p k_p} \cdot \alpha_{k_1 \cdots k_p}. \tag{2.3}$$

We are now in a position to define the *inner product* of two p forms α and β in the case where M is compact.

A direct calculation gives: $\alpha \wedge * \beta = \beta \wedge * \alpha$.

Define

$$(\alpha, \beta) = \int_M \alpha \wedge * \beta .$$

For the definition of integrals over a manifold, see Choquet-Bruhat *et al.* (1977) and references therein.

The product (α, β) is a scalar product. It verifies

(i) $(\alpha, \beta) = (\beta, \alpha)$,

(ii) $(\alpha, \alpha) \geq 0$ and $(\alpha, \alpha) = 0 \Rightarrow \alpha = 0$,

(iii) $(*\alpha, *\beta) = (\alpha, \beta)$.

5. Linear differential

Let M and N be two differentiable manifolds (of dimensions m and n, respectively), and let ϕ be a mapping from M to N. We will define the *linear differential* ϕ_* of ϕ *at* p ($p \in M$) as a linear mapping ϕ_*: $T_p(M) \to T_{\phi(p)}(N)$ by

$$(\phi_* X) \cdot f = X(f \circ \phi)$$

for any vector $X \in T_p(M)$ and any real-valued function f on M.

The *transpose* of ϕ_* at p is a linear mapping of $T^*_{\phi(p)}(N)$ into $T^*_p(M)$, defined as follows: given an r form ω' on N, we define an r form $\phi^*\omega'$ on M by

$$(\phi^*\omega')(X_1, \ldots, X_r) = \omega'(\phi_* X_1, \ldots, \phi_* X n)$$

$$\forall X_1 \cdots X_r \in T_p(M).$$

$\phi^*\omega'$ is sometimes called the *pull back* of ω' by ϕ. Note that the pull-back operation has the following properties:

$$\phi^*(\omega_1 \wedge \omega_2) = \phi^*(\omega_1) \wedge \phi^*(\omega_2),$$

$$d(\phi^*\omega) = \phi^*(d\omega).$$

B. Gauge group G and its Lie algebra $\mathfrak{a}(G)$

We will use the above differential-geometric notions in the case of the gauge group G, which is supposed to be a Lie group: G is a finite-dimensional differentiable manifold and the group operation acts differentiably. We denote by L_a the left action by a, that is to say, the mapping from G to G defined by

$$L_a(g) = a \cdot g .$$

e is the identity (unit) element of G.

Given any vector $A \in T_e(G)$, we can construct a vector belonging to $T_a(G)$ by applying the linear differential $(L_a)_*$ on A. When a runs through G, we thus get a vector field on G. Such a vector field is by construction left invariant. By definition, the Lie algebra of G, denoted by $\mathfrak{a}(G)$, is the set of left-invariant vector fields on G. If G is an n-dimensional group, $\mathfrak{a}(G)$ is an n-dimensional vector space isomorphic to $T_e(G)$.

It so happens that the Lie algebra is closed under the Lie bracketing operation. In other words, if X and Y are two left-invariant vector fields on G, then $[X,Y]$ is also a left-invariant vector field on G. As a consequence, given a basis $(A_1, \ldots, A_n)$ of $\mathfrak{a}(G)$, there exist structure constants C^k_{ij} such that

$$[A_i, A_j] = \sum_{k=1}^{n} C^k_{ij} A_k .$$

An important action of G on itself is the *adjoint action* (action by inner automorphism). To any group element a we can associate the *adjoint map* Int_a, which is a map from G onto itself defined by $\mathrm{Int}_a(g) = aga^{-1}$.

Via its linear differential Ad_a, this map induces an automorphism of $\mathfrak{a}(G)$, i.e., Ad_a is an isomorphism of $\mathfrak{a}(G)$ onto itself. Moreover, the mapping Ad from G to Aut $\mathfrak{a}(G)$ [the group of automorphisms of $\mathfrak{a}(G)$ is a homomorphism of groups, and defines the so-called (linear) adjoint representation of G.

Note that the kernel of this homomorphism is the *center* Z *of* G ($Z = \{a \in G \mid ab = ba, \forall\, b \in G\}$).

A form ω on G is said to be left invariant if $L^*_a \omega = \omega$ for every $a \in G$. The vector space $\mathfrak{a}^*(G)$ formed by all left invariant 1 forms on G (the *Maurer-Cartan forms*) is the dual space of $\mathfrak{a}(G)$. If $A \in \mathfrak{a}(G)$ and $\omega \in \mathfrak{a}^*(G)$, then $\omega(A)$ is constant on G.

C. Theory of connections on a principal fiber bundle

Let us now return to the principal fiber bundle $P(M,G,\pi)$.

To any element A of the Lie algebra of G, we will associate a vector field $\Sigma(A)$ on P, the *fundamental vector field corresponding to* A. $\Sigma(A)$ is actually generated by the action of G on P: if $A \in \mathfrak{a}(G)$, then $\exp(tA)$ is a one-parameter subgroup of G. This subgroup acts on P. Through any point $u \in P$, we can draw the curve $u_t = R_{\exp(tA)}(u)$ utilizing the right action of G on the bundle space P. Then for any real-valued function f on P, we define:

$$\Sigma(A)_u \cdot f = \frac{d}{dt} f(u_t)\Big|_{t=0} .$$

Clearly, $\Sigma(A)_u$ is a vector tangent to P at u. In fact, it is tangent to the fiber through u, at u. (The fiber is considered here as a submanifold of P.)

Call G_u the subspace of $T_u(P)$ of vectors tangent to the fiber through u, at u.

Σ *is an isomorphism of* $\mathfrak{a}(G)$ *onto* G_u.

A *connection* in P is a choice of a supplementary linear subspace Q_u, in $T_u(P)$ to G_u:

$$T_u(P) = Q_u \oplus G_u .$$

Q_u verifying: (i) $Q_{ua} = (R_a)_* Q_u$. (ii) Q_u depends differentiably on u.

Q_u is called *horizontal subspace* at u (space of *horizontal vectors*) and has the same dimensions as M.

G_u is called the *vertical subspace* at u (space tangent to the fiber).

Note that the field (see Chevalley, 1946) of horizontal subspaces is not, in general, integrable.

Although to choose a linear subspace Q_u at every point u in P is a completely natural thing to do [it amounts to choosing a basis for $T_u(P)$], and a very easy one, the nonintegrability of the distribution so determined reveals the rich geometric structure of a principal fiber bundle.

The physical content of this nonintegrability is best explained in the presence of matter fields and is related to the *holonomy group* (see below Sec. II.G), which reflects the nontriviality of the parallel transport along a closed path (Loos, 1967).

The connection form: The above splitting of the tangent space to P into horizontal and vertical subspaces can be realized in terms of the *connection form*. This is a Lie-algebra-valued 1 form ω such that:

(i) ω applied on any fundamental vector field $\Sigma(A)$ reproduces A, i.e., $\omega(\Sigma(A))=A$.

(ii) $(R_a^*\omega)(X)=Ad_{a^{-1}}\cdot\omega(X)$. The horizontal subspace Q_u is the kernel of ω, that is to say, X_u is horizontal if and only if $\omega(X_u)=0$.

We shall now express ω, the connection form on P, by a family of local forms, each one being defined in an open subset of the *base-space manifold* M. We will see that the local forms verify necessary compatibility relations involving the transition functions defined in Sec. I, and, conversely, that any set of local forms obeying these conditions determines a unique connection form on P.

Let $\{U_\alpha\}$ be a covering of M, as in (3) of the definition of the principal fiber bundle (see Sec. I.A).

We choose in P the preferred set of local sections σ_α and the corresponding transition functions (see Sec. I.C).

For each α, we define a Lie-algebra-valued 1 form on U_α by

$$\omega_\alpha=\sigma_\alpha^*\omega\,.$$

Theorem: The local forms ω_α verify the compatibility condition

$$\omega_\beta=Ad_{\psi_{\alpha\beta}^{-1}}\cdot\omega_\alpha+\psi_{\alpha\beta}^{-1}d\psi_{\alpha\beta}$$

in $U_\alpha\cap U_\beta$ (where d denotes the exterior derivative on M).

Converse theorem (or another reconstruction theorem): Given a collection of local forms ω_α (on M) verifying the above compatibility conditions, there exists a unique connection form ω on P, giving rise to this family of local forms on M (i.e., $\omega_\alpha=\sigma_\alpha^*(\omega)\ \forall\,\alpha$).

Proof: Suppose ω is given. We can then construct the local forms ω_α on M. Recalling that we have on each open $\pi^{-1}(U_\alpha)$ a map φ_α: $\pi^{-1}(U_\alpha)\to G$, we may define on $\pi^{-1}(U_\alpha)$ the 1 form

$$\phi=Ad_{\varphi_\alpha^{-1}}(\pi^*(\omega_\alpha))+\varphi_\alpha^{-1}d_P\varphi_\alpha$$

(where d_P denotes the exterior derivative on the bundle space).

Suppose now that X is a vector tangent to the bundle space, at some point u with $u=\sigma_\alpha(\pi(x))$ (i.e., $\varphi_\alpha(u)=e$), as in Fig. 15:

The vector X can be decomposed in a unique way as a sum $X=Y+Z$, where Y is tangent at u to the section σ_α, Z being vertical. We have $Y=\sigma_*(\pi_*(Y))$ and $\pi_*(Z)=0$. Moreover, there exists a unique Lie algebra element A such that Z coincides with the fundamental vector field associated to A at u. Consequently,

$$\phi(X)=\omega_\alpha(\pi_*(Y))+\varphi_\alpha^{-1}d_P\varphi_\alpha(Z)\,.$$

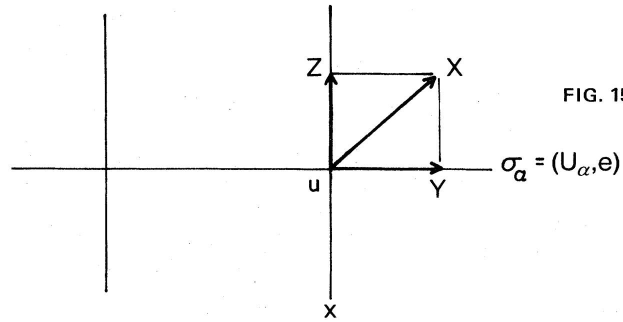

FIG. 15.

Due to the transformation properties of φ_α (i.e., $\varphi_\alpha(ua)=\varphi_\alpha(u)\cdot a$), we have

$$\varphi_\alpha^{-1}d_P\varphi_\alpha(Z)=A=\omega(Z)\,.$$

Then

$$\phi(X)=\omega(Y)+\omega(Z)=\omega(X)\,.$$

ϕ and ω coincide at every point of σ_α. In addition, ϕ and ω have the same transformation properties under the action of R_a. They coincide on $\pi^{-1}(U_\alpha)$.

By construction, ω_α acts on vectors tangent to the base space in U_α. If $V\in T_x(M)$ and $x\in U_\alpha\cap U_\beta$, then both ω_α and ω_β act on V. The above formula relates $\omega_\alpha(V)$ and $\omega_\beta(V)$ for any such V. Note that $\psi_{\alpha\beta}^{-1}d\psi_{\alpha\beta}$ is a 1 form on $U_\alpha\cap U_\beta$ and takes values in the space tangent to the group at e, identified with $\mathfrak{G}(G)$.

In $\pi^{-1}(U_\alpha\cap U_\beta)$ we have two descriptions of ω:

$$\omega=Ad_{\varphi_\alpha^{-1}}(\pi^*(\omega_\alpha))+\varphi_\alpha^{-1}d_P\varphi_\alpha$$
$$=Ad_{\varphi_\beta^{-1}}(\pi^*(\omega_\beta))+\varphi_\beta^{-1}d_P\varphi_\beta\,.$$

Then necessarily

$$\pi^*(\omega_\beta)=Ad_{\psi_{\alpha\beta}^{-1}}(\pi^*(\omega_\alpha))+\varphi_\beta(\varphi_\alpha^{-1}d_P\varphi_\alpha-\varphi_\beta^{-1}d_P\varphi_\beta)\varphi_\beta^{-1}\,,$$

i.e.,

$$\pi^*(\omega_\beta)=Ad_{\psi_{\alpha\beta}^{-1}}(\pi^*(\omega_\alpha))+\psi_{\alpha\beta}^{-1}d_P\psi_{\alpha\beta}\,.$$

From the fact that $\psi_{\alpha\beta}$ is invariant by right translation ($\psi_{\alpha\beta}$ is a function defined on the base space), we see that on $U_\alpha\cap U_\beta$

$$\omega_\beta=Ad_{\psi_{\alpha\beta}^{-1}}(\omega_\alpha)+\psi_{\alpha\beta}^{-1}d_M\psi_{\alpha\beta}\,.$$

We have then shown that this compatibility relation is necessary and sufficient for the existence of a well-defined connection form on P such that $\omega_\alpha=\sigma_\alpha^*(\omega)$ on any U.

Note that the relation we have obtained between ω_α and ω_β looks very much like the usual gauge transformation formula but has appeared as a *compatibility* relation.

However, if ω is a connection form on $P=R^4\times G$, we can construct from a global section σ_1 of P, the form on R^4:

$$\omega_1=\sigma_1^*(\omega)\,.$$

If we now use a G-valued function g on R^4 to transform σ_1 into σ_2 by $\sigma_2(x)=\sigma_1(x)\cdot g(x)$, we can define a *new* 1 form on R^4:

$$\omega_2=\sigma_2^*(\omega)\,.$$

We have

$$\omega_2 = Ad_{g^{-1}}\omega_1 + g^{-1}\,dg \quad \text{on } R^4 .$$

This last relation appears as a *transformation formula* rather than a compatibility relation (for a definition of gauge transformation, see below Sec. III.B).

D. The geometrical meaning of gauge potentials

Suppose M is a four-dimensional manifold ($R^4, S^4, P^4, \ldots$) with local coordinates x_μ ($\mu=1,2,3,4$) in U_α.

ω_α being a 1 form in U_α, it can be written in terms of its components (Lie-algebra-valued functions), $A^\mu_\alpha(x)$:

$$\omega_\alpha = \sum_\mu A^\mu_\alpha(x)\,dx_\mu . \tag{2.4}$$

Suppose now we transform σ_α into σ'_α by the action of some g:

$$\sigma'_\alpha(x) = \sigma_\alpha(x)\cdot g(x) .$$

Then

$$\omega'_\alpha = \sigma'^{*}_\alpha \omega = A'^\mu_\alpha\,dx_\mu .$$

We have

$$A'_\mu = g^{-1}A_\mu g + g^{-1}\partial_\mu g .$$

This exactly reproduces the gauge transformation formula for gauge potentials (we have not yet defined what a gauge transformation is).

Suppose $M=R^4$ and $P=R^4\times G$. The choice of a global section σ in P coordinatizes P, and gives a one-to-one correspondence between the projected forms $A^\mu\,dx_\mu$ and the connection forms ω: if σ is given $\sigma^*\omega$ is well defined and can be written $A^\mu\,dx_\mu$; if A^μ is given, there exists a unique connection form ω on P such that $A^\mu\,dx_\mu = \sigma^*\omega$ (cf. the reconstruction theorem).

A change of σ by the action of some G-valued function g on R^4 can be viewed as a change of coordinates in P, and it induces a transformation of the components A^μ similar to the usual gauge transformation of potentials.

If we then relate gauge transformation and change of section in the principal fiber bundle P, *the gauge potential naturally becomes the component of a geometrical object of a definite type: a connection form on P* (this word *component* is abusive, but we will use it).

Moreover, the *connection form has a global meaning and is therefore of great interest when topology matters* (as it will).

We shall complete the geometrization of the gauge potentials in Sec. III, where we give a definition of gauge transformations.

E. Covariant derivative in a principal fiber bundle

We shall first introduce the notion of a *horizontal lift*, or simply lift of vectors tangent to the base space, and then show that the covariant derivative $\mathfrak{D}_\mu$ is the lift of the derivative ∂_μ.

Definition: The lift $\tilde{X}$ of a vector field X on M is the unique horizontal field on P which projects onto X, that is:

$$\pi_*(\tilde{X}_u) = X_{\pi(u)} \quad \forall u \in P .$$

This definition presupposes, of course, the choice of a connection ω on P.

Let x^μ be the local coordinates in a neighborhood U_α as in (3) of the definition of a principal fiber bundle, Sec. I.A. The vector field $\partial_\mu = \partial/\partial x^\mu$ on U_α has a lift $\tilde{\partial}_\mu$ on $\pi^{-1}(U_\alpha)$.

Suppose σ_α is a section over U_α. Then from Eq. (2.4), we have

$$\omega_\alpha(\partial_\mu) = \omega(\sigma_{\alpha*}\partial_\mu) = A_{\mu\alpha} = \omega(\Sigma(A_{\mu\alpha})) ,$$

where $\Sigma(A)$ denotes the fundamental vector field associated to A. Hence

$$\omega(\sigma_{\alpha*}\partial_\mu - \Sigma(A_{\mu\alpha})) = 0 .$$

Consequently, $\sigma_{\alpha*}\partial_\mu - \Sigma(A_{\mu\alpha})$ is horizontal and clearly projects onto ∂_μ by π_*, since $\pi_* \circ \Sigma = 0$.

Therefore

$$\tilde{\partial}_\mu\big|_u = \sigma_{\alpha*}\partial_\mu - \Sigma(A_{\mu\alpha})$$

with $u=\sigma_\alpha(x)$.

We can identify $\sigma_{\alpha*}\partial_\mu$ with ∂_μ and the fundamental vector field $\Sigma(A_\mu)$ with the Lie algebra element A_μ and thus recover the usual covariant derivative, $\mathfrak{D}_\mu = \partial_\mu - A_\mu$, as follows.

The giving of a local section σ_α is equivalent to the trivialization of P over U_α. $\pi^{-1}(U_\alpha)$ can be viewed as a product $U_\alpha\times G$. Points on σ_α have coordinates (x,e), as shown in Fig. 16. σ_α reproduces U_α in the bundle

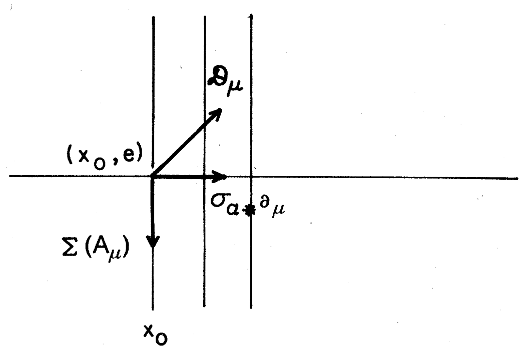

FIG. 16.

space. Consequently, we can identify the tangent vectors ∂_μ with $\sigma_{\alpha*}\partial_\mu$.

To see the action of $\Sigma(A_\mu)$ (which is a vector on the bundle) at the point $u_0=\sigma_\alpha(x_0)=(x_0,e)$, we have to draw the curve

$$u_t = u_0 \exp(tA_\mu) = (x_0, e^{tA_\mu})$$

in the bundle space and use it to compute a directional derivative.

Actually, u_t lies in the fiber $\pi^{-1}(x_0)$. Consider any function f defined on $\pi^{-1}(U_\alpha)$. The restriction of this function to the fiber $\pi^{-1}(x_0)$ is a function F defined on G, if we use the coordinates. The directional derivative along u_t clearly is the action of the Lie algebra element A_μ on F, at e (see the definition of the Lie algebra).

Thus $\mathfrak{D}_\mu = \partial_\mu - A_\mu$ is a shorthand notation for the lift of ∂_μ at the point $\sigma_\alpha(x)$. Note that this is section dependent; at some other point $\sigma_\alpha(x)\cdot g(x)$ on the same fiber, $\tilde{\partial}_\mu$ can be written $\mathfrak{D}'_\mu = \partial_\mu - A'_\mu$, with

$$A'_\mu = g^{-1}A_\mu g + g^{-1}\partial_\mu g . \tag{2.5}$$

It is true that the commutator of two fundamental vector fields is a fundamental vector field and that the mapping Σ respects not only the vector space structure (linear structure), but also the Lie algebra structure. It is not true, however, that the commutator of two horizontal vector fields (even lifts of vector fields on the base space) is horizontal. In fact, if X and Y are commuting vector fields on the base space, then $[\tilde{X},\tilde{Y}]$ is vertical.

As an example, $[\tilde{\partial}_\mu,\tilde{\partial}_\nu]$ is vertical. In order to compute this commutator we can use the local expression $\mathfrak{D}_\mu=\partial_\mu-A_\mu$, keeping in mind that A_μ varies along the fiber according to Eq. (2.5). Then

$$[\mathfrak{D}_\mu,\mathfrak{D}_\nu]=-(\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu]) .$$

The right-hand side is a fundamental vector field on the bundle space written as a Lie algebra element. The above derivation uses extensively the interplay between fundamental vector fields and Lie algebra elements provided by the isomorphism Σ.

In order to clarify this point, let us give a useful example of the action of $\mathfrak{D}_\mu$. Suppose ψ is a function with values in $G=SU(N)$, defined on P, and verifying $\psi(ua)=a^{-1}\cdot\psi(u)\cdot a$. Suppose also that G and its Lie algebra are given in a matrix representation (of the same dimension):

$$\mathfrak{D}_\mu\psi\,|_{u_0}=\partial_\mu\psi-\lim\frac{1}{t}\left[e^{-tA_\mu}\psi(u_0)e^{+tA_\mu}-\psi(u_0)\right],$$

$$\mathfrak{D}_\mu\psi\,|_{u_0}=\partial_\mu\psi+[A_\mu,\psi] .$$

We see that the action of the fundamental vector field $-\Sigma(A_\mu)$ can be identified with the action $+[A_\mu,\cdot]$, where A_μ is considered a Lie algebra element.

The actual action of $-\Sigma(A_\mu)$ depends essentially on the nature of the functions on which it is applied. In most cases, these functions will have a definite transformation law and will consequently be considered as sections of some associated bundle, where the action of $\mathfrak{D}_\mu$ will be the action of a *covariant derivative in an associated bundle* (see the definition below Sec. II.H).

The Lie algebra element

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu]$$

is the usual gauge field. It appears to be defined on any coordinate patch U_α, provided local sections over the U_α's are given. Strictly speaking, $F_{\mu\nu}$ carries an index α and ought to be written $F_{\alpha\mu\nu}$. Moreover, $F_{\alpha\mu\nu}$ and $F_{\beta\mu\nu}$ are related by:

$$F_{\beta\mu\nu}=Ad_{\psi_{\alpha\beta}^{-1}}\cdot F_{\alpha\mu\nu} . \tag{2.6}$$

F. Curvature form

1. Construction of the curvature form (Popov, 1975 and Chern, 1967).

From the gauge fields $F_{\mu\nu}$ we can construct a 2 form *on the bundle space* with values in $\mathfrak{a}(G)$, the curvature form Ω.

In fact, we first define a set of Lie-algebra-valued 2 forms Ω_α on the coordinate neighborhoods *of the base space*. Ω_α is defined by

$$\Omega_\alpha=\tfrac{1}{2}F_{\alpha\mu\nu}dx^\mu\wedge dx^\nu .$$

Note that from Eq. (2.6) we have

$$\Omega_\beta=Ad_{\psi_{\alpha\beta}^{-1}}\Omega_\alpha . \tag{2.7}$$

Recall that on $\pi^{-1}(U_\alpha)$ we have a map φ_α onto G which coordinatizes every fiber. φ_α takes the constant value e on the section σ_α (see Sec. I.C). Any point $u\in\pi^{-1}(U_\alpha)$ verifies

$$u=\sigma_\alpha(\pi(u))\cdot\varphi_\alpha(u) .$$

We use φ_α to construct Ω in $\pi^{-1}(U_\alpha)$:

Let $\Omega=Ad_{\varphi_\alpha^{-1}}(\pi^*\Omega_\alpha)$.

Relation (2.7) ensures the absence of discrepancy between the various possible definitions on any overlap. More precisely, if Y and Z are two vectors, tangent to the bundle in $\pi^{-1}(U_\alpha\cap U_\beta)$, we have

$$Ad\varphi_\alpha^{-1}(\pi^*\Omega_\alpha)(Y,Z)=Ad\varphi_\beta^{-1}(\pi^*\Omega_\beta)(Y,Z) .$$

Consequently, Ω is a well-defined 2 form in the bundle space.

2. Relations between the curvature form and the connection form. The exterior covariant differentiation

The curvature form Ω can also be expressed in terms of the connection form ω (globally) by

$$\Omega=d\omega+\tfrac{1}{2}[\omega,\omega] .$$

If X and Y are vectors tangent to the bundle, then

$$\Omega(X,Y)=d\omega(X,Y)+\tfrac{1}{2}[\omega(X),\omega(Y)] ,$$

where the bracket [,] denotes the Lie bracket in $\mathfrak{a}(G)$.

If we decompose X and Y in the above relation into their horizontal and vertical parts (hX,hY,vX,vY) and use the bilinearity of Ω, we have

$$\Omega(X,Y)=d\omega(vX,vY)+\tfrac{1}{2}[\omega(vX),\omega(vY)]$$
$$+d\omega(vX,hY)+d\omega(hX,vY)+d\omega(hX,hY) .$$

On the right-hand side of the above relation, only the last term survives. Consequently,

$$\Omega(X,Y)=d\omega(hX,hY) . \tag{2.8}$$

Definition: If ϕ is an r form on a principal fiber bundle P endowed with a connection, then $D\phi$ is the $r+1$ form on P such that

$$D\phi(X_1,\ldots,X_{r+1})=d\phi(hX_1,\ldots,hX_{r+1}) .$$

D is called the *exterior covariant differentiation*.

We can see from Eq. (2.8) that $\Omega=D\omega$. It is important to note that $D^2\neq0$, although $d^2=0$. However, $D\Omega=0$ [this relation is true for any connection and is called the Bianchi identity (Kobayashi *et al.*, 1963)].

3. Curvature form as a cross section valued form on space-time

Consider E the bundle associated to P, with standard fiber $\mathfrak{a}(G)$, on which G acts by the adjoint action. Call $\Gamma(E)$ the set of sections of E. We can associate to any connection on P a 2 form on M with values in $\Gamma(E)$:

Let X,Y be two vector fields on M. The lifts $\tilde{X}$ and $\tilde{Y}$ are right-invariant vector fields on P. $\Omega(\tilde{X},\tilde{Y})$ is then a function from P to $\mathfrak{a}(G)$ with the property that

$$\Omega(\tilde{X},\tilde{Y})\,|_{ua}=Ad_{a^{-1}}\cdot\Omega(\tilde{X},\tilde{Y})\,|_u .$$

From the lemma of Sec. I, we know that we can associate a section $\mathcal{S}$ of E to such a function.

Then let us define the 2 form R on M with values in $\Gamma(E)$ by:

$$R(X,Y)=\mathcal{S}\ .$$

[*Nota bene*: strictly speaking, R has values in E, but when applied to a vector *field* it has values in $\Gamma(E)$.]

This link between Ω and R is very simple, since: locally we can describe Ω by its projection

$$\Omega_\alpha=\sigma_\alpha^*\Omega=F_{\alpha\mu\nu}dx^\mu\wedge dx^\nu\ .$$

On the other hand, $\pi_E^{-1}(U_\alpha)\approx U_\alpha\times\mathfrak{A}(G)$. We could as well write R locally as $F_{\alpha\mu\nu}dx^\mu\wedge dx^\nu$. We can extend the $*$ operation and the inner product to the forms on M with values in $\Gamma(E)$.

The components of such forms are Lie-algebra-valued functions.

Formulas (2.1)–(2.3) are valid without any change.

We use a matrix (adjoint) representation of $\mathfrak{A}(G)$.

To define the scalar product of two p forms α and β, replace $\int_M\alpha\wedge*\beta$ by $\int_M \mathrm{tr}(\alpha\wedge*\beta)$, with the convention that if α is an r form and α' an r' form, then $\alpha\wedge\alpha'$ is the $(r+r')$ form defined by

$$\alpha\wedge\alpha'(X_1,\ldots,X_{r+r'})$$

$$=\frac{1}{(r+r')!}\sum\varepsilon(j,k)\,\alpha(X_{j_1}\cdots X_{j_1})\cdot\alpha'(X_{k_1}\cdots X_{k_{r'}})\ ,$$

where the sum runs over all partitions of $1\cdots(r+r')$ into $j_1\cdots j_r$ and $k_1\cdots k_{r'}$, and $\varepsilon(j,k)$ stands for the sign of the permutation

$$1\cdots(r+r')\to(j_1\cdots j_r,k_1\cdots k_{r'})\ .$$

The operation "tr" means trace of the matrix.

The product $(\alpha,\beta)=\int_M\mathrm{tr}(\alpha\wedge*\beta)$ is a scalar product. This product makes essential use of the Riemannian structure of M and the inner product on $\mathfrak{A}(G)$.

Note that we may extend this scalar product to the Sobolev completion of the set of C^∞ forms on M by using a modified integral (Choquet-Bruhat *et al.*, 1977).

As an example, we will use the form $*R$ and the scalar product

$$(R,R)=\int_M \mathrm{tr}(R\wedge*R)\ .$$

In terms of components, $(R,R)=\int F^2$. (R,R) is the *action* of the gauge field. Note that $*R$ has components

$$\tilde F_{\mu\nu}=\tfrac12\sqrt{|g|}\,\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}\ .$$

G. Holonomy group of a connection

A connection in a G bundle $P(M,G)$ can be used to define the notion of parallel transport.

Suppose (x_t) is a curve on M. A lift of this curve in P is a curve (u_t) in P such that $\pi(u_t)=x_t$ and such that all vectors tangent to (u_t) are horizontal.

Given any curve (x_t) on M, and any point u_0 [with $\pi(u_0)=x_0$], there exists a unique lift of (x_t) starting from u_0.

Let u be a point of P and $x=\pi(u)$. We can draw all closed loops on M starting and ending at x. All these curves have a lift starting from u, and ending at some point $v\in\pi^{-1}(x)$. Necessarily, $v=ug$ for some $g\in G$. The set of these g's clearly forms a subgroup of G.

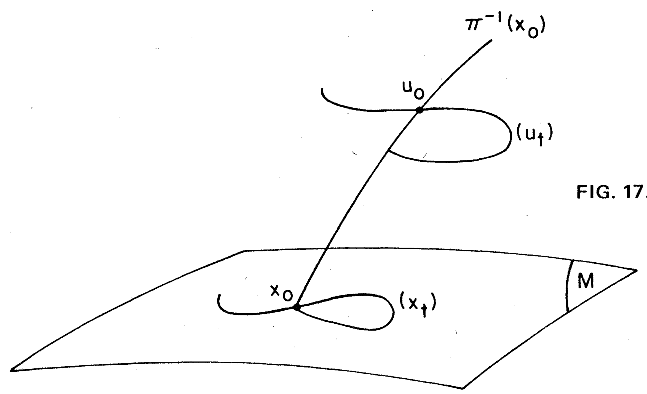

FIG. 17.

This group is called the holonomy group (of the connection) with reference point u: $\Phi(u)$.

It happens that if $\Phi(u)=G$ for some point and if M is connected, then Φ is independent of u. The connection is then said to be *irreducible*, and any two points of P can be joined by a horizontal curve in P.

The Lie algebra of the holonomy group at u is generated by $F_{\mu\nu}(u)$ and all its covariant derivatives $\mathfrak{D}_\rho F_{\mu\nu}(u)$, $\mathfrak{D}_\rho\mathfrak{D}_\sigma F_{\mu\nu}(u)$, . . . (see Loos, 1967).

H. Covariant derivative and exterior covariant differentiation in an associated bundle

Suppose P is a G bundle over M endowed with a connection and ε is a bundle associated to P, with standard fiber F. Let $\Gamma(\varepsilon)$ be the set of sections of ε.

1. Covariant derivative ∇

To any section $\varphi\in\Gamma(\varepsilon)$ we can associate an $\Gamma(\varepsilon)$-valued 1 form on M, denoted by $\nabla\varphi$, defined as follows:

Let X be a vector field on M. X has a unique lift $\tilde X$ on P. We know (from the lemma of Sec. I.E) that we can associate an F-valued function f on P to the section φ [f verifies $f(ua)=a^{-1}f(u)$]. $f'=\tilde Xf$ is also a F-valued function on P and verifies $f'(ua)=a^{-1}f'(u)$. Again by the lemma of Sec. I.E we can associate to f' a section of ε, which we denote $\nabla\varphi(X)$.

The mapping $X\to\nabla\varphi(X)$ so defined is linear in X. Thus we have constructed from the section φ a 1 form $\nabla\varphi$ on M, with values in $\Gamma(\varepsilon)$. $\nabla\varphi(X)$ is also denoted by $\nabla_X(\varphi)$. ∇ is called the covariant derivative, and it associates an $\Gamma(\varepsilon)$-valued 1 form on M to any element of $\Gamma(\varepsilon)$.

For example, if $\varepsilon=E$ (see above), and $X=\partial_\mu$, then $\nabla_\mu=\partial_\mu+[A_\mu,\cdot\,]$, locally.

2. Exterior covariant differentiation $\mathfrak{D}$

Define $\mathfrak{D}$ as follows: $\mathfrak{D}$ takes $\Gamma(\varepsilon)$-valued r forms into $\Gamma(\varepsilon)$-valued $r+1$ forms.

$\mathfrak{D}$ takes sections φ of ε into their covariant derivative $\nabla\varphi$.

If α is an r form,

$$\mathfrak{D}\alpha(X_1\cdots X_{r+1})=\sum(-1)^{j+1}\nabla_{X_j}\alpha(X_1\cdots X_{j-1},X_{j+1}\cdots X_{r+1})$$
$$+\sum(-1)^{i+j}\alpha([X_i,X_j],X_1\cdots X_{i-1},$$
$$X_{i+1}\cdots X_{j-1},X_{j+1}\cdots X_{r+1})$$

where the covariant derivative is defined using the connection on ε and the Riemannian connection on M.

Remark: We have seen that when M is a compact manifold equipped with a Riemannian metric, the set of forms on M with values in $\Gamma(\varepsilon)$ possesses a scalar product (,).

The operator $\mathfrak{D}$ has an adjoint $\mathfrak{D}^*$ such that, if α is any r form and β any $r+1$ form,

$$(\mathfrak{D}\alpha,\beta)=(\alpha,\mathfrak{D}^*\beta)\ .$$

If α is an r form, we have

$$\mathfrak{D}^*\alpha(X_1\cdots X_{r-1})=-\sum_{i=1}^{n}\nabla_{e_i}\alpha(e_i,X_1,\ \ldots\ ,X_{r-1})\ ,$$

where $\{e_i\}$ is an orthogonal basis of $T(M)$. Note also that the Bianchi identity $D\Omega=0$ reads $\mathfrak{D}R=0$.

III. IMPORTANCE OF BEING GLOBAL

A. Equivalence between gauge potentials and connection forms

The gauge potential is usually not given as a connection in a principal fiber bundle but rather as Lie-algebra-valued functions (A_μ) defined on R^4. However, it is always possible to consider $A_\mu(x)$ as the "components" of a connection form on the trivial principal fiber bundle $P=R^4\times G$.

We define first $\omega_\sigma=A_\mu\,dx^\mu$ as a 1 form on R^4: Define on P the unique connection form ω such that $\omega=\sigma^*(\omega)$. ω is a connection form on P but clearly depends on the choice of the section σ. This dependence is related to the gauge freedom.

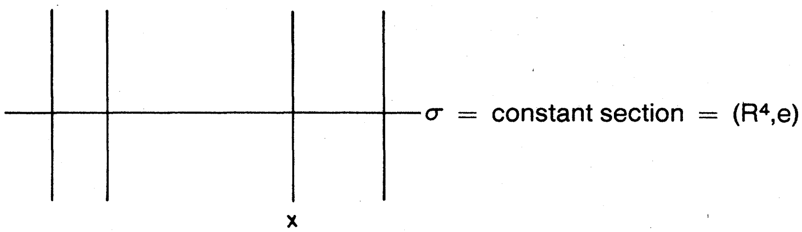

FIG. 18.

B. Gauge transformations on R^4

The basic notion of gauge transformation is that of a mapping γ: $R^4\to G$ which determines isospin rotations at every point in space-time. We will use the actual action of γ in $P=R^4\times G$ to define the gauge transformation of ω.

1. Coordinate-dependent definition of the gauge transformation

Suppose we have a global section of P. Call σ' the section of P defined by $\sigma'(x)=\sigma(x)\cdot\gamma(x)$. From $A_\mu(x)$ and σ we can construct ω (as above). However, from the same A_μ, but using σ', we can construct another connection form ω', with $\sigma^*(\omega)=\sigma'^*(\omega')$.

Provisory definition: We will say that ω' is the gauge transformation of ω by γ^{-1} and denote it by ${}^{\gamma^{-1}}\omega$.

In fact, σ (respectively, σ') coordinatizes P (in the same way a frame would define components for vectors in three-dimensional space). ω' has in the coordinate system transformed by γ (i.e., σ') the same components as ω in the original one (i.e., σ) (in analogy with rotations when a rotated vector has the components of the original vector, but in a frame transformed by the inverse rotation).

Note that the components of ω, with respect to σ, transform according to

$$A'_\mu=Ad_\gamma\cdot A_\mu+\gamma\,\partial_\mu\gamma^{-1}\ .$$

With such a definition, the gauge transformation could be viewed as a transformation of the section σ into another section of P, which induces a transformation of the connections.

2. The need to make the definition independent of the choice of σ

The same connection form ω has, with respect to a section $\Sigma=\sigma\cdot\varphi$, "coordinates" B_μ:

$$B_\mu=Ad_{\varphi^{-1}}\cdot A_\mu+\varphi^{-1}\partial_\mu\varphi\ .$$

The form ${}^\gamma\omega$, defined as above, has, with respect to Σ, the coordinates B'_μ:

$$B'_\mu=Ad_{\varphi^{-1}\gamma\varphi}\cdot B_\mu+(\varphi^{-1}\gamma\varphi)\partial_\mu(\varphi^{-1}\gamma\varphi)\ .$$

We must therefore associate to a different section Σ (related to σ by $\Sigma=\sigma\varphi$) *a different group-valued function* Γ, with

$$\Gamma(x)=\varphi^{-1}(x)\cdot\gamma(x)\cdot\varphi(x)\ ,$$

in order to induce the same transformation of the connection form.[2]

It appears that the gauge transformation can be described by a *section-dependent* group-valued function on R^4.

We can now give a *global* definition of the gauge transformation.

Definition: A gauge transformation in P is a mapping f: $P\to P$ such that (1) $\forall u\in P$, $\exists g(u)\in G\,|\,f(u)=u\cdot g(u)$ and (2) $g(ua)=a^{-1}\cdot g(u)\cdot a$ $\forall u\in P$, $\forall a\in G$. Note that $f(ua)=f(u)\cdot a$. f is said to be *equivariant*.

Equivalent definition (Atiyah, Hitchin, and Singer, 1978): A gauge transformation in P is an equivariant bundle isomorphism which induces the identity on the base space.

[2]Actually, $\Gamma(x)=(\phi^{-1}\gamma\phi)\cdot c(\Sigma,\sigma)$, where c belongs to the center of G and verifies $c(\sigma(x),\Sigma(x))\cdot c(\Sigma(x),\sigma(x))=e$ [Eq. (3.1)].

The action of the gauge transformation on a connection form is the one induced by the automorphism of P (pull back).

The gauge-valued function γ we started with is just $\gamma(x)=g(\sigma(x))$. The equivariance of f is the assumption that

$$\Gamma(x)=\varphi^{-1}(x)\cdot\gamma(x)\cdot\varphi(x)$$

(see above).

The interest of this global (coordinate-independent) definition is that it can be used for nontrivial bundles.

C. The finiteness of the action. Topology on the set of connections

The set $\mathfrak{C}$ of connection forms on $P=R^4\times G$ is an affine (convex) space and is consequently topologically trivial. Notice that if ω_0 and ω_1 are connection forms on P, then $\omega_t=(1-t)\omega_0+t\omega_1$ is a connection form on P, although $\mathfrak{C}$ is *not* a vector space.

However, the requirement of the finiteness of the action $\int \mathrm{tr}F^2<\infty$ leads to a restricted set of connections, which possesses a nontrivial topology.

We assume that a section σ of P is known, to describe the connections by using gauge potentials.

Since

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu],$$

the problem is to study the set of A's verifying

$$\int \mathrm{tr}[\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu]]^2<\infty,$$

This is clearly a nasty problem, but we can approach it in the following way.

We shall assume that $F_{\mu\nu}$ falls off at infinity in R^4 (faster than $1/|x|^2$). There might be $F_{\mu\nu}$'s which do not exhibit such behavior and such that $\int \mathrm{tr}F^2$ is finite. Our assumption means that F is approximately zero at large distances.

It happens that we know the solution of the equation (in A):

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu]$$

when $F=0$. This general solution is $A_\mu=g\cdot\partial_\mu g^{-1}$, where g is a mapping $R^4\to G$ (A_μ is a pure gauge). This ought to to give an approximate expression for A_μ at large distances. We shall then write, at least outside a finite region:

$$A_\mu=g\partial_\mu g^{-1}+B_\mu,$$

where B_μ is "small" at large distances.

Clearly, we do not have a canonical procedure to split A into two such parts. The point, however, is that the term $g\,\partial_\mu g^{-1}$ gives an accidental cancellation in the expression of $F_{\mu\nu}$. Thus the decrease properties of $F_{\mu\nu}$ are directly related to the smallness of B_μ at infinity.

As an example, consider the one-instanton solution of the equation of motion (Belavin *et al.*, 1975).

$$A_\mu=\frac{x^2}{x^2+a^2}\,g\,\partial_\mu g^{-1}$$

with

$$g(x)=\frac{x_0+i\vec{x}\cdot\vec{\sigma}}{|x|};$$

[$\vec{\sigma}$ are the Pauli matrices, $G=\mathrm{SU}(2)$].

$$A_\mu=g\partial_\mu g^{-1}+B_\mu$$

with

$$B_\mu=\frac{1}{x^2+a^2}\,g\,\partial_\mu g^{-1}.$$

B_μ is singular at $x=0$.

$$A_\mu\sim\frac{1}{|x|}\ \text{and}\ B_\mu\sim\frac{1}{|x|^3}\quad\text{when}\ |x|\to\infty.$$

If it were not of the form $g\,\partial_\mu g^{-1}$, the first term would not give a finite action.

In any case, we will consider g to be defined (and smooth) outside a finite sphere, that is to say, in a region of R^4 which is topologically like $S^3\times R$.

Once g is known, it determines an element of $\pi_3(G)$ (winding number) in the following way: Suppose we know g for distances (to the origin) larger than ρ_0. Then for any $\rho\geq\rho_0$ the restriction g_ρ of g to a sphere of radius ρ determines a mapping: $S^3\to G$. From the continuity of g, we see that any change in ρ induces a continuous deformation of g_ρ. Consequently, the homotopy class of g_ρ does not depend on ρ, and can be attributed to g and hence to A. (We avoid the sphere at infinity and do not suppose that g_ρ has a limit as $\rho\to\infty$.)

In the example given above, g is even independent of ρ.

In order to attribute the element of $\pi_3(G)$ so constructed *to a connection form on* P, we have to see what the effect of a change of section in P is.

Suppose σ' is another section of P, with $\sigma'(x)=\sigma(x)\cdot\gamma(x)$. If A_μ are the components of ω with respect to σ, then ω has, with respect to σ', the components

$$A'_\mu=(g\gamma)^{-1}\partial_\mu(g\gamma)+\gamma^{-1}B_\mu\gamma.$$

The element of $\pi_3(G)$ associated with $g\gamma$ by the above procedure is the same as the one associated with g, since γ is continuous on R^4. If G is a compact, simple Lie group, $\pi_3(G)\approx Z$.

We have then introduced a splitting of the space of connections on $P=R^4\times G$ into an infinite countable number of subsets (Belavin *et al.*, 1975).

D. Finiteness of the action. Compactification of R^4

What we have described so far is the asymptotic behavior of the Euclidean gauge potentials imposed by the finiteness of the action under a reasonable (from the physics point of view) assumption of the behavior of $F_{\mu\nu}$ at large distances.

In short, this asymptotic behavior singles out a region V in R^4 defined by $|x|>1/\varepsilon$ (ε is sufficiently small) and defines a map from V into G.

In what follows we shall make use of these asymptotic data in order to *construct* a possibly nontrivial fiber bundle.

Since g^{-1} acts naturally on σ in V, it induces a change of the components of ω:

$$A\to A'=gBg^{-1}.$$

A' has the same limit at infinity in all directions, namely, $A'(x)\to 0$ as $|x|\to\infty$.

It is natural to consider A' to be defined on the set

$$V_\infty = \{|x| > 1/\varepsilon\} \cup \{\infty\},$$

where ∞ is an extra point added to R^4 in such a way that its open neighborhoods are the regions $|x| > 1/\eta$. A' is then defined in the neighborhood of ∞.

$R^4 \cup \infty$ with the above topology is a compact space.

If we also demand differentiable structure, this compact space is equivalent to S^4 (Lüscher and Mack, 1975).

We can construct a principal fiber bundle $P = (S^4, G)$ over S^4 using two patches R^4 and V_∞, and g as a transition function defined on the overlap $V_\infty \cap R^4 = \{|x| > 1/\varepsilon\}$. P is reconstructed from the coordinate bundle $(S^4, G, \{R^4, V_\infty\}, g)$.

Moreover, A and A' are the components of a connection form in $P(S^4, G)$ corresponding to the natural local sections (associated with the above covering of S^4).

To get such a compactification of the base space is satisfactory, *but it might contain more than the original requirement, namely, the finiteness of the action.* On S^4, due to compactness, the integrability of $\int \mathrm{tr} F^2$ is, for example, a consequence of the continuity of the integrand. However, we shall use locally C^∞ functions on S^4. In particular, C^∞ at the point ∞ ($\in S^4$) is a strong constraint on the behavior of the functions at large distances in R^4. To see this we can use the stereographic projection from S^4 to R^4 (Jackiw and Rebbi, 1976).

Moreover, the equations of motion (see Sec. IV) are conformally invariant. The use of S^4 provides us with a space-time manifold where the conformal transformations act.

Actually, S^4 *justifies the use of bundle theory* because the topology of the set of gauge potentials with finite action is retained by the classification of the principal G bundles over S^4.

It is known (Steenrod, 1951, and Bott and Matheu, 1968) that the set of inequivalent classes of principal G bundles over S^4 is in one-to-one correspondence with $\pi_3(G)$. *It is remarkable that the above construction of principal coordinate bundles* $(S^4, G, \{R^4, V_\infty\}, g)$ *provides representatives for every one of these equivalence classes* (see below Sec. III.F).

E. Gauge transformation on S^4

The global definition given in Sec. III.B can be taken without any change for the bundle $P(S^4, G)$: a gauge transformation is an equivariant automorphism f of P, inducing the identity map on the base space [to such an f is associated a mapping $\gamma : P \to G$].

Since in general there is no global section of P, a gauge transformation is described by a pair of section-dependent group-valued functions.

Suppose $\{U_1, U_2\}$ is a covering of S^4, and $\{\sigma_1, \sigma_2\}$ are local sections over U_1 and U_2 (for example, $U_1 = R^4$, $U_2 = V_\infty$).

A gauge transformation will be described by two functions γ_1 and γ_2, defined, respectively, on U_1 and U_2:

$$\gamma_1(x) = \gamma(\sigma_1(x)), \quad \gamma_2(x) = \gamma(\sigma_2(x));$$

σ_1 and σ_2 are related by a transition function ψ_{12} on the overlap $U_1 \cap U_2$.

As a consequence of the equivariance of f, γ_1 and γ_2 are also related on $U_1 \cap U_2$:

$$\gamma_2(x) = \psi_{12}^{-1}(x) \cdot \gamma_1(x) \cdot \psi_{12}(x).$$

We shall study the group of gauge transformations in Sec. IV.

F. S^4 vs R^4

Note that the gauge transformation f would not change the coordinate bundle defined by the two local sections σ_1 and σ_2, since the equivariance of f implies the conservation of the transition function:

$$\sigma_2(x)\gamma_2(x) = \sigma_1(x)\gamma_1(x) \cdot \psi_{12}(x) \forall x \in U_1 \cap U_2.$$

In a sense, the gauge transformations are transformations which preserve the coordinate bundles.

Let us return to the case of $P = R^4 \times G$. Suppose σ is a global section of $P = R^4 \times G$. The one-instanton gauge potential (see above) provides us with a function g which is well defined outside the origin. Call V_1 the open $R^4 - \{0\}$, and V_0 the interior of the sphere of radius 1. We can construct a section σ_1 over V_1 with the action of g on σ: $\sigma_1(x) = \sigma(x) g(x) \;\; \forall x \in V_1$. P can be described by the coordinate bundle $(R^4, G, \{V_0, V_1\}, \psi_{01} = g)$. We know we can associate a "winding number" to any function defined on $V_0 \cap V_1$ (like ψ_{01}), as above, by considering the restriction to a sphere of radius ρ $(0 < \rho < 1)$. We also know that we get equivalent coordinate bundles by changes of the local sections σ (over V_0) and σ_1 (over V_1):

$$\sigma \to \sigma'(x) = \sigma(x) g_0(x) \;\; \forall x \in V_0,$$

$$\sigma_1 \to \sigma_1'(x) = \sigma_1(x) g_1(x) \;\; \forall x \in V_1.$$

The transition functions are then related by

$$\psi_{01}' = g_0^{-1} \cdot \psi_{01} \cdot g_1.$$

Over V_1 it is possible to choose $g_1 = g$, *but this becomes impossible when we add an extra point* (∞) *to* R^4, *since* g_1 *would not be continuous at that point.*

Actually, when we replace R^4 by S^4, we impose the condition that g_1 be smooth on $V_\infty = V_1 \cup \infty$.

The "winding number" associated with g_1 is necessarily zero, since any sphere of radius ρ can be continuously shrunk to one point in V. Since this number adds (algebraically) in the product $g_0^{-1} \cdot \psi_{01} \cdot g_1$, we attribute the same number to ψ_{01}' and to ψ_{01}.

What happens with the compactified version is that the topology of S^4 is such that the equivalence between coordinate bundles preserves as much of the topology of the bundle as the gauge transformation does.

It is the reason—inside the preexisting theory of fiber bundles—G bundles over S^4 retain more topological information than bundles over R^4.

G. Classification of principal bundles

The problem of classifying G bundles over a given manifold M, that is to say, the description of all nonequivalent principal bundles having G as a structure group and M as a base space, involves a delicate interplay between the topologies of M and G.

We give here some of the properties and the constructions used in the theory of classification. We introduce the necessary definitions but do not give the proofs of the theorems, supposing that the notions of *homotopy* and *exact sequence* are known, and that when homotopy matters, the sets are supposed to be pointed sets [see (Steenrod, 1951)].

Preliminary definition: Consider first a principal G bundle $\xi(B,G)$. For any map f: $M \to B$ it is possible to construct an *induced bundle* $f^*(\xi)$ over M, by gluing over any point $x \in M$ a copy of the fiber over $y=f(x)$. The bundle $f^*(\xi)$ is also called *pull-back* of ξ by f.

It so happens that the description of all equivalence classes of G bundles over M can be computed by inducing (as above) all possible principal bundles from the so-called universal bundle.

Theorem (Milnor construction) (Milnor, 1956a and 1956b; Husemoller, 1966): For each principal G bundle over a paracompact space M there exists a universal bundle $EG(BG,G)$ with the following properties: (i) For each principal G bundle ξ over M there exists a map f: $M \to BG$ such that ξ and the pull-back $f^*(EG)$ are isomorphic. (ii) If f_1, f_2: $M \to BG$ are two maps such that $f_1^*(EG)$ and $f_2^*(EG)$ are isomorphic, then f_1 and f_2 are homotopic.

Thus the classification scheme reduces to the study of homotopy classes of maps $M \to BG$.

The space BG, which is in general not a manifold, is called the classifying space of G.

If we denote by $\mathcal{B}_G(M)$ the set of equivalence classes of G bundles over M and by $[M,BG]$ the set of homotopy classes of maps f: $M \to BG$, we have, due to the above theorem, a one-to-one correspondence between $[M,BG]$ and $\mathcal{B}_G(M)$.

1. G bundles over S^4 We now specialize M to be S^4

The existence of the fibration of EG by G with base space BG gives rise to the bundle sequence of EG (Steenrod, 1951; Kobayashi and Nomizu, 1969).

$$\to \pi_k(EG) \to \pi_k(BG) \to \pi_{k-1}(G) \to \pi_{k-1}(EG) \to .$$

For G a compact, simple, and simply connected Lie group (e.g., $SU(n), Sp(n)$, Spin$(n) n \geq 7, G_2, F_4, F_8, \overline{E}_6, \overline{E}_7$) $\pi_1(G) = \pi_2(G) = 0$, and $\pi_3(G) \neq 0$. In fact, $\pi_3(G) \simeq Z$. Hence, from the universality of EG, we get the following short exact sequence:

$$0 \to \pi_4(BG) \to \pi_3(G) \to 0 .$$

The exactness of the sequence implies that $\pi_4(G) \simeq \pi_3(G) \simeq Z$. However, $\pi_4(BG) \equiv [S^4, BG]$. Hence

$$\mathcal{B}_G(S^4) \simeq \pi_3(G) \simeq Z .$$

There exists, therefore, a countable infinity of inequivalent G bundles over S^4.

The above classification has also a cohomological description. It is clear from the above analysis that the homotopy groups of BG beyond the fourth do not play an important role. The idea then is to approximate BG, a topologically complicated space, by what is called a Postnikov system (see Avis and Isham, 1978, and references therein). As a result, when G is a Lie group of the type described above, one could show that

$$\mathcal{B}_G(S^4) \simeq H^4(S^4, Z) ,$$

where $H^4(S^4,Z)$ is the fourth cohomology group of S^4 with values in Z.

In conclusion, we see that the classification scheme is related to the cohomology of the base space, with values in a group (Z), which depends on the homotopy properties of the structure group. This ties up with the remarks we made in Sec. I.D, where we claimed that the topology of the bundle space comes from the topology of the base space and the topology of the group.

2. Computation of the class: the Chern-Weil theory of characteristic classes

We know from de Rham's theorem (de Rahm, 1960) that real cohomology classes [i.e., elements in $H^*(M,R)$] can be represented by closed forms on M. With the help of the connections defined on a given principal bundle $P(M,G)$ it is possible to construct a privileged set of closed forms of even degree on M (using the Weil homomorphism). Their cohomology classes do not depend on the choice of the connection on P but only on P itself. Consequently it is possible to associate an element of $H^*(M,R)$ to a given bundle $P(M,G)$.

Moreover, these elements of $H^*(M,R)$ happen to verify axioms of definition of the so-called Chern classes. They belong to $H^*(M,Z)$ and classify the bundles $P(M,G)$ (when G is one of the examples given above).

3. Weil Homomorphism (Kobayashi and Nomizu, 1969; Dupond, 1978)

G is assumed to be a closed subgroup of $GL(n,C)$. Consider a real-valued symmetric multilinear function $W_k(A_1,\ldots,A_k)$ such that

$$\forall g \in G ,$$

$$W_k(Ad_g \cdot A_1, \ldots, Ad_g \cdot A_k) = W_k(A_1,\ldots,A_k) .$$

and whose arguments belong to the Lie algebra $\mathcal{G}(G)$.

W_k is said to be *adG* invariant.

We call $I^k(G)$ the set of all such functions.

$$I(G) = \sum_{k=0}^{\infty} I^k(G)$$

possesses a natural structure of algebra with the product defined by

$$\forall f \in I^\kappa(G) , \ \forall g \in I^\rho(G) , \ fg \in I^{\kappa+\rho}(G)$$

and

$$f \cdot g(t_1 \cdots t_{\kappa+\rho}) = \frac{1}{(\kappa+\rho)!} \sum_\sigma f(t_{\sigma(1)} \cdots t_{\sigma(\kappa)}) \times g(t_{\sigma(\kappa+1)} \cdots t_{\sigma(\kappa+\rho)}) .$$

Suppose we have a connection ω in P. Since the curvature form Ω takes values in $\mathcal{G}(G)$, it is possible to define the $2k$ form $W_k(\Omega)$ on the bundle space by:

$$W_k(\Omega)(X_1,\ldots,X_{2k}) = \frac{1}{(2k)!} \sum \varepsilon_\sigma W_k(\Omega(X_{\sigma(1)}, X_{\sigma(2)}), \ldots, \Omega(X_{\sigma(2k-1)}, X_{\sigma(2k)})),$$

where the summation is over all permutations of

$(1, 2, \ldots, 2k)$ and ε_σ denotes the signature of σ.

Theorem: 1. For any invariant function W_k, the $2k$ form $W_k(\Omega)$ on P projects to a unique closed $2k$ form $\overline{W}_k$ on M with $W_k(\Omega) = \pi^* \cdot \overline{W}_k$ (π is the projection defined in P).

2. The de Rham cohomology class of the closed form $\overline{W}_k$ does not depend on the choice of the connection on P. Moreover, the map W: $I(G) \to H^*(M, R)$ is an algebra homomorphism (Weil homomorphism).

It can be shown that $I(G)$ can be identified with the algebra of (adG) invariant polynomial functions on $\mathfrak{G}(G)$, we can thus introduce a basis of the algebra.

Define the functions $f_0 \cdots f_n$ on $\mathfrak{G}(G)$ by

$$\det\left(\lambda I_n - \frac{1}{2i\pi}X\right) = \sum_{k=0}^{n} f_k(X)\lambda^{n-k}$$

$(X \in \mathfrak{G}(G))$.

The function f_k belongs to $I^k(G)$. For any k, there exists a unique closed $2k$ form λ_k on M such that $\pi^*\gamma_k = f_k(\Omega)$.

Theorem: The forms γ_k generate the algebra of characteristic classes of $P(M, G)$. We can express γ_k by using a matrix-valued 2 form Ω^i_j:

$$\pi^*\gamma_k = \frac{(-1)^k}{(2i\pi)^k \cdot k!} \sum \delta^{j_1 \cdots j_k}_{i_1 \cdots i_k} \Omega^{i_1}_{j_1} \wedge \cdots \wedge \Omega^{i_k}_{j_k}, \tag{3.2}$$

where the summation is taken over all the ordered subsets $i_1 \cdots i_k$ of k elements of $1 \cdots n$, and all permutations $\binom{j_1 \cdots j_k}{i_1 \cdots i_k}$. Notice that the $2k$ forms in (3.2) are all gauge invariant.

It is instructive to look at (3.2) locally. Let U be one of the open neighborhoods belonging to the covering of M. On $\pi^{-1}(U)$ we know there is a group-valued function φ_U (cf. definition of the principal G bundle):

$$\pi^*(\gamma_k|_U) = f_k(\Omega|_{\pi^{-1}(U)}),$$

$$\Omega|_{\pi^{-1}(U)} = Ad_{\varphi_U^{-1}} \cdot \pi^*\Omega_U .$$

We see that

$$\gamma_k|_U = \frac{(-1)^k}{(2i\pi)^k \cdot k!} \sum \delta^{j_1 \cdots j_k}_{i_1 \cdots i_k} (\Omega_U)^{i_1}_{j_1} \wedge \cdots \wedge (\Omega_U)^{i_k}_{j_k},$$

where

$$(\Omega_U)^i_j = \tfrac{1}{2} F^a_{\mu\nu} (T_a)^i_j \, dx^\mu \wedge dx^\nu \, (T_a \in \mathfrak{G}(G)).$$

Example $M = S^4$:

$$\gamma_1|_U = -(1/2i\pi)\mathrm{tr}\Omega_U ,$$

$$\gamma_1 = 0 \quad \text{if } G = \mathrm{SU}(n),$$

$$\gamma_2 = \frac{1}{8\pi^2} \mathrm{tr}\, \Omega_U \wedge \Omega_U$$

$$= \frac{1}{32\pi^2} \sqrt{g} \cdot \varepsilon_{\mu\nu\alpha\beta} F^a_{\mu\nu} F^{a\alpha\beta} d^4x ,$$

$$\gamma_k = 0 \quad \forall k \geq 3 .$$

The bundles $P(S^4, \mathrm{SU}(n))$ are characterized by γ_2 considered as representing an element of $H^4(M, Z)$. Due to its gauge invariance, γ_2 could, in principle, be used as part of the Lagrangian density in the YM action.

The bundles $P(S^4, \mathrm{SU}(n))$ are characterized by the integer $n = \int \gamma_2 \cdot dM$ (= Chern number = instanton number). n can be expressed as an integral over R^4:

$$n = \frac{1}{32\pi^2} \int \mathrm{tr} \varepsilon_{\mu\nu\alpha\beta} F^a_{\mu\nu} F^a_{\alpha\beta} d^4x$$

$$= \frac{1}{16\pi^2} \int \mathrm{tr}\, F \cdot \tilde{F} \, d^4x .$$

It remains to show that n is the winding number associated to the connection which enabled us to construct the bundle over S^4. This is done in Belavin et al. (1975) and Crewther (1978).

Note: It is crucial to note in which sense the use of S^4 as a space-time allows the retention of the topology of the space of connections. We have seen that there exists a splitting of the set of connections on $P = R^4 \times G$ into a countable infinity of subsets (provided $F \to 0$ at infinity). We see now that all the connections pertaining to one of these subsets can be transported onto *one* G bundle over S^4. Two connections with different "winding numbers" *are not defined on the same* bundle over S^4.

We will see that the use of S^4 also affects the set of allowed gauge transformations, which acquires a topology that the set of gauge transformations on R^4 does not have *a priori*.

Remark: We implicitly admitted that a connection on $P = R^4 \times G$ giving rise to a sufficiently fast decreasing gauge field F would give an integer value to the integral over R^4:

$$n = \frac{1}{16\pi^2} \int \mathrm{tr}\, F \cdot \tilde{F} \, d^4x .$$

A priori the mere convergence of the action $a = \int \mathrm{tr} F^2 d^4x$ implies 1. the finiteness of the integral $\int \mathrm{tr} F \cdot \tilde{F} d^4x$, and 2. $|n| \leq a$.

But it is still an open problem to clarify what supplementary hypotheses are to be added to the finiteness of the action to lead to the construction of some bundle over S^4 and *a fortiori ensure* that n is an integer. Probably a fast decrease of F at infinity is sufficient, but we have not proved it.

IV. GROUP OF GAUGE TRANSFORMATIONS

A. Group operation on the set $\mathcal{G}$ of gauge transformations

We have defined gauge transformations as equivariant automorphisms of some G bundle P, inducing the identity map on the base space of P. The interesting geometrical objects defined on P are the connection 1 forms, the components of which are the gauge potentials. It has to be emphasized that the choice of a definite G bundle (over S^4) amounts to choosing a given Chern class (instanton number). In what follows, P should read P_k ($k \in Z$).

We can define the product of two gauge transformations as their composition as mappings: $P \to P$.

This composition gives a gauge transformation, and the set $\mathcal{G}$ of gauge transformations acquires the structure of a group.

Any gauge transformation f: $P \to P$ can be equivalently described by a mapping γ: $P \to G$ [recalling that

$$f(u) = u \cdot \gamma(u) \quad \forall u \in P$$

and that $\gamma(ua) = a^{-1}\gamma(u)a$]. The group operation in $\mathcal{G}$

yields a pointwise product for the γ's: if $f_1(u) = u \cdot \gamma_1(u)$ and $f_2(u) = u \cdot \gamma_2(u)$, then $f_2 \circ f_1(u) = u \cdot (\gamma_1 \cdot \gamma_2)(u)$.

Finally, the mapping γ: $P \to G$ itself may be given in terms of a family of local mappings: γ_α: $U_\alpha \to G$ (from the base space to the structure group) with compatibility relations

$$\gamma_\beta(x) = \psi_{\alpha\beta}^{-1}(x)\gamma_\alpha(x)\psi_{\alpha\beta}(x) \;\; \forall\, x \in U_\alpha \cap U_\beta \,.$$

The latter description is equivalent to the following one: let us construct the bundle $\mathfrak{B}$ associated to P, with standard fiber G, G acting on itself by the adjoint map $[a(g) = \mathrm{Int}_a(g) = aga^{-1}]$. We know from the lemma of Sec. I.C that there is a one-to-one correspondence between the mappings γ: $P \to G$, which describe the gauge transformations, and the sections of $\mathfrak{B}$. Consequently, the group $\mathcal{G}$ of gauge transformations can be identified with the set $\Gamma(\mathfrak{B})$ of sections of $\mathfrak{B}$. In order to visualize the product operation in $\Gamma(\mathfrak{B})$ we have to use a local trivialization of $\mathfrak{B}$ (related to a covering $\{U_\alpha\}$ of the base space) and perform a pointwise product in G. Note that the absence of discrepancy between the two possible definitions on any overlap $U_\alpha \cap U_\beta$ comes from the property

$$\mathrm{Int}_{\psi_{\alpha\beta}^{-1}}(\gamma_1\gamma_2) = \mathrm{Int}_{\psi_{\alpha\beta}^{-1}}(\gamma_1) \cdot \mathrm{Int}_{\psi_{\alpha\beta}^{-1}}(\gamma_2) \,.$$

Note also that, although its fiber is the group G, the bundle $\mathfrak{B}$ is not a principal fiber bundle: the action of G on itself is not free. $\mathfrak{B}$ will have global sections. Moreover, suppose that some point $b \in \pi_{\mathfrak{B}}^{-1}(U_\alpha \cap U_\beta)$ has coordinates (x, e) "over U_α." Then it necessarily has coordinates (x, e) "over U_β." Consequently, the word unit element has a meaning globally, in the bundle space. Actually, the same applies to all elements of the center Z of G: we may speak of sections of $\mathfrak{B}$ with value $z\,(z \in Z)$. The unit section (constantly equal to the unit element e) is the unit element of $\mathcal{G}$. The set z of Z-valued sections is the center of $\mathcal{G}$. We shall denote it by $\mathfrak{z}$.

B. Lie algebra of $\mathcal{G}$

Consider the constant unit section s of $\mathfrak{B}$. Through any point of $\mathfrak{B}$ passes one fiber. Using the local triviality of $\mathfrak{B}$ over patches U_α, we may identify this fiber with the group G. We can draw vectors tangent to the fiber at any point of s. The group operation and its linear differential allow us to transport these vectors to any point of $\mathfrak{B}$ and to define vector fields on $\mathfrak{B}$. The restriction of such a vector field to a fiber $\pi_{\mathfrak{B}}^{-1}(x)$ can be identified with an element A_α of $\mathfrak{A}(G)\,(x \in U_\alpha)$. If $x \in U_\alpha \cap U_\beta$, the same vector field over $\pi_{\mathfrak{B}}^{-1}(x)$ can be identified with a different element A_β of the Lie algebra if we use the trivialization of $\pi_{\mathfrak{B}}^{-1}(U_\beta)$.

We have

$$A_\beta = Ad_{\psi_{\alpha\beta}^{-1}} \cdot A_\alpha \,.$$

As a consequence, the vector field we have determined on $\mathfrak{B}$ can be viewed as a section of the bundle E associated to P, with standard fiber $\mathfrak{A}(G)$ with the adjoint action of G on $\mathfrak{A}(G)$ (cf. Sec. II.F).

Call $\Gamma(E)$ the set of sections of E. $\Gamma(E)$ is the Lie algebra of $\mathcal{G} \equiv \Gamma(\mathfrak{B})$. $\Gamma(E)$ is an infinite-dimensional module.

Any section of $\mathfrak{B}$ can be written as $\exp(\sigma)$, where $\sigma \in \Gamma(E)$. In order to define the operation of taking the exponential of a section of E, we again have to use the local triviality of the associated bundle: locally σ determines mappings: $U_\alpha \to \mathfrak{A}(G)$. We can define mappings $U_\alpha \to G$ by using the exp map of G. These mappings verify the proper compatibility relations on any overlap $U_\alpha \cap U_\beta$; thus they define a section s of $\mathfrak{B}$. We will say that $s = \exp(\sigma)$. If G is connected (this will be the case), any element of G can be written as the exponential of some element in $\mathfrak{A}(G)$. In this case, any section s of $\mathfrak{B}$ can be written as $s = \exp(\sigma)$ $(\sigma \in \Gamma(E))$.

C. Influence of the topology of space-time on the group of gauge transformations

Suppose that space-time is S^4. We have seen that a gauge transformation is described by two mappings: γ_1: $U_1 \to G$ and γ_2: $U_2 \to G$ (with $U_1 \approx R^4, U_2 \approx V_\infty$; see above). Clearly, γ_1 determines a gauge transformation over R^4, *but not all gauge transformations over R^4 can be obtained in such a way:* γ_1 verifies the compatibility relation

$$\gamma_1(x) = \psi_{12}^{-1}(x) \cdot \gamma_2(x) \cdot \psi_{12}(x) \;\; \forall x \in U_1 \cap U_2 \,.$$

The mere continuity of γ_2 at the point ∞ is already a strong restriction on the behavior "at infinity" of γ_1.

Suppose, for example, that the bundle P is trivial. Then $\psi_{12} = e$, and if $g = \gamma_2(\infty)$, we have $\gamma_1(x) \to g$ as $|x| \to \infty$. γ_1 is bound to be defined on S^4. The topology of the set of functions from S^4 to G is much richer than the topology of the set of functions from R^4 to G. We shall investigate more deeply the topology of $\mathcal{G}$ when the base space is S^4, but the method is usual in bundle theory and could be applied to other base spaces. This study is given in Singer (1978), when $G = \mathrm{SU}(n)$ (see later Sec. VI.C).

V. EQUATIONS OF MOTION OF A PURE YANG-MILLS THEORY

A. Equations of motion and (anti-) self-duality

The classical equations of motion of the gauge fields (or rather the gauge potentials) are the Euler-Lagrange equations obtained by minimizing the action $a = \int \mathrm{tr} F^2 d^4x$.

Suppose P is a principal G bundle over S^4. Once the choice of P is made, we have restricted ourselves to a given instanton number.

The action is a functional defined on the set of connections on P.

In Sec. II.F we have constructed a form $R(\omega)$ on M, with values in $\Gamma(E)$, and we have seen that $a = (R, R)$:

$$a = \int_{S^4} (\mathrm{tr} R \wedge {*R}) dM \,.$$

The solutions of the classical equations of motion are critical points of the functional a (in the sense of Morse (Milnor, 1963)).

Note that the search for critical points of a function usually invites the study of the topology of the space on which it is defined (Milnor, 1963). However, the space $\mathcal{C}$ of connection forms on P is an affine space and is then contractible and has a trivial topology

$[\pi_j(\mathcal{C}) = 0\ \forall j \in N]$. What happens is that the functional $a(\omega)$ is gauge invariant.

Suppose $\omega \in \mathcal{C}$ and ω' = the gauge transformation of ω. Then

$$a(\omega') = a(\omega) .$$

a is thus actually defined on the quotient $\mathcal{C}/\mathcal{G}$.

The quotient $\mathcal{C}/\mathcal{G}$ has a nontrivial topology[3] since $\mathcal{G}$ has a nontrivial homotopy (see Sec. VI).

The equation of motion can be computed locally from the expression of $a = \int \mathrm{tr} F_{\mu\nu}^2$. These equations are known to be $\mathfrak{D}^\mu F_{\mu\nu} = 0$, where $\mathfrak{D}_\mu$ is the covariant derivative acting on $F_{\mu\nu}$.

It is very easy to confirm that these equations have a global expression using the form R introduced earlier and the adjoint $\mathfrak{D}^*$ of the exterior covariant differentiation $\mathfrak{D}$:

$$\mathfrak{D}^* R = 0 ,$$

or equivalently

$$\mathfrak{D}(*R) = 0 .$$

It so happens that the operation $*$ verifies $*^2 = 1$ (as is easy to check) on S^4.

The space A^2 of 2 forms on S^4 [with values in $\Gamma(E)$] can be decomposed into the sum of two supplementary linear subspaces $(A^2 = A_+^2 \oplus A_-^2)$, the spaces of self-dual (respectively, antiself-dual) forms. A self-dual form verifies $*\varphi = \varphi$ (respectively, an antiself-dual form verifies $*\varphi = -\varphi$).

We immediately see that, if R is self-dual (respectively, antiself-dual), the Bianchi identity ensures that the equation of motion is verified. It is still an open problem to decide whether or not there exist nonself-dual (respectively, nonantiself-dual) solutions of the equations of motion (Bourguignon, *et al.*, 1979; Daniel *et al.*, 1978; Flume, 1978).

A large literature exists on the geometrical meaning of the self-duality equations, leading to a complete study of the set of their solutions, and even to an explicit construction of these solutions. We will not get into the study of this widely discussed subject here. See Atiyah, Hitchin, and Singer (1978), Atiyah and Ward (1977); Atiyah, Hitchin, Drinfield, and Manin (1978); Corrigan, Fairlie, Goddard, and Yates (1978), Corrigan, Fairlie, Templeton and Goddard (1978), Christ et al. (1978), Bernard *et al.* (1977), and Madore *et al.* (1979).

B. First and second variations of the Yang-Mills action on the Euclidean sphere

In the previous section we saw that the pure Yang-Mills action on the Euclidean sphere is a function on the affine space $\mathcal{C}$. Fixing a point in $\mathcal{C}$ turns it into an infinite-dimensional vector space isomorphic to the space of smooth $\Gamma(E)$ valued 1 forms on S^4 (see Sec. II.D.3). Indeed, consider a family, A^t, of connections on a straight line through A:

$$A^t = A + t\eta .$$

Locally, $(A^t_{\alpha\mu} - A_{\alpha\mu})dx^\mu$ is a 1 form taking values in the Lie algebra of the gauge group in the adjoint representation, and the above assertion follows.

We shall now use the above family of connections in order to calculate the first and second variations of the action.

Clearly, such variations are of great importance in physics. The first variation leads to the dynamical equations of motion, whereas the second determines a fluctuation operator which governs the quantum fluctuations about the background field (i.e., the vector potential associated with the connection A).

We remark here that it is sufficient for the action to vary along straight lines, because $\mathcal{C}$ is an affine space.

Let R^t be the curvature corresponding to A^t. Then

$$R^t = R + t\mathfrak{D}\eta + \tfrac{1}{2}t^2[\eta, \eta] ,$$

where $\mathfrak{D}$ is the exterior covariant derivative acting on 1 forms taking values in the Lie algebra (see Sec. II.H.2). Consequently (Atiyah and Bott, 1978; Bourguignon *et al.*, 1979),

$$a(A^t) = a(A) + t(\mathfrak{D}\eta, R) + t^2\{(\mathfrak{D}\eta, \mathfrak{D}\eta) + (\eta, *[*R, \eta])\} + O(\eta^3) .$$

Hence,

$$\left.\frac{da(A^t)}{dt}\right|_{t=0} = (\mathfrak{D}\eta, R) .$$

Thus, if A corresponds to a stationary point, the first variation must vanish. Consequently,

$$(\mathfrak{D}\eta, R) = 0 \quad \text{or} \quad (\eta, \mathfrak{D}^* R) = 0 .$$

The equations of motion now read

$$\mathfrak{D}^* R = 0$$

where $\mathfrak{D}^*$ is the adjoint of $\mathfrak{D}$. It can be shown that, on S^4, $\mathfrak{D}^* = -*\mathfrak{D}*$.

The second variation gives

$$\frac{1}{2}\left.\frac{d^2a(A^t)}{dt^2}\right|_{t=0} = (\eta, \mathfrak{D}^*\mathfrak{D}\eta + *[*R, \eta]) .$$

This yields immediately the Hessian of the action as a quadratic form on the tangent space to the space of connections at A. Consequently, we have an explicit expression for the fluctuation operator (which governs quantum fluctuations around the background field A).

It is of interest to take A to be a k-instanton configuration. More explicitly, A gives rise to self-dual curvature $R^* = R$, corresponding to instanton number k. In this case, the fluctuation operator, $\tilde{\Delta}^1_A$, (the superscript 1 indicates that it is acting on 1 forms) reduces to

$$\tilde{\Delta}^1_A = 2\mathfrak{D}^* P \mathfrak{D} ,$$

where $P = \tfrac{1}{2}(1 - *)$ is the projection operator on the antiself-dual 2 forms with values in the Lie algebra.

Now, if the variation $A^t = A + t\eta$ is such that R^t is a

[3] This has nothing to do with the topology introduced in Sec. III B. In Sec. III we had a splitting of the set of connections on $P = R^4 \times G$ into an infinity of subsets. This splitting led to different bundles over S^4. Here we have chosen a bundle over S^4, and the topology appears when we quotient out the redundancy of the description of the fields due to the gauge invariance. We have seen in Sec. IV how much S^4 matters.

solution of the equations of motion, then

$$\tilde{\Delta}_A^1 \eta = 0 .$$

The solutions of this equation describe the *tangent space* to the space of selfdual solutions of the Yang-Mills equations. However, the Yang-Mills action is gauge invariant. A variation along an orbit (see Sec. VI) through A will certainly yield a solution which is gauge related to A. We must eliminate such variations. This can be done by using the background gauge (Schwarz, 1977; Daniel and Viallet, 1978). Then the above equation must be replaced by a pair of equations, namely,

$$\tilde{\Delta}_A^2 \eta = 0 \text{ and } \mathfrak{D}^* \eta = 0$$

or equivalently by

$$\Delta_A^1 \eta = 0$$

where

$$\Delta_A^1 = \mathfrak{D}^*\mathfrak{D} + \mathfrak{D}\mathfrak{D}^* - \mathfrak{D}^* * \mathfrak{D} .$$

Now, $\mathfrak{D}^*\mathfrak{D} + \mathfrak{D}\mathfrak{D}^*$ is the covariant Laplacian acting on 1 forms taking values in the Lie algebra, and $\mathfrak{D}^* * \mathfrak{D}$ is a degree zero operator. Hence, Δ_A^1 is an elliptic operator. As such it has a finite number of zero modes. For example, if $G = SU(N)$, and A is a k instanton, then the null space of Δ_A^1 is $(4Nk - N^2 + 1)$ dimensional.

Note that Δ_A^1 is one of the Laplacians which can be constructed from the Atiyah-Hitchin-Singer complex (Atiyah, Hitchin, and Singer, 1978):

$$0 \to A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A_-^2 \to 0 ,$$

where

$d^0 = \mathfrak{D}$ acting on $\Gamma(E)$,

$d^1 = \mathfrak{D}$ acting $\Gamma(E)$ valued 1 forms,

A_-^2 = Hilbert space of $\Gamma(E)$ valued antiself-dual 2 forms.

Let $g_{\mu\nu}$ be the metric on S^4 obtained from sterographic coordinates. $g_{\mu\nu}$ is, of course, conformally flat, and is given explicitly by

$$g_{\mu\nu}(x) = \Omega(x)\delta_{\mu\nu} ,$$

with $\Omega(x) = 4a^4[(a^2 + x^2)^2]^{-1}$, where a is the radius of the 4 sphere. In this system of local coordinates we obtain, by making use of the formulas given in Sec. II.H.2,

$$(\Delta_A^1)_{\mu\nu} = -\Omega^{-1}\{\delta_{\mu\nu}\mathfrak{D}_\sigma\mathfrak{D}_\sigma + (\Omega\partial_\mu\Omega^{-1})\mathfrak{D}_\nu - (\Omega\partial_\nu\Omega^{-1})\mathfrak{D}_\mu - \Omega\partial_\nu\partial_\mu\Omega^{-1} + F_{\mu\nu} + {}^*F_{\mu\nu}\} .$$

For the importance of the operator Δ_A^1 for the problem of quantum fluctuations around multi-instanton configurations we refer to a recent paper of Schwarz (1979) and references therein.

VI. GAUGE FIXING PROBLEM

A. Notion of gauge fixing

Since all physically relevant quantities are gauge invariant, the objects of interest are the families of gauge related connections (on some G bundle P), rather than the connections themselves.

If $\mathcal{C}$ denotes the set of connections on P, these families—the orbits under the group $\mathcal{G}$ of gauge transformations—are the elements of the quotient set $\eta = \mathcal{C}/\mathcal{G}$. There exists a canonical projection p: $\mathcal{C} \to \eta$.

A natural problem arising at this point is the problem of the choice of a representative in $\mathcal{C}$ of all orbits. We will say that we fix the gauge if we choose a unique representative (which is a connection on P) in any equivalence class of gauge-ralated connections.

The usual realization of gauge fixing is essentially local and is performed by imposing a condition on the components A_μ (such as $\partial_\mu A_\mu = 0$ or other conditions...).

Strictly speaking, gauge fixing is the construction of a mapping φ: $\eta \to \mathcal{C}$ such that $p \circ \varphi$ = identity map of $\mathcal{C}$.

This closely resembles constructing a section of the (infinite-dimensional) principal bundle $\mathcal{C}$ over η with group $\mathcal{G}$. Unfortunately, in addition to the mere fact that we have not yet defined any distance in $\mathcal{C}$, we will have to restrict ourselves to irreducible connections and to replace $\mathcal{G}$ by a restricted group of gauge transformations in order to have a free group action and to get a nice (infinite-dimensional) principal fiber bundle.

B. A restricted group of gauge transformations and a restricted set of connections

Locally, the gauge transformations act on the components A_μ^α by:

$$A_\mu^\alpha \to A_\mu'^\alpha = g_\alpha A_\mu^\alpha g_\alpha^{-1} + g_\alpha \cdot \partial_\mu g_\alpha^{-1} .$$

We see that if g is a constant Z-valued transformation, then

$$A_\mu'^\alpha = A_\mu^\alpha \quad (\forall\, \alpha) .$$

Any such gauge transformation leaves *all* connections unchanged.

Note that $A_\mu' = A_\mu$ can be written

$$\partial_\mu g_\alpha + [A_\mu^\alpha, g_\alpha] = \mathfrak{D}_\mu^\alpha g_\alpha = 0 ,$$

i.e., $\nabla g = 0$.

Actually, $\nabla g = 0$ implies that g belongs to the center of the holonomy group of the connection under consideration. To see this, it suffices to apply locally $\mathfrak{D}_\mu$ and $\mathfrak{D}_\nu$; and if we antisymmetrize in μ and ν, we get $[F_{\mu\nu}, g] = 0$. Further applications of the covariant derivatives show that g commutes with the whole algebra of the holonomy group of the connection (Loos, 1967). In particular, if the connection is irreducible, then g necessarily belongs to the center of G (G is supposed to be connected).

As we noted earlier, the Z-valued sections of $\mathcal{B}$ are well defined: the set of these sections forms an Abelian subgroup $\mathfrak{z}$ of $\mathcal{G}$. We shall denote by $\tilde{\mathfrak{z}}$ the set of *constant* Z-valued sections of $\mathcal{B}$. *A priori*, a Z-valued section of $\mathcal{B}$ is not necessarily constant; but in the case of $SU(N)$, the fact that the center is a discrete (finite) subgroup Z_N, together with the mere continuity of the sections, implies that the center-valued sections are constant and that $\tilde{\mathfrak{z}} = \mathfrak{z}$.

The quotient $\tilde{\mathcal{G}} = \mathcal{G}/\tilde{\mathfrak{z}}$ is a well-defined group.

$\tilde{\mathcal{G}}$ acts freely on the set $\mathcal{C}'$ of irreducible connections on P.

Remark: The replacement of $\mathcal{G}$ by $\tilde{\mathcal{G}}$ was natural from the very moment we noted that the transformation formula of the potentials did not determine completely the global gauge transformations [see Eq. (3.1)]. Nevertheless, we have to point out that the role of the center is not completely understood, although it might have fundamental consequences on the interpretation of gauge theories (Mack, 1978; t'Hooft, 1978). Note also that if the structure group G is Abelian, the center of G is G itself, and $\tilde{\mathcal{G}}$ is a trivial group with one element.

C. A universal bundle for the group of gauge transformations

The set of (C^∞) connections on P can be equipped with a distance if we note that, given two connection forms ω and ω' on P, their difference $\tau=\omega'-\omega$ is a Lie-algebra-valued form on P which can be viewed as a 1 form on M with values in $\Gamma(E)$.

The scalar product (τ,τ) is well defined and naturally gives rise to a distance between ω and ω' $[d(\omega,\omega')=\sqrt{(\tau,\tau)}]$.

$\mathcal{C}'$ inherits the distance defined on $\mathcal{C}$.

Claim: $\mathcal{C}'$ is a principal fiber bundle over $\eta'=\mathcal{C}'/\tilde{\mathcal{G}}$ with group $\tilde{\mathcal{G}}$. Moreover, $\pi_j(\mathcal{C}')=0\ \forall j\in N$.

We will not give any proof of this and refer to Singer (1978).

Note: It is clear that $\mathcal{C}'(\eta',\tilde{\mathcal{G}})$ is a ∞-universal $\tilde{\mathcal{G}}$ bundle.

What is interesting about this result for the gauge fixing problem is that the existence of a continuous mapping $\eta\to\mathcal{C}$ (that is to say, a gauge fixing as we defined it) would imply the existence of a global section of $\mathcal{C}'$, and, as a corollary, would imply the triviality of the bundle $\mathcal{C}'(\eta',\tilde{\mathcal{G}})$. We would consequently have

$$\pi_j(\mathcal{C}')\approx\pi_j(\eta')\oplus\pi_j(\tilde{\mathcal{G}})(\forall j)\,.$$

This last relation is impossible to fulfill because of the nonvanishing of some of the homotopy groups of $\tilde{\mathcal{G}}$, as proved in Singer (1978).

Singer introduces the group $\mathcal{G}_\infty$ of gauge transformations that are identity at ∞, and the two following bundle sequences:

$$0\to\tilde{\mathfrak{z}}\to\mathcal{G}\to\tilde{\mathcal{G}}\to 0\,,$$

$$0\to\mathcal{G}_\infty\to\mathcal{G}\to\mathrm{SU}(N)\to 0\,.$$

These two sequences relate the homotopy groups of $\tilde{\mathcal{G}}$ with the ones of $\mathcal{G}_\infty$. The latter are directly related to the homotopy groups of SU(N). To be more precise,

$$\pi_j(\mathcal{G}_\infty)\approx\pi_{j+4}(\mathrm{SU}(N))\,.$$

For $N>2$, we have

$$\pi_0(\mathcal{G})\approx\pi_0(\mathcal{G}_\infty)\approx\pi_4(\mathrm{SU}(N))\approx 0\,.$$

The sequence

$$\pi_1(\tilde{\mathcal{G}})\to\pi_0(\tilde{\mathfrak{z}})=Z_N\to\pi_0(\mathcal{G})=0$$

is exact. Therefore, $\pi_1(\tilde{\mathcal{G}})\neq 0$.

For SU(2) we see that $\pi_j(\tilde{\mathcal{G}})=\pi_j(\mathcal{G})$ $(j>1)$ from the first sequence.

The sequence

$$\pi_3(\mathcal{G})\to\pi_3(\mathrm{SU}(2))\to\pi_2(\mathcal{G}_\infty)\to\pi_2(\mathcal{G})$$

is exact, i.e.,

$$\pi_3(\mathcal{G})\to Z\to Z_{12}\to\pi_2(\mathcal{G})$$

is exact.

Since $Z\neq Z_{12}$, we cannot have $\pi_2(\mathcal{G})$ and $\pi_3(\mathcal{G})$ equal to zero at the same time. Consequently, $\pi_2(\tilde{\mathcal{G}})\neq 0$ or $\pi_3(\tilde{\mathcal{G}})\neq 0$.

The nontriviality of the homotopy of $\tilde{\mathcal{G}}$ forbids the triviality of $\mathcal{C}'\to\eta'$ and then forbids any *continuous* gauge fixing.

This very elegant "no-go theorem" due to I. M. Singer is an example of the results that bundle theory can yield: essentially, they are *global* results (valid about smooth objects), and typically about homotopy.

The following remarks are now in order: The so-called Gribov ambiguity (Gribov, 1977) is related to the topological obstruction that one meets in trying to prolong a local section in the fibration $\mathcal{C}'\to\eta'$ to a global one. It is clear from Singer's analysis that the obstruction is due to the nontrivial topology of $\tilde{\mathcal{G}}$. We have seen that $\tilde{\mathcal{G}}$ has nontrivial homotopy groups. We would like to stress that this topology of $\tilde{\mathcal{G}}$ is due to the compactification of R^4 into S^4. *It has nothing to do with the presence of instantons.* The nontrivial topology of $\tilde{\mathcal{G}}$ is there even in the zero-instanton sector. This can be seen directly from the following argument: In the zero-instanton sector we have a trivial bundle $P=S^4\times G$. In this case, $\mathcal{G}_\infty$ can be identified with the set of maps g: $(S^4,\infty)\to(G,e)$. These maps fall into two classes. Hence $\tilde{\mathcal{G}}$ acquires some nontrivial topology.

D. Orbit space η'

In the previous section we have claimed that the set of irreducible connections is a principal fiber bundle with group $\tilde{\mathcal{G}}$. The base space of this bundle $\eta'=\mathcal{C}'/\tilde{\mathcal{G}}$ is the set of gauge inequivalent irreducible connections.

Our interest in η' is not merely academic: the Feynman path integral in gauge theories is essentially an integral over η. We could, however, consider η' instead of η because η' is dense in η (Singer, 1978).

To investigate what sort of space η' is, we can make use of the bundle sequence of the fibration $\mathcal{C}'\to\eta'$ by $\tilde{\mathcal{G}}$:

$$\to\pi_k(\mathcal{C}')\to\pi_k(\eta')\to\pi_{k-1}(\tilde{\mathcal{G}})\to\pi_{k-1}(\mathcal{C}')\to$$

Since $\pi_k(\mathcal{C}')=0$ for all k, we obtain the following exact sequence:

$$0\to\pi_k(\eta')\to\pi_{k-1}(\tilde{\mathcal{G}})\to 0\,.$$

Hence

$$\pi_k(\eta')\approx\pi_{k-1}(\tilde{G})\,.$$

An immediate consequence is that the set of gauge inequivalent irreducible connections (or irreducible orbit space) is topologically nontrivial.

We know we can give local coordinates to η' by locally fixing the gauge, that is to say, by giving local sections of the bundle $\mathcal{C}'\to\eta'$ (Daniel and Viallet, 1978).

A given orbit may have different coordinates if it belongs to the intersection of different coordinate neighborhoods, all of them being gauge related points in $\mathcal{C}'$.

CONCLUDING REMARK

In all we have said, we have been dealing with C^∞ field configurations, that is to say, with a rather restricted set of fields. What justifies the use of smooth objects is the fact that we are interested in a sourceless classical theory.

In dealing with quantum theory (defining the Feynman path integral), it will be necessary to enlarge the configuration space. A possible extension is to replace C^∞ configurations by some Sobolev completion H_k of this space. *The structure of the classical theory survives this extension* (Narashiman and Ramadas, 1979).

Consequently, we hope that the structural results of the classical theory may give new insights for the quantum Yang-Mills theory—at least at the semiclassical approximation. For instance, we may use the rich topology of the classical theory in order to get an understanding of a function space measure of the form

$$d\mu = (1/Z)e^{-a(A)}[dA],$$

$a(A)$ being the Yang-Mills action (see Sec. V.A), and Z a normalization factor. Now, the interest in $d\mu$ is that it can be used to define the statistical mechanics of the gauge potentials (in Euclidean space). With the statistical mechanics on the space of connections at hand, one could use the Osterwalder-Schrader axioms to obtain an appropriate quantum field theory in Minkowski space.

In any case, bundle theory remains the correct language to deal with the global (and local) aspects of classical gauge field theories. The implications for physics of the results that it already yields still have to be investigated, but it is by now clear that the geometrization shall invaluably deepen our understanding of gauge field theories.

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