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WHAT IS TOPOLOGICAL ABOUT TOPOLOGICAL DYNAMICS?

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Abstract. We consider various notions from the theory of dynamical systems from a topological point of view. Many of these notions can be sensibly defined either in terms of (finite) open covers or uniformities. These Hausdorff or uniform versions coincide in compact Hausdorff spaces and are equivalent to the standard definition stated in terms of a metric in compact metric spaces.

We show for example that in a Tychonoff space, transitivity and dense periodic points imply (uniform) sensitivity to initial conditions. We generalise Bryant's result that a compact Hausdorff space admitting a c-expansive homeomorphism in the obvious uniform sense is metrizable. We study versions of shadowing, generalising a number of well-known results to the topological setting, and internal chain transitivity, showing for example that ω-limit sets are (uniform) internally chain transitive and weak incompressibility is equivalent to (uniform) internal chain transitivity in compact spaces.

1. **Introduction.** A discrete dynamical system usually consists of a compact metric space X and a continuous function f from X to itself. A number of properties of interest in such systems are defined in purely topological terms, for example transitivity, recurrence, nonwandering points (see below for the definitions). Others are defined in terms of the metric or the existence of an equivalent metric on the space, for example sensitive dependence on initial conditions, chain transitivity and recurrence, shadowing and positive and c-expansivity.

A case in point is Devaney chaos. A system is Devaney chaotic if it is topologically transitive, has a dense set of periodic points and is sensitive to initial conditions. Transitivity and periodicity are topological notions preserved by topological conjugacy, whereas sensitivity depends on the particular metric. However,

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Banks *et al* [11], Glasner and Weiss [24] and Silverman [36] show that sensitivity follows from transitivity and dense periodic points. A full characterization of sensitive transitive maps is given in [2] (for recent progress, see also [26]). Blokh (see [35]) and Vellekoop and Berglund [37] show that, on the interval, transitivity implies Devaney chaos. Wang *et al* [39] state that non-metric spaces do not admit a notion of sensitivity. In [38], Wang *et al* consider various versions of Devaney chaos in the light of various notions related to transitivity.

Motivated by these ideas we show that in fact that many dynamical properties can be defined in a natural way on Hausdorff (but not necessarily compact or metric) spaces. It turns out that there are two sensible ways to do this, either in terms of finite open covers and in terms of uniformities (compatible with the topology). In the presence of compactness, where there is a unique uniformity, these two approaches are equivalent and in compact metric spaces they coincide exactly with the standard definition. Each has its advantages: using uniformities allows one to mimic existing metric proofs; using open covers can throw up new ideas, for example allowing one to use graph approximations to systems. The uniform approach has been studied in a number of cases: Hood [25] defines entropy for uniform spaces (see also [41]); Morales and Sirvent [32] consider positively expansive measures for measurable functions on uniform spaces, extending results from the literature; Devaney chaos for uniform spaces is considered in [20] (for group actions) and in [9]; Auslander, Greschonig and Nagar [8] generalise many known results about equicontinuity to the uniform world; notions of expansion are considered in [18], [19], [34]; contractions on uniform spaces are considered in [21], [29], [28]. Less appears to have been done via the open cover approach, although it has been used explicitly by Brian [17], considering chain transitivity in compact Hausdorff spaces, and implicitly in, for example, [15] where Bernardes and Darji study the genericity of shadowing in Cantor sets.

In the context of generalisations of dynamics on compact metric spaces, we should also mention Akin's book [1], which generalises the notion of iterations of a function to iterations of a relation and proves some results in more general contexts than compact metric. Again, the authors of [5] point out that many of their results can be recast in the uniform setting.

Section 2 introduces some basic notions used throughout the paper. In Section 3, we introduce the notions of uniform and Hausdorff sensitivity. Theorem 3.2 shows that the two notions are equivalent in the class of compact Hausdorff spaces and coincide with the standard notion of sensitivity in compact metric spaces. The proof of 3.2, together with the proofs of all other such equivalences, is relegated to the Appendix. We then show that in a Tychonoff space transitivity and dense periodic points imply uniform sensitivity from which it follows that in a compact Hausdorff space transitivity and dense periodic points imply Hausdorff sensitivity. We end the section with two examples which show that unlike the case in compact metric spaces without isolated points transitivity is not equivalent to existence of a dense orbit.

In Section 4, we consider positively expansive and c-expansive maps. It turns out that a proof due to Bryant [19] shows that if a compact space admits a positively expansivity map or a c-expansive homeomorphism in the obvious uniform sense, then the space is metrizable. We extend this result to semi-open c-expansive maps. It follows that a compact space admitting a uniform expanding map must also be metrizable.

There are very natural definitions of pseudo-orbits and shadowing in terms of uniformities and finite open covers. Section 5 looks at shadowing in compact Hausdorff spaces. We show that the identity map on a compact space has uniform shadowing if and only if the space is totally disconnected. Uniform shadowing is preserved by topological conjugacy. For sequentially compact and for compact spaces, uniform shadowing is equivalent to finite uniform shadowing. In compact spaces, a map has uniform shadowing if and only if it has uniform shadowing on a dense invariant set. A map g on a compact Hausdorff space has uniform shadowing if and only if g^n has uniform shadowing for any/all n. A homeomorphism g has uniform full shadowing if and only if g^{-1} has uniform full shadowing. If g has uniform periodic shadowing, then the periodic points are dense in the uniform nonwandering points. In Section 6 we show that uniform internal chain transitivity is equivalent to weak incompressibility. If g has uniform shadowing, then the set of uniform nonwandering points is equivalent to the set of uniform chain recurrent points. In Section 7, we show that the induced map on the hyperspace of compact subsets of a compact Hausdorff space has uniform shadowing if and if the map itself does.

Most of our results were proved originally for metric spaces; in each case we give an appropriate reference to the original result in the statement of the generalised version.

2. **Preliminaries.**

All of our spaces are Hausdorff unless otherwise indicated. The symbol $\mathbb{N}$ stands for the set of positive integers, $\mathbb{Z}$ represents the set of integers and $\mathbb{R}$ stands for the set of real numbers.

If A is a set, then $|A|$ denotes the cardinality of A.

If X is a metric space, x is a point of X and $\varepsilon > 0$, then $\mathcal{V}_\varepsilon(x) = \{x' \in X \mid d(x,x') < \varepsilon\}$.

If X is metric space and $\mathcal{U}$ is a family of subsets of X, then the *mesh of* $\mathcal{U}$, denoted $\mathrm{mesh}(\mathcal{U})$, is $\sup\{\mathrm{diam}(U) \mid U \in \mathcal{U}\}$.

A *compactum* is a compact metric space.

We work with uniformities, we follow [22, Chapter 8]. We present the basic notation. Let Z be a Hausdorff space. If V and W are subsets of $Z \times Z$, then

$$-V = \{(z',z) \mid (z,z') \in V\}$$

and

$$V + W = \{(z,z'') \mid \text{there exists } z' \in Z \text{ such that } (z,z') \in V \text{ and } (z',z'') \in W\}.$$

We write $1V = V$ and for each positive integer n, $(n+1)V = nV + 1V$.

The diagonal of Z is the set $\Delta_Z = \{(z,z) \mid z \in Z\}$. An *entourage* of the diagonal of Z is a subset V of $Z \times Z$ such that $\Delta_Z \subset V$ and $V = -V$. The family of entourages of the diagonal of Z is denoted by $\mathfrak{D}_Z$. If $V \in \mathfrak{D}_Z$ and $z \in Z$, then $B(z,V) = \{z' \in Z \mid (z,z') \in V\}$. By [22, 8.1.3], $Int(B(z,V))$ is a neighbourhood of z. If A is a subset of Z and $V \in \mathfrak{D}_Z$, then $B(A,V) = \bigcup\{B(a,V) \mid a \in A\}$. If $V \in \mathfrak{D}_Z$ and $(z,z') \in V$, then we write $\rho(z,z') < V$. If $(z,z') \notin V$, then we write $\rho(z,z') \geq V$. If for any two elements a_1 and a_2 of A, $\rho(a_1,a_2) < V$, then we write that $\delta(A) < V$. If A and A' are nonempty subsets of Z and $U \in \mathfrak{D}_Z$, then we write $\rho(A,A') \geq U$ if $\rho(a,a') \geq U$ for all $(a,a') \in A \times A'$. We also have that if z, z' and z'' are points of Z, and V and W belong to $\mathfrak{D}_Z$ then the following hold [22, p. 426]:

(i) $\rho(z,z) < V$.

(ii) $\rho(z,z') < V$ if and only if $\rho(z',z) < V$.

(iii) If $\rho(z,z') < V$ and $\rho(z',z'') < W$, then $\rho(z,z'') < V + W$.

Let Z be a Tychonoff space. A *uniformity* on Z is a subfamily $\mathfrak{U}$ of $\mathfrak{D}_Z$ such that:

(1) If $V \in \mathfrak{U}$, $W \in \mathfrak{D}_Z$ and $V \subset W$, then $W \in \mathfrak{U}$.

(2) If V and W belong to $\mathfrak{U}$, then $V \cap W \in \mathfrak{U}$.

(3) For every $V \in \mathfrak{U}$, there exists $W \in \mathfrak{U}$ such that $2W \subset V$.

(4) $\bigcap\{V \mid V \in \mathfrak{U}\} = \Delta_Z$.

Remark 2.1. Let Z be a Tychonoff space and let $\mathfrak{U}$ a uniformity of Z that induces its topology. If $V \in \mathfrak{U}$, then we define the cover $\mathfrak{C}(V) = \{B(z, V) \mid z \in Z\}$.

Remark 2.2. Note that, by [22, 8.3.13], for every compact Hausdorff space Z, there exists a unique uniformity $\mathfrak{U}_Z$ on Z that induces the original topology of Z.

Note the following:

Theorem 2.3. *Let Z and W be compact Hausdorff spaces and let $g\colon Z \to W$ be a function. Then g is continuous if and only if g is uniformly continuous.*

Proof. By [40, 35.11], every uniformly continuous function is continuous. By [40, 36.20], every continuous function between compact Hausdorff spaces is uniformly continuous. $\square$

A *map* is a uniformly continuous function with respect to the uniformities of its domain and range. Note that, by Theorem 2.3, for functions defined between compact Hausdorff spaces, a map is just a continuous function.

Notation. Let Z be a compact Hausdorff space and let $\mathfrak{U}_Z$ be the unique uniformity of Z that induces its topology. If Y is a subspace of Z and $U \in \mathfrak{U}_Z$, then $U_Y = U \cap (Y \times Y)$.

We need the following result [22, 8.3.G]:

Theorem 2.4. *Let Z be a compact Hausdorff space and let $\mathfrak{U}_Z$ be the unique uniformity of Z that induces its topology. Then for every open cover $\mathcal{U}$ of Z, there exists $V \in \mathfrak{U}_Z$ such that $\mathfrak{C}(V)$ refines $\mathcal{U}$.*

The following simple lemma is useful.

Lemma 2.5. *Let Z be a Tychonoff space, let $\mathfrak{U}$ be a uniformity of Z that induces its topology, let $V \in \mathfrak{U}$ and let $z \in Z$. Then $B(z, nV) \subset Int_Z(B(z, (n+1)V))$ for all $n \in \mathbb{N}$.*

Proof. Let $n \in \mathbb{N}$ and let $z' \in B(z, nV)$ and let $z'' \in B(z', V)$. Since $\rho(z'', z') < V$ and $\rho(z', z) < nV$, we have that $\rho(z'', z) < (n+1)V$. Thus, $B(z', V) \subset B(z, (n+1)V)$. Therefore, by [22, 8.1.2], $B(z, nV) \subset Int_Z(B(z, (n+1)V))$. $\square$

Lemma 2.6. *Let Z be a compact Hausdorff space and let $\mathcal{U}$ be an open cover of Z. If $U_0 = \bigcup\{U \times U \mid U \in \mathcal{U}\}$, then $U_0 \in \mathfrak{U}_Z$.*

Proof. Let $U_0 = \bigcup\{U \times U \mid U \in \mathcal{U}\}$. Note that $\Delta_Z \subset U_0$ and $-U_0 = U_0$. Thus, $U_0 \in \mathfrak{D}_Z$. Since Z is a compact Hausdorff space, by Theorem 2.4, there exists $W \in \mathfrak{U}_Z$ such that $\mathfrak{C}(W)$ refines $\mathcal{U}$. Since $\mathfrak{C}(W)$ refines $\mathcal{U}$, we have that $W \subset U_0$. This implies that $U_0 \in \mathfrak{U}_Z$. $\square$

3. **Transitivity and Devaney chaos.** The various related notions of transitivity in the literature are clearly topological in nature (see, for example, [3], [4], [10], [16]). Akin and Carlson [4] identify 7 notions of transitivity: let Z be a Hausdorff topological space and let $g\colon Z \to Z$ be a map.

- (IN) Z is not the union of two proper closed subsets A and B such that $g(A) \subset A$ and $g(B) \subset B$.
- (TT) For every pair of nonempty open subsets U and V of Z, the set $N(U,V) = \{k \in \mathbb{Z} \mid U \cap g^{-k}(V) \neq \emptyset\}$ is nonempty.
- (TT_+) For every pair of nonempty open subsets U and V of Z, the set $N_+(U,V) = N(U,V) \cap \mathbb{N}$ is nonempty.
- (TT_{++}) For every pair of nonempty open subsets U and V of Z, the set $N_+(U,V)$ is infinite.
- (DO) There exists an orbit sequence $\{z_k\}_{k=-\infty}^{\infty}$ or $\{z_k\}_{k \geq n}$ dense in Z.
- (DO_+) There exists a point z in Z such that $\{g^n(z)\}_{n=0}^{\infty}$ is dense in Z.
- (DO_{++}) There exists a point z in Z such that the ω-limit set $\omega(z,g) = Z$

Akin and Carlson show that for any space Z: DO_{++} implies DO_+, which implies DO; DO_{++} implies TT_{++}, which implies TT_+, which in turn implies TT; IN and TT are equivalent. They demonstrate that none of the other implications hold in general. For perfect, Hausdorff Z, IN implies TT_{++} and DO_+ implies DO_{++}. Moreover for second countable, non-meagre, Hausdorff Z, TT_{++} implies DO_{++}. Hence for perfect, second countable, non-meagre, Hausdorff spaces all 7 notions coincide. The notions also coincide in the case that the system is minimal (i.e. the forward orbit of every point is dense on the space). They give a complete picture of transitivity in the case that the space has an isolated point.

Banks and Brett [10] show that, for Hausdorff Z, the following are equivalent for a map $g\colon Z \to Z$: TT; any invariant subset of Z is either dense or nowhere dense; every proper closed invariant subset of Z is nowhere dense; every backward invariant subset of Z with non-empty interior is dense. In [3] the authors also consider several strengthenings of the concept of topological transitivity for the class of compact metric spaces.

Let Z be a Hausdorff space and let $g\colon Z \to Z$ be a continuous function. In the sequel, we will say that g is *topologically transitive* if it satisfies TT; i.e., if for each pair of open subsets U and V of Z, there exists $n \in \mathbb{N}$ such that $g^n(U) \cap V \neq \emptyset$. In the case that Z is a compact metric space with no isolated points this is equivalent to g having a dense forward orbit (i.e. the set $\{g^n(x) \mid 0 < n\}$ is dense for some $x \in X$ [36]). For infinite Hausdorff Z, if g has a dense orbit, then Z is separable and g is topologically transitive. We say that g is *exact* or *locally everywhere onto* if for each open subset U of Z, there exists $n \in \mathbb{N}$ such that $g^n(U) = Z$. Also, g is *mixing* if for each pair of open subsets U and V of Z, there exists $N \in \mathbb{N}$ such that $g^n(U) \cap V \neq \emptyset$ for all $n \geq N$ and g is *weakly mixing* provided that $g \times g$ is transitive on $Z \times Z$. Exact functions are mixing, mixing functions are weakly mixing, and weakly mixing functions are transitive.

A point z in Z is a *periodic point of* g with *(fundamental) period* n provided that there exists $n \in \mathbb{N}$ such that $g^n(z) = z$ and $g^m(z) \neq z$ for any $m < n$. The set of periodic points of g is denoted by $Per(g)$.

Example 3.1. There exists a compact Hausdorff space Z and exact, hence (weakly) mixing and transitive map $g\colon Z \to Z$ such that no point of Z has a dense orbit with respect to g. The set of periodic points of g is dense in Z, so that g is Devaney chaotic (see below).

Proof. Let $f\colon [0,1] \to [0,1]$ be the usual tent map of gradient 2 on the interval given by $f(t) = 1 - |2t-1|$. Then f is an exact function. Let κ be any cardinal greater than the continuum, let $Z = [0,1]^\kappa$, with the product topology, and let $g = \prod_{\alpha \leq \kappa} f_\alpha$, where $f_\alpha = f$ for each $\alpha \leq \kappa$. Then Z is a compact Hausdorff space. By [30, first Théorèm, p. 139], Z is not a separable space. Hence, there does not exist a point in Z with a dense orbit with respect to g. Note that g is an exact function. We show the set of periodic points of g is dense. Let $\mathcal{U}$ be a basic open set of Z. Then there exist $\beta_1, \dots, \beta_n$ in κ and n open subsets $I_{\beta_1}, \dots, I_{\beta_n}$ of $[0,1]$ such that $\mathcal{U} = \bigcap_{j=1}^n \pi_{\beta_j}^{-1}(I_{\beta_j})$. Since the set of periodic points of f is dense in $[0,1]$, for each $j \in \{1, \dots, n\}$, there exists a periodic point t_{β_j} of f in I_{β_j}. Let $\{z_\beta\}_{\beta \leq \kappa}$ be the point of Z defined as follows: $z_\beta = 0$ if $\beta \notin \{\beta_1, \dots, \beta_n\}$ and $z_{\beta_j} = t_{\beta_j}$ for all $j \in \{1, \dots, n\}$. Then $\{z_\beta\}_{\beta \leq \kappa}$ is periodic point of g and belongs to $\mathcal{U}$. By Theorem 3.3, g has uniform Devaney chaos. $\square$

One of the most frequently discussed notions of chaos is Devaney's; a continuous function, $f\colon X \to X$, on a metric space is said to be *Devaney Chaotic* provided it is topologically transitive, has a dense set of periodic points and has sensitive dependence on initial conditions; i.e., there exists $\delta > 0$ such that for every point x_1 of X and every $\varepsilon > 0$, there exist $x_2 \in \mathcal{V}_\varepsilon(x_1)$ and $k \in \mathbb{N}$ such that $d(f^k(x_1), f^k(x_2)) \geq \delta$. Banks *et al* [11] and Silverman [36] prove that if $f\colon X \to X$ is a transitive continuous function on an infinite compact metric space X that has a dense set of periodic points, then f has sensitive dependence on initial conditions. Although the notion of sensitivity as defined clearly depends on the specific metric, it turns out that there are two natural definitions of sensitivity for arbitrary topological spaces which are equivalent in compact Hausdorff spaces.

Let Z be a Hausdorff space and let $g\colon Z \to Z$ be a continuous function. Then g has *Hausdorff sensitive dependence on initial conditions*, if there exists a finite open cover $\mathcal{U}$ of Z such that for every point z of Z and every open subset V of Z containing z, there exist $v \in V$ and $k \in \mathbb{N}$ such that $|\{g^k(z), g^k(v)\} \cap U| \leq 1$ for all $U \in \mathcal{U}$.

If Z is a Tychonoff space, $\mathfrak{U}'$ is a uniformity of Z that induces its topology, and let $g\colon Z \to Z$ be a map with respect to $\mathfrak{U}'$, then g *has uniform sensitive dependence on initial conditions* if there exists a compatible uniformity $\mathfrak{U}$ such that there exists $V \in \mathfrak{U}$ such that for every point z in Z and every $W \in \mathfrak{U}$, there exist $z' \in B(z, W)$ and $k \in \mathbb{N}$ such that $\rho(g^k(z), g^k(z')) \geq V$.

The two notions coincide in compact spaces, where there is a unique uniformity generating the topology (Remark 2.2), so that the notion of sensitivity is indeed topological. They are equivalent to the standard definition in compact metric spaces (in the sense that there is a metric with respect to which g is sensitive).

Theorem 3.2. *Let Z be a compact Hausdorff space, and let $g\colon Z \to Z$ be a map. Then the following are equivalent:*

(1) g has Hausdorff sensitive dependence on initial conditions;

(2) g has uniform sensitive dependence on initial conditions.

If Z is metric, then (1) *and* (2) *are equivalent to*

(3) g has sensitive dependence on initial conditions.

The proof of Theorem 3.2 is left to the Appendix.

In the following theorem, we show that in Tychonoff spaces, transitive maps with dense periodic points are uniform sensitive, extending the result in [11, 36].

Theorem 3.3. *Let Z be a Tychonoff space, let $\mathfrak{U}$ be a uniformity of Z that induces its topology and let $g\colon Z \to Z$ be a map with respect to $\mathfrak{U}$. If g is transitive and its set of periodic points is dense in Z, then g has uniform sensitive dependence on initial conditions with respect to $\mathfrak{U}$.*

Proof. Let q_1 and q_2 be two periodic points of g such that their orbits $\mathcal{O}(q_1, g)$ and $\mathcal{O}(q_2, g)$ do not intersect. Since $\mathcal{O}(q_1, g) \cup \mathcal{O}(q_2, g)$ is a finite set, there exists $V_0 \in \mathfrak{U}$ such that $B(\mathcal{O}(q_1, g), V_0) \cap B(\mathcal{O}(q_2, g), V_0) = \emptyset$. Let $V_1 \in \mathfrak{U}$ be such that $2V_1 \subset V_0$. Observe that for each point z of Z, either $B(z, V_1) \cap B(\mathcal{O}(q_1, g), V_1) = \emptyset$ or $B(z, V_1) \cap B(\mathcal{O}(q_2, g, V_1)) = \emptyset$. Let $V_2 \in \mathfrak{U}$ be such that $2V_2 \subset V_1$. We show that V_2 satisfies the definition of uniform sensitivity dependence on initial conditions.

Let z be an arbitrary point of Z and let $W \in \mathfrak{U}$. Let $U = B(z, W) \cap B(z, V_2)$. Since the set of periodic points of g is dense in Z, there exists a periodic point p in U. Let n be the period of p. By the previous paragraph, without loss of generality, we assume that $B(z, V_1) \cap B(\mathcal{O}(q_1, g), V_1) = \emptyset$. Let $U' = \bigcap_{j=1}^{n} g^{-j}(B(g^j(q_1), V_2))$. Then U' is a neighbourhood of q_1. Since g is transitive, there exist a point z' in U and $k \in \mathbb{N}$ such that $g^k(z') \in U'$. Let $\ell = \left[\frac{k}{n} + 1\right]$. Then $1 \leq n\ell - k \leq n$. Thus,

$$g^{n\ell}(z') = g^{n\ell - k}(g^k(z')) \in g^{n\ell - k}(U') \subset B(g^{n\ell - k}(q_1), V_2).$$

Since p has period n, we obtain that $g^{n\ell}(p) = p$. Note that, since $B(z, V_1) \cap B(\mathcal{O}(q_1, g), V_1) = \emptyset$, we have that $\rho(g^{n\ell}(p), g^{n\ell}(z')) = \rho(p, g^{n\ell}(z')) \geq V_1$. We claim that either $\rho(g^{n\ell}(z), g^{n\ell}(p)) \geq V_2$ or $\rho(g^{n\ell}(z), g^{n\ell}(z')) \geq V_2$. Suppose that both $\rho(g^{n\ell}(z), g^{n\ell}(p)) < V_2$ and $\rho(g^{n\ell}(z), g^{n\ell}(z')) < V_2$. Then $\rho(p, g^{n\ell}(z')) = \rho(g^{n\ell}(p), g^{n\ell}(z')) < V_2 + V_2 = 2V_2$. Since $2V_2 \subset V_1$, we obtain that $\rho(p, g^{n\ell}(z')) < V_1$, a contradiction. Therefore, either $\rho(g^{n\ell}(z), g^{n\ell}(p)) \geq V_2$ or $\rho(g^{n\ell}(z), g^{n\ell}(z')) \geq V_2$ and g has uniform sensitive dependence on initial conditions. □

From Theorems 3.3 and 3.2, we obtain:

Corollary 3.4. *Let Z be a compact Hausdorff space, and let $g\colon Z \to Z$ be a map. If g is transitive and its set of periodic points is dense in Z, then g has Hausdorff sensitive dependence on initial conditions.*

Theorem 3.5. *There exists a compact Hausdorff topological group Z and an exact map $g\colon Z \to Z$ such that the set of periodic points of g is dense in Z. Hence, g has uniform Devaney chaos.*

Proof. Let Λ be an uncountable directed set. For each $\lambda \in \Lambda$, let X_λ be the unit circle in the plane and for every two elements γ and λ of Λ such that $\gamma \geq \lambda$, let $f_\lambda^\gamma\colon X_\gamma \to X_\lambda$ be given by $f_\lambda^\gamma(z) = z^2$. Then $\Sigma = \varprojlim\{X_\lambda, f_\lambda^\gamma, \Lambda\}$ is a Hausdorff continuum that is a topological group. Let $g\colon \Sigma \to \Sigma$ be given by $g(\{z_\lambda\}_{\lambda\in\Lambda}) = \{z_\lambda^2\}_{\lambda\in\Lambda}$. Note that g is well defined and continuous. Also, observe that g is an exact map and the set of periodic points of g is dense. Thus, by Theorem 3.3, g has uniform Devaney chaos. □

4. **Expansivity and metrizability.** B. F. Bryant proves in [19, Corollary, p. 1164] that if a compact Hausdorff space admits a uniform c-expansive homeomorphism, then that space is metrizable. It is clear that his proof applies to uniform positively expansive maps. We extend his result to semi-open maps.

Let X be a metric space and let $f\colon X \to X$ be a continuous function. Then f is *c-expansive* (*positively expansive*) if there exists a topologically equivalent metric d on the space and a $\delta > 0$ such that for any two points x_0 and y_0 of X and any two full orbits $\{x_n\}_{n=-\infty}^{\infty}$ and $\{y_n\}_{n=-\infty}^{\infty}$ through x and y, respectively (for any two points x_0 and y_0 of X and any two orbits $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$), if $d(x_n, y_n) < \delta$ for all $n \in \mathbb{Z}$ ($n \in \mathbb{N} \cup \{0\}$), then $x = y$.

Let Z be a Tychonoff space, let $\mathfrak{U}'$ be a uniformity that induces the topology of Z and let $g\colon Z \to Z$ be a map with respect to $\mathfrak{U}'$. Then g is *uniform c-expansive* (*positively expansive*) if there exists a compatible uniformity $\mathfrak{U}$ on Z and a $U \in \mathfrak{U}$ such that for any two points z_0 and w_0 of Z, and any two full orbits $\{z_n\}_{n=-\infty}^{\infty}$ and $\{w_n\}_{n=-\infty}^{\infty}$ through z_0 and w_0, respectively (for any two points z_0 and w_0 of Z and any two orbits $\{z_n\}_{n=0}^{\infty}$ and $\{w_n\}_{n=0}^{\infty}$), if for each $n \in \mathbb{Z}$ ($n \in \mathbb{N} \cup \{0\}$), $\rho(z_n, w_n) < U$, we have that $z_0 = w_0$.

A map $g\colon Z \to W$ between compact Hausdorff spaces is *semi-open* provided that for each open subset U of Z, $Int_W(g(U)) \neq \emptyset$. Clearly every open map is semi-open. However, in [27], an example is given where the converse is not true.

Theorem 4.1. *Let Z be a compact Hausdorff space, which is not metric, and let $g\colon Z \to Z$ be a semi-open map. If $U \in \mathfrak{U}_Z$ (Remark 2.2), then there exist two distinct points z_0 and w_0 in Z and full orbits $\{z_k\}_{k=-\infty}^{\infty}$ $\{w_k\}_{k=-\infty}^{\infty}$ of z_0 and w_0, respectively, such that for every $k \in \mathbb{Z}$, $\rho(z_k, w_k) < U$.*

Proof. Let $U_0 = U$. Since g is a map, there exists $U_1 \in \mathfrak{U}_Z$ such that $U_1 \subset U$ and $g \times g(U_1) \subset U$. Inductively, suppose we have $U_k \subset U_{k-1} \subset \cdots \subset U_1$ such that $g \times g(U_j) \subset U_{j-1}$ for all $j \in \{1, \ldots, k\}$. Since g is a map, there exists $U_{k+1} \in \mathfrak{U}_Z$ such that $U_{k+1} \subset U_k$ and $g \times g(U_{k+1}) \subset U_k$. Observe that for all $k \in \mathbb{N}$, $(g \times g)^k(U_k) \subset U_0$.

Since g is a map, there exists $U_{-1} \in \mathfrak{U}_Z$ such that $U_{-1} \subset U_1$ and $g \times g(Cl(U_{-1})) \subset U_0 \cap U_2$. Inductively, assume that we have $U_{-k} \subset U_{-k+1} \subset \cdots \subset U_{-1}$ such that $g \times g(Cl(U_{-j})) \subset U_{-j+1} \cap U_{2j}$ for every $j \in \{1, \ldots, k\}$. Since g is a map, there exists $U_{-k-1} \in \mathfrak{U}_Z$ such that $U_{-k-1} \subset U_{-k}$ and $g \times g(Cl(U_{-k-1})) \subset U_{-k} \cap U_{2(k+1)}$. Note that $\bigcap_{k=1}^{\infty}(g \times g)^k(Cl(U_{-k})) \neq \emptyset$. Also observe that for each $k \in \mathbb{N}$, $(g \times g)^k(Cl(U_{-k})) \subset U_k$. This implies that $\bigcap_{k=1}^{\infty}(g \times g)^k(Cl(U_{-k})) \subset \bigcap_{k=1}^{\infty} U_k$. Since g is semi-open, by [22, 8.1.21], $\bigcap_{k=1}^{\infty}(g \times g)^k(Cl(U_{-k})) \neq \Delta_Z$. Let $(z_0, w_0) \in \bigcap_{k=1}^{\infty}(g \times g)^k(Cl(U_{-k})) \setminus \Delta_Z$. Then for each $k \in \mathbb{N}$, $(g \times g)^{-k}((z_0, w_0)) \cap Cl(U_{-k}) \neq \emptyset$. For every $k \in \mathbb{N}$, let $(z_k, w_k) \in (g \times g)^{-k}((z_0, w_0)) \cap Cl(U_{-k})$ and let $(z_k, w_k) = g \times g((z_0, w_0))$. Hence, $\{z_k\}_{k=-\infty}^{\infty}$ and $\{w_k\}_{k=-\infty}^{\infty}$ are full orbits of z_0 and w_0, respectively, such that $\rho(z_k, w_k) < U$ for all $k \in \mathbb{N}$. $\square$

Corollary 4.2. *Let Z be a compact Hausdorff space on which it is possible to define either a uniform positively expansive map or a uniform c-expansive semi-open map, then Z is metrizable.*

One might try to extend the definition of metric expanding to the uniform case. First note the following result:

Theorem 4.3. [33, Theorem 1] *Let X be a compact metric space and let $f\colon X \to X$ be an onto map. Then the following statements are equivalent:*

(1) f expands small distances; i.e., there exist a metric d for X and numbers $\varepsilon > 0$ and $\lambda > 1$ such that $0 < d(x, x') < \varepsilon$ implies that $d(f(x), f(x')) > \lambda d(x, x')$.

(2) f increases small distances; i.e., there exist a metric d for X and $\varepsilon > 0$ such that $0 < d(x, x') < \varepsilon$ implies that $d(f(x), f(x')) > d(x, x')$.

(3) f is positively expansive.

We use (2) of Theorem 4.3 to define uniform expanding maps: Let Z be a compact Hausdorff space and let $g\colon Z \to Z$ be an onto map. Then g is *uniform expanding* if there exist $V, U \in \mathfrak{U}_Z$ (Remark 2.2) such that for all distinct points z and z' of Z such that $\rho(z, z') < V$, we have that $\rho(g(z), g(z')) \geq U$.

Observe the following:

Theorem 4.4. *Let Z be a compact Hausdorff space and let $g\colon Z \to Z$ be an onto map. If g is uniform expanding, then g is uniform positively expansive.*

Proof. Suppose g is uniform expanding. Then there exist $V, U \in \mathfrak{U}_Z$ (Remark 2.2) such that if z and z' are distinct points of Z such that $\rho(z, z') < V$, then $\rho(g(z), g(z')) \geq U$. Without loss of generality, we assume that $V \subset U$. Let $V' \in \mathfrak{U}_Z$ be such that $2V' \subset V$. Suppose that there exist two distinct points z and z' in Z such that $\rho(g^n(z), g^n(z')) < V'$ for all $n \geq 0$. Let $B = Cl_{Z\times Z}(\{(g^n \times g^n)(z, z') \mid n \geq 0\})$. Then B is a $(g \times g)$-invariant closed subset of $Z \times Z$. Observe that $B \subset Cl_{Z\times Z}(V') \subset 2V' \subset V$ (for the second inclusion see [22, p. 428]). Let $(z_0, z_0') \in B$. Then $\rho(z_0, z_0') < V$ and $\rho(g(z_0), g(z_0')) < V$. Since $V \subset U$, $\rho(g(z_0), g(z_0')) < U$, a contradiction. Therefore, g is uniform positively expansive. $\square$

Corollary 4.5. *Let Z be a compact Hausdorff space on which it is possible to define a uniform expanding map, then Z is metrizable.*

5. **Uniform shadowing.** There are very natural definitions of pseudo-orbits and shadowing in terms of uniformities and finite open covers. We look at shadowing in compact Hausdorff spaces. We show that the identity map on a compact space has uniform shadowing if and only if the space is totally disconnected. We also prove that uniform shadowing is preserved by topological conjugacy. Then we see that for sequentially compact and for compact spaces, uniform shadowing is equivalent to finite uniform shadowing. In compact spaces, a map has uniform shadowing if and only if it has uniform shadowing on a dense invariant set. A map g on a compact Hausdorff space has uniform (h-)shadowing if and only if g^n has uniform (h-)shadowing for any/all n. A homeomorphism g has uniform full shadowing if and only if g^{-1} has uniform full shadowing. If g has uniform periodic shadowing then the periodic points are dense in the uniform nonwandering points.

Let X be a metric space and let $f\colon X \to X$ be a continuous function. Then

- For $\delta > 0$, the (finite or infinite) sequence $\{x_0, x_1, \dots\}$ of points of X is a *δ-pseudo-orbit* if $d(f(x_j), x_{j+1}) < \delta$ for all $j \geq 0$.
- For $\varepsilon > 0$, a point x of X *ε-shadows* the (finite or infinite) sequence $\{x_0, x_1, \dots\}$ of points of X if $d(f^j(x), x_j) < \varepsilon$ for all $j \geq 0$.
- f has *(finite) shadowing* if for each $\varepsilon > 0$, there exists $\delta > 0$ such that every (finite) δ-pseudo-orbit is ε-shadowed by a point of X.
- f has *h-shadowing* if for each $\varepsilon > 0$, there exists $\delta > 0$ such that for every finite uniform δ-pseudo-orbit $\{x_0, \dots, x_r\}$ of f, there exists a point x' in X such that $\rho(f^l(x'), x_l) < \varepsilon$ for all $l \in \{0, \dots, r-1\}$ and $f^r(x') = x_r$.

Let Z be a Hausdorff space, let $\mathfrak{U}$ is a uniformity that generates the topology of Z, and let $g\colon Z \to Z$ be a map with respect to $\mathfrak{U}$ (recall that uniformly continuous functions are continuous [40, 35.11]). Then

- For a finite open cover $\mathcal{V}$ of Z, the (finite or infinite) sequence $\{z_0, z_1, \ldots\}$ of points of Z is a *Hausdorff $\mathcal{V}$-pseudo-orbit* if for each $j \geq 0$, there exists $V_j \in \mathcal{V}$ such that $\{g(z_j), z_{j+1}\} \subset V_j$.
- For an element $V \in \mathfrak{U}$, the (finite or infinite) sequence $\{z_0, z_1, \ldots\}$ of points of Z is a *uniform V-pseudo-orbit* if for each $j \geq 0$, $\rho(g(z_j), z_{j+1}) < V$.

• For a finite open cover $\mathcal{U}$ of Z, a point z of X *Hausdorff* $\mathcal{U}$*-shadows* the (finite or infinite) sequence $\{z_0, z_1, \ldots\}$ of points of Z if for each $j \geq 0$, there exists $U_j \in \mathcal{U}$ such that $\{g^j(z), z_j\} \subset U_j$.

• g has *Hausdorff shadowing* if for every finite open cover $\mathcal{U}$ of Z, there exists a finite open cover $\mathcal{V}$ of Z such that every Hausdorff $\mathcal{V}$-pseudo-orbit is Hausdorff $\mathcal{U}$-shadowed by a point of Z.

• A point z of Z *uniform* U*-shadows* the (finite or infinite) sequence $\{z_0, z_1, \ldots\}$ if for each $j \geq 0$, $\rho(g^j(z), z_j) < U$.

• *g has uniform (finite) shadowing* if for every $U \in \mathfrak{U}$, there exists $V \in \mathfrak{U}$ such that every (finite) uniform V-pseudo-orbit is uniform U-shadowed by a point of Z.

• *g has Hausdorff h-shadowint* provided that for every finite open cover $\mathcal{U}$ of Z, there exists a finite open cover $\mathcal{V}$ of Z such that every finite Hausdorff $\mathcal{V}$-pseudo-orbit $\{z_0, \ldots, z_r\}$ of g, there exists a point z' in Z such that $\{g^j(z), z_j\} \subset U_j$ for all $j \in \{0, \ldots, r-1\}$ and $g^r(z') = z_r$.

• g has *uniform h-shadowing* if for each $U \in \mathfrak{U}$, there exists $V \in \mathfrak{U}$ such that for every finite uniform V-pseudo-orbit $\{z_0, \ldots, z_r\}$ of g, there exists a point z' in Z such that $\rho(g^l(z'), z_l) < U$ for all $l \in \{0, \ldots, r-1\}$ and $g^r(z') = z_r$.

The proof of the following theorem is in the Appendix.

Theorem 5.1. *Let Z be a compact Hausdorff space, and let $g \colon Z \to Z$ be a map. Then the following are equivalent:*

(1) g has Hausdorff shadowing;

(2) g has uniform shadowing.

If Z is metric, then (1) *and* (2) *are equivalent to*

(3) g has shadowing.

Theorem 5.2. [7, Theorem 2.3.2] *Let Z be a compact Hausdorff space. Then the identity map, 1_Z, has uniform shadowing if and only if Z is totally disconnected.*

Proof. Suppose Z is totally disconnected. Let $U \in \mathfrak{U}_Z$ (Remark 2.2). Since Z is totally disconnected, for each $z \in Z$, there exists an open and closed subset W_z of Z such that $z \in W_z \subset B(z, U)$. Since Z is compact, there exist $z_1, \ldots, z_n$ in Z such that $Z = \bigcup_{j=1}^{n} W_{z_j}$. Let $W_1 = W_{z_1}$ and for each $j \in \{2, \ldots, n\}$, let $W_j = W_{z_j} \setminus \bigcup_{k=1}^{j-1} W_{z_k}$. Then $\mathcal{W} = \{W_1, \ldots, W_n\}$ is a finite open cover of Z such that $W_j \cap W_k = \emptyset$ if $j \neq k$. By Theorem 2.4, there exists $V \in \mathfrak{U}_Z$ such that $\mathfrak{C}(V)$ refines $\mathcal{W}$. Let $\{w_0, w_1, \ldots\}$ be a uniform V-pseudo-orbit for 1_Z. Note that, by construction, there exists $j \in \{1, \ldots, n\}$ such that $\{w_0, w_1, \ldots\} \subset W_j$. Hence, if $w \in W_j$, then $\rho(w, w_j) < U$ for all $j \geq 0$. Therefore, 1_Z has uniform shadowing.

Assume that Z is not totally disconnected. Then Z has a nondegenerate component K. Let z and z' be two distinct points of K. Since Z is Hausdorff, there exists $U \in \mathfrak{U}_Z$ such that $\rho(z, z') \geq U$. Let $V \in \mathfrak{U}_Z$ be such that $2V \subset U$. Since K is connected, there exist $w_0 = z, w_1, \ldots, w_n = z'$ in K such that $\rho(w_j, w_{j+1}) < V$ for each $j \in \{0, \ldots, n-1\}$. Suppose there exists w in Z such that $\rho(w, w_j) < V$ for every $j \in \{0, \ldots, n\}$. Since $\rho(w_0, w) < V$, $\rho(w, w_n) < V$ and $2V \subset U$, we obtain that $\rho(w_0, w_n) < U$. Since $w_0 = z$ and $w_n = z'$, we have that $\rho(z, z') < U$, a contradiction. Therefore, Z is totally disconnected. □

Theorem 5.3. [7, Theorem 2.3.6] *Let Z and Z' be compact Hausdorff spaces, let $g \colon Z \to Z$ be a map, and let $h \colon Z \to Z'$ be a homeomorphism. Then g has uniform shadowing if and only if $g' = h \circ g \circ h^{-1}$ has uniform shadowing.*

Proof. Suppose g has uniform shadowing. Let $U' \in \mathfrak{U}_{Z'}$ (Remark 2.2). Since h is uniformly continuous [40, 36.20], there exists $U \in \mathfrak{U}_Z$ such that if z_1 and z_2 are points of Z such that $\rho_Z(z_1, z_2) < U$, then $\rho_{Z'}(h(z_1), h(z_2)) < U'$. Since g has uniform shadowing, there exists $V \in \mathfrak{U}_Z$ such that each uniform V-pseudo-orbit of g is uniform U shadowed by a point of Z. Since h^{-1} is uniformly continuous, there exists $V' \in \mathfrak{U}_{Z'}$ such that if z_1' and z_2' are points of Z' such that $\rho_{Z'}(z_1', z_2') < V'$, then $\rho_Z(h^{-1}(z_1'), h^{-1}(z_2')) < V$.

Let $\{z_0', z_1', \ldots\}$ be a uniform V'-pseudo-orbit of g'. For each n, let $z_n = h^{-1}(z_n')$. Since $\rho_{Z'}(g'(z_n'), z_{n+1}') < V'$, we have that $\rho_Z(h^{-1} \circ g'(z_n'), h^{-1}(z_{n+1}')) < V$. Note that $h^{-1} \circ g'(z_n') = g \circ h^{-1}(z_n') = g(z_n)$. Hence, $\rho_Z(g(z_n), z_{n+1}) < V$. Thus, $\{z_0, z_1, \ldots,\}$ is a uniform V-pseudo-orbit of g. Hence, there exists a point z in Z such that $\rho_Z(g^n(z), z_n) < U$ for all n. This implies that $\rho_{Z'}(h \circ g^n(z), h(z_n)) < U'$. Observe that $h \circ g^n(z) = (g')^n \circ h(z)$. Thus, $\rho_{Z'}((g')^n(h(z)), z_n') < U'$ for every n. Therefore, g' has uniform shadowing.

The proof of the reverse implication is similar. □

Theorem 5.4. *Let Z be a Tychonoff space that is sequentially compact, let $\mathfrak{U}$ be a uniformity that generates the topology of Z, and let $g\colon Z \to Z$ be a map with respect to $\mathfrak{U}$. Then g has uniform shadowing if and only if g has finite uniform shadowing*

Proof. Clearly, if g has uniform shadowing, then g has finite uniform shadowing.

Suppose g has finite uniform shadowing. Let $U \in \mathfrak{U}$ and let $U' \in \mathfrak{U}$ be such that $2U' \subset U$. Since g has uniform finite shadowing, there exists $V \in \mathfrak{U}$, such that each finite uniform V-pseudo-orbit is uniform U'-shadowed by a point of Z. Let $\{z_n\}_{n=1}^{\infty}$ be an infinite uniform V-pseudo-orbit. For each $n \in \mathbb{N}$, there exists $z_n' \in Z$ such that z_n' uniform U'-shadows $\{z_1, \ldots, z_n\}$. Since Z is sequentially compact, $\{z_n'\}_{n=1}^{\infty}$ has a convergent subsequence $\{z_{n_k}'\}_{k=1}^{\infty}$. Suppose $\{z_{n_k}'\}_{k=1}^{\infty}$ converges to z_0. Let $m \in \mathbb{N}$. Then there exists $n_k > m$ such that $\rho(g^m(z_{n_k}'), g^m(z_0)) < U'$. Since z_{n_k}' uniform U'-shadows $\{z_1, \ldots, z_{n_k}\}$ and $n_k > m$, we have that $\rho(g^m(z_{n_k}'), z_m) < U'$. Hence, $\rho(g^m(z_0), z_m) < 2U'$. Since $2U' \subset U$, we obtain that $\rho(g^m(z_0), z_m) < U$. Thus, z_0 uniform U-shadows $\{z_n\}_{n=1}^{\infty}$. Therefore, g has uniform shadowing. □

The next two results are used in the proof of Theorem 7.4.

Theorem 5.5. *Let Z be a compact Hausdorff space, and let $g\colon Z \to Z$ be a map. Then g has uniform shadowing if and only if g has finite uniform shadowing*

Proof. Clearly, if g has uniform shadowing, then g has finite uniform shadowing.

Suppose g has finite uniform shadowing. Let $U \in \mathfrak{U}_Z$ (Remark 2.2) and let $U' \in \mathfrak{U}_Z$ be such that $2U' \subset U$. Since g has uniform finite shadowing, there exists $V \in \mathfrak{U}_Z$, such that each finite uniform V-pseudo-orbit is uniform U'-shadowed by a point of Z. Let $\{z_n\}_{n=1}^{\infty}$ be an infinite uniform V-pseudo-orbit. For each $n \in \mathbb{N}$, there exists $z_n' \in Z$ such that z_n' uniform U'-shadows $\{z_1, \ldots, z_n\}$. If $\{z_n'\}_{n=1}^{\infty}$ is a finite set, then it is clear that there exists an element $z_k' \in \{z_n'\}_{n=1}^{\infty}$ such that z_k' uniform U'-shadows $\{z_n\}_{n=1}^{\infty}$. In particular, z_k' uniform U-shadows $\{z_n\}_{n=1}^{\infty}$. Assume that $\{z_n'\}_{n=1}^{\infty}$ is an infinite set. Let z_0 be a limit point of $\{z_n'\}_{n=1}^{\infty}$. Let $m \in \mathbb{N}$. Note that $(Int_Z(g^{-m}(B(g^m(z_0), U'))) \setminus \{z_1', \ldots, z_m'\}) \cup \{z_0\}$ is an open subset of Z containing z_0. Hence, since z_0 is a limit point of $\{z_n'\}_{n=1}^{\infty}$, there exists $k > m$ such that $z_k' \in g^{-m}(B(g^m(z_0), U')) \cap \{z_n'\}_{n=1}^{\infty}$. Thus, $\rho(g^m(z_k'), g^m(z_0)) < U'$. Since z_k' uniform U'-shadows $\{z_1, \ldots, z_k\}$ and $k > m$, $\rho(g^m(z_k'), z_m) < U'$. Hence, $\rho(g^m(z_0), z_m) < 2U'$. Since $2U' \subset U$, we obtain that $\rho(g^m(z_0), z_m) < U$. Thus, z_0 uniform U-shadows $\{z_n\}_{n=1}^{\infty}$. Therefore, g has uniform shadowing. □

Theorem 5.6. [23, Lemma 3.1] *Let Z be a compact Hausdorff space, let $g\colon Z \to Z$ be a map and let Y be a dense subset of Z that is invariant under g. Then g has uniform finite shadowing if and only if $g|_Y$ has uniform finite shadowing.*

Proof. Suppose g has uniform finite shadowing and let $U \in \mathfrak{U}_Z$ (Remark 2.2). Let $U' \in \mathfrak{U}_Z$ be such that $2U' \subset U$. Since g has uniform finite shadowing, there exists $V \in \mathfrak{U}_Z$ such that every uniform finite V-pseudo-orbit is uniform U'-shadowed by an element of Z. Let $\{y_0, \dots, y_r\}$ be a uniform finite V_Y-pseudo-orbit (Notation 2) of $g|_Y$. Note that $\{y_0, \dots, y_r\}$ is a uniform V-pseudo-orbit of g. Since g has uniform finite shadowing, there exists an element z of Z such that for each $j \in \{0, \dots, r\}$, $\rho(g^j(z), y_j) < U'$. Since g is a map, there exists $U_{r-1} \in \mathfrak{U}_Z$ such that $2U_{r-1} \subset U'$ and $g(B(g^{r-1}(z), U_{r-1})) \subset B(g^r(z), U')$. Since g is a map, there exists $U_{r-2} \in \mathfrak{U}_Z$ such that $2U_{r-2} \subset U_{r-1}$ and $g(B(g^{r-2}(z), U_{r-2})) \subset B(g^{r-1}(z), U_{r-1})$. Continuing with this process, we obtain a finite sequence $\{U_0, \dots, U_{r-1}\}$ of elements of $\mathfrak{U}_Z$ such that for every $j \in \{0, \dots, r-2\}$, $2U_j \subset U_{j+1}$, $g(B(g^j(z), U_j)) \subset B(g^{j+1}(z), U_{j+1})$ and $g(B(g^{r-1}(z), U_{r-1})) \subset B(g^r(z), U')$. Let $y \in Int_Z(B(g^0(z), U_0)) \cap Y = Int_Z(B(z, U_0)) \cap Y$. Let $j \in \{0, \dots, r\}$. Observe that $\rho(g^j(y), g^j(z)) < U_j$ and $\rho(g^j(z), y_j) < U'$. Then $\rho(g^j(y), y_j) < U_j + U'$. Since $U_j \subset U'$ and $2U' \subset U$, we obtain that $\rho(g^j(y), y_j) < U$. Thus, y uniform U_Y-shadows $\{y_0, \dots, y_r\}$. Therefore, $g|_Y$ has uniform finite shadowing.

Assume $g|_Y$ has uniform finite shadowing. Let $U \in \mathfrak{U}_Z$ and let $U' \in \mathfrak{U}_Z$ be such that $2U' \subset U$. Since $g|_Y$ has uniform finite shadowing, there exists $V' \in \mathfrak{U}_Z$ (Notation 2) such that each uniform finite V'_Y-pseudo-orbit of $g|_Y$ is uniform U'_Y-shadowed by an element of Y. Let $V \in \mathfrak{U}_Z$ be such that $4V \subset V'$. Let $\{z_0, \dots, z_r\}$ be a uniform finite V-pseudo-orbit of g. Since g is a map, there exist W and W' in $\mathfrak{U}_Z$ such that $W + W' \subset V$, $W \subset U'$, and for every $j \in \{0, \dots, r\}$, if $\rho(y, z_j) < W$, then $\rho(g(y), g(z_j)) < W'$. Since $\{z_0, \dots, z_r\}$ is a uniform finite V-pseudo-orbit of g, $\rho(g(z_j), z_{j+1}) < V$ for all $j \in \{0, \dots, r-1\}$. For every $j \in \{0, \dots, r\}$, let $y_j \in Int_Z(B(z_j, W)) \cap Y$. Let $j \in \{0, \dots, r-1\}$. Then, since $\rho(g(y_j), g(z_j)) < W'$, $\rho(g(z_j), z_{j+1}) < V$ and $\rho(y_{j+1}, z_{j+1}) < W$, we have that $\rho(g(y_j), y_{j+1}) < W' + V + W$. Since $W + W' \subset V$ and $4V \subset V'$, we obtain that $\rho(g(y_j), y_{j+1}) < V'$. Hence, $\{y_0, \dots, y_r\}$ is a uniform finite V'_Y-pseudo-orbit of $g|_Y$. Since $g|_Y$ has uniform finite shadowing, there exists $y \in Y$ such that for each $j \in \{0, \dots, r\}$, $\rho(g^j(y), y_j) < U'_Y$. Let $j \in \{0, \dots, r\}$. Then, since $\rho(g^j(y_j), y_j) < U'$ and $\rho(y_j, z_j) < W$, we have that $\rho(g^j(y), z_j) < U' + W$. Since $W \subset U'$ and $2U' \subset U$, we obtain that $\rho(g^j(y), z_j) < U$. Thus, y uniform U-shadows $\{z_0, \dots, z_r\}$. Therefore, g has uniform finite shadowing. □

Lemma 5.7. [13, Lemma 3.8] *Let Z be a compact Hausdorff space and let $g\colon Z \to Z$ be an onto map, let $U \in \mathfrak{U}_Z$ (Remark 2.2) and let $n \in \mathbb{N}$. Then there exists $V \in \mathfrak{U}_Z$ such that if $\{z_0, \dots, z_n\}$ is a uniform V-pseudo-orbit for g and z is a point of Z such that $\rho(z, z_0) < V$, then $\rho(g^k(z), z_k) < U$ for all $k \in \{1, \dots, n\}$.*

Proof. Let $V_{n+1} \in \mathfrak{U}_Z$ be such that $2V_{n+1} \subset U$. Let $V'_n \in \mathfrak{U}_Z$ be such that $2V'_n \subset V_{n+1}$ Since g is a map, there exists $V_n \in \mathfrak{U}_Z$ such that $V_n \subset V'_n$ and if $\rho(z, z') < V_n$, then $\rho(g(z), g(z')) < V'_n$. Suppose we have found $V_{n+1}, V_n, V'_n \dots, V_{k+1}, V'_{k+1}$ such that $V_{k+1} \subset V'_{k+1} \subset 2V_{k+1} \cdots \subset V_n \subset V'_n \subset 2V_n \subset V_{n+1} \subset 2V_{n+1} \subset U$, and if $j \in \{k+1, \dots, n\}$ and $\rho(z, z') < V'_j$, then $\rho(g(z), g(z)) < V_j$. Let $V'_k \in \mathfrak{U}_Z$ be such that $2V'_k \subset V_{k+1}$ Since g is a map, there exists $V_k \in \mathfrak{U}_Z$ such that $V_k \subset V'_k$ and if $\rho(z, z') < V_k$, then $\rho(g(z), g(z)) < V'_k$.

Let $\{z_0, \ldots, z_n\}$ be a uniform V_1-pseudo-orbit of g and let z be a point of Z such that $\rho(z, z_0) < V_1$. Then $\rho(g(z), g(z_0)) < V_1'$. Since $\rho(g(z), g(z_0)) < V_1$ and $\rho(g(z_0), z_1) < V_1'$, we obtain that $\rho(g(z), z_1) < V_1 + V_1'$. Thus, since $V_1 + V_1' \subset 2V_1$ and $2V_1 \subset V_2$, we have that $\rho(g(z), z_1) < V_2$. This implies that $\rho(g^2(z), g(z_1)) < V_2'$. Since $\rho(g(z_1), z_2) < V_0$, $\rho(g^2(z), g(z_1)) < V_2'$ and $V_0 + V_2' \subset 2V_2' \subset V_3$, we obtain that $\rho(g^2(z), z_2) < V_3$. Continuing with this process, we have that for each $k \in \{1, \ldots, n-1\}$, $\rho(g^k(z), z_k) < V_{k+1}$, and $\rho(g^n(z), z_n) < 2V_{n+1}$. Since $V_1 \subset V_1' \subset 2V_2 \cdots \subset V_n' \subset V_n \subset 2V_n \subset V_{n+1} \subset 2V_{n+1} \subset U$, we conclude that $\rho(g^k(z), z_k) < U$ for all $k \in \{1, \ldots, n\}$. □

The next theorem extends [7, 2.3.3] to uniform shadowing and uniform h-shadowing.

Theorem 5.8. [13, Theorem 3.9] *Let be a compact Hausdorff space and let $g\colon Z \to Z$ be an onto map. The following are equivalent:*

(1) g has uniform shadowing;

(2) g^n has uniform shadowing for some $n \in \mathbb{N}$;

(3) g^n has uniform shadowing for all $n \in \mathbb{N}$.

The same holds for uniform h-shadowing.

Proof. We prove the result for uniform h-shadowing; the proof for uniform shadowing is similar.

It is clear that (3) implies (2). The fact that (1) implies (3) is also clear, since for every $V \in \mathfrak{U}_Z$ (Remark 2.2) and each $n \in \mathbb{N}$, if $\{y_0, \ldots, y_m\}$ is a uniform V-pseudo-orbit for g^n, then $\{y_0, g(y_0), \ldots, g^{n-1}(y_0), y_1, g(y_1), \ldots, g^{n-1}(y_1), y_2, \ldots, y_{m-1}, g(y_{m-1}), \ldots, g^{n-1}(y_{m-1}), y_m\}$ is a uniform V-pseudo-orbit for g.

Now, we show that (2) implies (1). Let $n \in \mathbb{N}$ and suppose that g^n has uniform h-shadowing. Let $U \in \mathfrak{U}_Z$. By Lemma 5.7, there exists $V' \in \mathfrak{U}_Z$ such that if $\{z_0, \ldots, z_n\}$ is a uniform V'-pseudo-orbit for g and z is a point of Z such that $\rho(z, z_0) < V'$, then $\rho(g^k(z), z_k) < U$ for all $k \in \{1, \ldots, n\}$.

Since g^n has uniform h-shadowing, there exists $W \in \mathfrak{U}_Z$ such that if $\{w_0, \ldots, w_r\}$ is a uniform W-pseudo-orbit for g^n, there exists a point w in Z such that $\rho(g^{nk}(w), w_k) < V'$ for each $k \in \{0, \ldots, r-1\}$ and $g^{nr}(w) = w_r$. By Lemma 5.7, there exists $W' \in \mathfrak{U}_Z$ such that whenever $\{z_0, \ldots, z_n\}$ is a uniform W'-pseudo-orbit for g, we have that $\rho(g^k(z_0), z_k) < W$ for every $k \in \{1, \ldots, n\}$.

Let $\{w_0, \ldots, w_m\}$ be a uniform W'-pseudo-orbit for g. Note that there exist $j \geq 0$ and $r < n$ such that $m = jn + r$. Since g is onto, there exists $z \in Z$ such that $g^{n-r}(z) = w_0$. Then $\{z, g(z), \ldots, g^{n-r}(z), w_1, \ldots, w_m\}$ is a uniform W'-pseudo-orbit for g, which we enumerate obtaining the sequence $\{y_0 \ldots, y_{(j+1)n}\}$. We show that $\{y_0, y_{2n}, \ldots, y_{(j+1)n}\}$ is a uniform W-pseudo-orbit for g^n. To this end, observe that $\{y_0, \ldots, g^{n-r}(y_0) = y_{n-r}, \ldots, y_n\}$ is a uniform W'-pseudo-orbit for g of length $n+1$. Hence, $\rho(g^n(y_0), y_n) < W$ (Lemma 5.7). Continuing with this process, we have that $\rho(g^n(y_{\ell n}), y_{(\ell+1)n}) < W$ for all $\ell \in \{1, \ldots, j\}$.

Since g^n has uniform h-shadowing, there exists a point w of Z such that $\rho(g^{\ell n}(w), y_{\ell n}) < V'$ for every $\ell \in \{0, \ldots, j\}$ and $g^{(j+1)n}(w) = y_{(j+1)n}$. By the definition of V', $\rho(g^{kn+\ell}(w), y_{kn+\ell}) < U$ for each $k \in \{0, \ldots, j+1\}$ and all $\ell \in \{0, \ldots, n-1\}$. Thus, the point w uniform U-shadows the uniform W'-pseudo-orbit $\{y_0, y_{2n}, \ldots, y_{(j+1)n}\} = \{z, g(z), \ldots, g^{n-r}(z), w_1, \ldots, w_m\}$ for g. As a consequence of this, the point $u = g^{n-r}(w)$ uniform U-shadows the uniform W'-pseudo-orbit $\{w_0, \ldots, w_m\}$ and $g^m(u) = g^m\left(g^{n-r}(w)\right) = g^{(j+1)n}(w) = y_{(j+1)n} = w_m$. Therefore, g has uniform h-shadowing. □

As a consequence of Theorems 5.1 and 5.8, we have that:

Theorem 5.9. [13, Theorem 3.9] *Let be a compact Hausdorff space and let $g\colon Z \to Z$ be an onto map. The following are equivalent:*

(1) g has Haudorff shadowing;

(2) g^n has Hausdorff shadowing for some $n \in \mathbb{N}$;

(3) g^n has Hausdorff shadowing for all $n \in \mathbb{N}$.

The same holds for Hausdorff h-shadowing.

Let Z be a compact Hausdorff space and let $g\colon Z \to Z$ be a homeomorphism. Then g has *full uniform shadowing* provided that for each $U \in \mathfrak{U}_Z$ (Remark 2.2), there exists $V \in \mathfrak{U}_Z$ such that if $\{z_n\}_{n=-\infty}^{\infty}$ is a uniform V-pseudo-orbit of g, then there exists a point z in Z such that $\rho(g^n(z), z_n) < U$ for all $n \in \mathbb{Z}$.

Theorem 5.10. [7, Theorem 2.3.4] *Let Z be a compact Hausdorff space and let $g\colon Z \to Z$ be a homeomorphism. If g has full uniform shadowing, then g^{-1} has full uniform shadowing.*

Proof. Let $U \in \mathfrak{U}_Z$ (Remark 2.2). Since g has full uniform shadowing, there exists $V \in \mathfrak{U}_Z$ such that if $\{z_n\}_{n=-\infty}^{\infty}$ is a uniform V-pseudo-orbit of g, then there exists a point z in Z such that $\rho(g^n(z), z_n) < U$ for all $n \in \mathbb{Z}$. Since g is a map, there exists $V' \in \mathfrak{U}_Z$ such that if z' and z'' are two points of Z such that $\rho(z', z'') < V'$, then $\rho(g(z'), g(z'')) < V$.

Let $\{z_n\}_{n=-\infty}^{\infty}$ be a uniform V'-pseudo-orbit of g^{-1}. Since $\rho(g^{-1}(z_n), z_{n+1}) < V'$ for every $n \in \mathbb{Z}$, we have that $\rho(z_n, g(z_{n+1})) < V$ for all $n \in \mathbb{Z}$. For each $n \in \mathbb{Z}$, let $w_n = z_{-n}$. Then $\{w_n\}_{n=-\infty}^{\infty}$ is a uniform V-pseudo-orbit of g. Thus, there exists a point z of Z such that $\rho(g^n(z), w_n) < U$ for every $n \in \mathbb{Z}$. This implies that $\rho(g^n(z), z_{-n}) < U$ for each $n \in \mathbb{Z}$. Hence, $\rho(g^{-n}(z), z_n) < U$ for all $n \in \mathbb{Z}$. Therefore, g^{-1} has full uniform shadowing. □

Let Z be a compact Hausdorff space and let $g\colon Z \to Z$ be a map. Then g satisfies the *uniform periodic shadowing property*, if for all $U \in \mathfrak{U}_Z$ (Remark 2.2), there exists $V \in \mathfrak{U}_Z$ such that for every $n \in \mathbb{N}$ and each uniform periodic V-pseudo-orbit $z_0, \dots, z_{n-1}$– that is a finite sequence of points $z_0, \dots, z_{n-1}$ such that $\rho(g(z_j), z_{(j+1)(\mathrm{mod\ n})}) < V$–, there exists a point z in Z, of period n, such that for all $j \in \{0, \dots, n-1\}$, $\rho(g^j(z), z_{j+1}) < U$.

Let Z be a compact Hausdorff space and let $g\colon Z \to Z$ be a map. A point z of Z is a *wandering point of* g, if there exists an open subset W of Z such that $z \in W$ and $g^n(W) \cap W = \emptyset$ for all $n \in \mathbb{N}$. If z is not a wandering point of g, then z is a *non-wandering point of* g. The set of non-wandering points of g is denoted by $\Omega(g)$.

Theorem 5.11. [7, Theorem 2.4.8] *Let Z be a compact Hausdorff space and let $g\colon Z \to Z$ be a map satisfying the uniform periodic shadowing property. Then $Per(g)$ is dense in $\Omega(g)$.*

Proof. Let $z \in \Omega(g)$ and let W be an open subset of Z such that $z \in W$. Then there exists $U \in \mathfrak{U}_Z$ (Remark 2.2) such that $B(z, U) \subset W$. Let $U' \in \mathfrak{U}_Z$ be such that $2U' \subset U$. Since g satisfies the uniform periodic shadowing property, there exists $V \in \mathfrak{U}_Z$ such that for each $N \in \mathbb{N}$ and every uniform periodic V-pseudo-orbit, $\{z_0, \dots, z_{n-1}\}$, there exists a point z' of Z, of period n, such that for all $j \in \{0, \dots, n-1\}$, $\rho(g^j(z'), z_j) < U'$. Without loss of generality, we assume that $V \subset U'$. Let $V' \in \mathfrak{U}_Z$ be such that $2V' \subset V$. Since $z \in \Omega(g)$, there exist $z' \in Int_Z(B(z, V'))$ and an integer $n \geq 1$ such that $g^n(z') \in Int_Z(B(z, V'))$.

Hence, $\rho(z', g^n(z')) < 2V'$. Thus, since $2V' \subset V$, $\rho(z', g^n(z')) < V$. This implies that there exists a point z'' in Z, of period $n+1$, such that $\rho(z'', z') < U'$, and for all $j \in \{1, \dots, n\}$, $\rho(g^j(z''), g^j(z')) < U'$. Since $\rho(z, z') < V'$, $\rho(z', z'') < U'$, $V' \subset U'$ and $2U' \subset U$, we have that $\rho(z, z'') < U$. In particular, $z'' \in W$. Therefore, the set of periodic points of g is dense in $\Omega(g)$. $\square$

6. **Uniform internal chain transitivity.** We prove that uniform internal chain transitivity is equivalent to weak incompressibility. We also show that if g has uniform shadowing, then the set of uniform nonwandering points is equivalent to the set of uniform chain recurrent points.

Let Z be a compact Hausdorff space, let Λ be a subset of Z and let $g\colon Z \to Z$ be a map. Then Λ is *uniform internally chain transitive* (or g is uniform internally chain transitive on Λ) if for every pair of points z and z' of Λ and every $U \in \mathfrak{U}_Z$ (Remark 2.2), there exists a uniform U-pseudo-orbit $\{z_0 = z, \dots, z_m = z'\} \subset \Lambda$ between z and z' of length greater than one. The set Λ is *weakly incompressible* (or has *weak incompressibility*) if $M \cap Cl_\Lambda(g(\Lambda \setminus M)) \neq \emptyset$, for all nonempty, proper, closed subsets M of Λ.

Remark 6.1. Let Z be a compact Hausdorff space, let Λ be a subset of Z and let $g\colon Z \to Z$ be a map. Then Λ is weakly incompressible if and only if $Cl_\Lambda(g(U)) \cap (\Lambda \setminus U) \neq \emptyset$ for any nonempty, proper, open subset U of Λ.

Let Z be a compact Hausdorff space and let $g\colon Z \to Z$ be a map. Then the *ω-limit set* of a point z in Z is the set

$$\omega(z, g) = \bigcap_{n=1}^{\infty} Cl(\{g^k(z) \mid k \geq n\}).$$

Note that in the non-metric case this does not necessarily coincide with the set of limit points of all subsequences of the sequence $\{g^n(z)\}_{n=1}^{\infty}$: consider the extension of the function $g\colon \mathbb{Z} \to \mathbb{Z}$ given by $g(n) = n+1$ to the Stone-Čech compactification $\beta\mathbb{Z}$. Then $\omega(0, g) = \beta\mathbb{N} \setminus \mathbb{N}$, but no point of $\beta\mathbb{N} \setminus \mathbb{N}$ is the limit of a sequence.

Theorem 6.2. [12, Lemma 3.2.7] *Let Z be a compact Hausdorff space and let $g\colon Z \to Z$ be a map. For any $z \in Z$, $\omega(z, g)$ is uniform internally chain transitive.*

Proof. Let $U \in \mathfrak{U}_Z$ (Remark 2.2), and let $U' \in \mathfrak{U}_Z$ be such that $3U' \subset U$. Since g is a map, there exists $V \in \mathfrak{U}_Z$ such that if z' and z'' are two points of Z such that $\rho(z', z'') < V$, then $\rho(g(z'), g(z'')) < U'$. Without loss of generality, we assume that $V \subset U'$.

Let w and w' be two elements of $\omega(z, g)$. Then there exists $m \geq 2$ such that $g^{m-1}(z) \in Int_Z(B(w, V))$. Hence, $\rho(g^m(z), g(w)) < U'$. Also, there exists $k > m$ such that $\rho(g^k(z), w') < U'$. Then the set $W = \{w_0 = w, w_1 = g^m(z), \dots, w_{k-m} = g^{k-1}(z), w_{k_m+1} = w'\} \subset B(\omega(z, g), U')$ is a uniform U'-pseudo-orbit between w and w'. For each $w_j \in W$, there exists $z_j \in \omega(z, g)$ such that $\rho(w_j, z_j) < U'$. Let $z_0 = w$ and $z_{k-m+1} = w'$. Then, for every $j \in \{0, \dots, k_m\}$, we have that $\rho(g(z_j), g(w_j)) < U'$, $\rho(g(w_j), w_{j+1}) < U'$ and $\rho(w_{j+1}, z_{j+1}) < U'$. Thus, $\rho(g(z_j), z_{j+1}) < 3U'$. Since $3U' \subset U$, we obtain that $\rho(g(z_j), z_{j+1}) < U$. Therefore, $\omega(z, g)$ is uniform internally chain transitive. $\square$

Lemma 6.3. [14, Proposition 1] *Let Z be a compact Hausdorff space, let Λ be a subset of Z and let $g\colon Z \to Z$ be a map. If Λ is a uniform internally chain transitive subset of Z, then $g(\Lambda) \subset \Lambda$; i.e., Λ is invariant.*

Proof. We need to show that $g(\Lambda) \subset \Lambda$. Let z be a point of Λ. Let z' be a point in $\Lambda \setminus \{z\}$, and let $U \in \mathfrak{U}_Z$ (Remark 2.2). Since Λ is uniform internally chain transitive, there exists a uniform U-pseudo-orbit for g $\{z_0 = z, \ldots, z_m = z'\} \subset \Lambda$. In particular, $g(z) \in \Lambda$. Therefore, Λ is invariant □

Theorem 6.4. [14, Theorem 2.1] *Let Z be a compact Hausdorff space, let Λ be a closed subset of Z and let $g\colon Z \to Z$ be a map. Then Λ is uniform internally chain transitive if and only if Λ is weakly incompressible.*

Proof. Suppose Λ is weakly incompressible. Let W be a nonempty proper open subset of Λ. Define $G(W) = Cl_\Lambda(g(W)) \cap (\Lambda \setminus W)$. Since Λ is weakly incompressible, $G(W)$ is nonempty.

Let $U \in \mathfrak{U}_Z$ (Remark 2.2) and let $U' \in \mathfrak{U}_Z$ be such that $2U' \subset U$. Since Λ is compact, there exist $w_1, \ldots, w_r$ in Λ such that $\{Int_Z(B(w_j, U'))\}_{j=1}^r$ is a finite open cover of Λ with no proper subcover. Let $\mathcal{B} = \{\Lambda \cap Int_Z(B(w_j, U'))\}_{j=1}^r$. Let $j \in \{1, \ldots, r\}$ and let $B_j = \Lambda \cap Int_Z(B(w_j, U'))$. Let $j_1 = 1$. Unless $B_{j_1} = \Lambda$, we have that $G(B_{j_1}) \neq \emptyset$, and there exists $j_2 \in \{1, \ldots, r\} \setminus \{j_1\}$ such that $B_{j_2} \cap Cl_\Lambda(g(B_{j_1})) \neq \emptyset$. Hence, $B_{j_2} \cap g(B_{j_1}) \neq \emptyset$. Suppose we have chosen $B_{j_1}, \ldots, B_{j_k}$ in $\mathcal{B}$ such that for each $t \in \{2, \ldots, k\}$, there exists $s < t$ such that $B_{j_t} \cap g(B_{j_s}) \neq \emptyset$. Assume that $\bigcup_{t=1}^k B_{j_t} \neq \Lambda$. Then $G\left(\bigcup_{t=1}^k B_{j_t}\right) \neq \emptyset$. Thus, there exists $B_{j_{k+1}} \in \mathcal{B}$ such that $B_{j_{k+1}} \cap g\left(\bigcup_{t=1}^k B_{j_t}\right) \neq \emptyset$. This implies that there exists $t \in \{1, \ldots, k\}$ such that $B_{j_{k+1}} \cap g(B_{j_t}) \neq \emptyset$. Since $\mathcal{B}$ is a minimal cover, it follows that for any two elements B and B' of $\mathcal{B}$, there exist $B_1, \ldots, B_n$ in $\mathcal{B}$ such that $B_1 = B$, $B_n = B'$ and $B_{j+1} \cap g(B_j) \neq \emptyset$.

Let z and z' be two points of Λ, and let B and B' elements of $\mathcal{B}$ such that $z \in B$ and $z' \in B'$. By the previous paragraph, there exist $B_1, \ldots, B_n$ in $\mathcal{B}$ such that $B_1 = B$, $B_n = B'$ and $B_{j+1} \cap g(B_j) \neq \emptyset$. For each $j \in \{1, \ldots, n-1\}$, let $z_j \in B_j \cap g^{-1}(B_{j+1})$. Let $z_0 = z$ and let $z_n = z'$. Then $\{z_0, \ldots, z_n\}$ is a uniform U-pseudo-orbit, in Λ, for g from z to z'. Therefore, Λ is uniform internally chain transitive.

Suppose Λ is uniform internally chain transitive. Let M be a nonempty, proper, closed subset of Λ. Let $z \in \Lambda \setminus M$, let $z' \in M$, and let $U_0 \in \mathfrak{U}_Z$. Since Λ is uniform internally chain transitive, there exists a uniform U_0-pseudo-orbit for g from z to z'. Let z_{U_0} be the last point of the uniform U_0-pseudo-orbit that is not in M and there exists a point w_{U_0} in M such that $\rho(g(z_{U_0}), w_{U_0}) < U_0$. Observe that $\mathfrak{U}_Z$ is a directed set with respect to reverse inclusion. Hence $\{z_U\}_{U \in \mathfrak{U}_Z}$ and $\{w_U\}_{U \in \mathfrak{U}_Z}$ are nets. Since Λ is compact, by [22, 3.1.23 and 1.6.1], we assume, without loss of generality, that $\{z_U\}_{U \in \mathfrak{U}_Z}$ and $\{w_U\}_{U \in \mathfrak{U}_Z}$ converge to z_0 and w_0, respectively. Since g is a map, $\{g(z_U)\}_{U \in \mathfrak{U}_Z}$ converges to $g(z_0)$. Let $U \in \mathfrak{U}_Z$ and let $V \in \mathfrak{U}_Z$ be such that $3V \subset U$. Then there exists $V' \in \mathfrak{U}_Z$ such that $V' \subset V$ and $\rho(w_{V'}, w_0) < V$, $\rho(g(z_{V'}), g(z_0)) < V$ and $\rho(w_{V'}, g(z_{V'})) < V'$. Hence, $\rho(w_0, g(z_0)) < 3V$. Since $3V \subset U$, we obtain that $\rho(w_0, g(z_0)) < U$. Thus, $w_0 = g(z_0)$. Therefore, $g(z_0) \in M \setminus Cl_\Lambda(g(\Lambda \setminus M))$, and Λ is weakly incompressible. □

Let Z be a compact Hausdorff space, let z be a point of Z and let $g\colon Z \to Z$ be a map. Then Z is *uniform chain recurrent at z* if for every $U \in \mathfrak{U}_Z$ (Remark 2.2), there exists a uniform U-pseudo-orbit $\{z_0 = z, z_1, \ldots, z_m = z\}$, and $m \geq 1$. Let

$$CR(g) = \{z \in Z \mid Z \text{ is uniform chain recurrent at } z\}.$$

Lemma 6.5. [7, p. 96] *Let Z be a compact Hausdorff space, and let $g\colon Z \to Z$ be a homeomorphism. Then $g(CR(g)) = CR(g)$.*

Proof. Let $z \in CR(g)$ and let $U \in \mathfrak{U}_Z$ (Remark 2.2). Since g is a map, there exists $V \in \mathfrak{U}_Z$ such that if z' and z'' are two points of Z such that $\rho(z', z'') < V$, then $\rho(g(z'), g(z'')) < U$. Since $z \in CR(g)$, there exists a uniform V-pseudo-orbit, of g, $\{z_0, \dots, z_m\}$ such that $z_0 = z = z_m$. Then $\{g(z_0), \dots, g(z_m)\}$ is a uniform U-pseudo-orbit such that $g(z_0) = g(z) = g(z_m)$. Hence, $g(z) \in CR(g)$. Thus, $g(CR(g)) \subset CR(g)$. Similarly, $g^{-1}(CR(g)) \subset CR(g)$. Therefore, $g(CR(g)) = CR(g)$. □

The following results are stated for homeomorphisms in [7, pp. 96 and 97]. They are true for onto maps.

Lemma 6.6. [7, p. 96] *Let Z be a compact Hausdorff space, and let $g\colon Z \to Z$ be a map. Then $\Omega(g) \subset CR(g)$.*

Proof. Let $z \in \Omega(g)$ and let $U \in \mathfrak{U}_Z$ (Remark 2.2). Since $z \in \Omega(g)$, there exists $n \in \mathbb{N}$ such that $g^n(Int_Z(B(z, U))) \cap Int_Z(B(z, U)) \neq \emptyset$. Then $\{z, g(z), \dots, g^n(z), z\}$ is a uniform U-pseudo-orbit of g. Therefore, $z \in CR(g)$. □

Lemma 6.7. [7, Lemma 3.1.1] *Let Z be a compact Hausdorff space, and let $g\colon Z \to Z$ be a map. Then $CR(g)$ is closed in Z.*

Proof. Let $z \in Cl_Z(CR(g))$ and let $U \in \mathfrak{U}_Z$ (Remark 2.2). Let $U' \in \mathfrak{U}_Z$ be such that $2U' \subset U$. Since g is a map, there exists $V \in \mathfrak{U}_Z$ such that $V \subset U'$ and if z' and z'' are points of Z such that $\rho(z', z'') < V$, then $\rho(g(z'), g(z'')) < U'$. Let $w \in Int_Z(B(z, V)) \cap CR(g)$. Since $w \in CR(g)$, there exists a uniform V-pseudo-orbit $\{w_0, \dots, w_m\}$ such that $w_0 = w = w_m$. Then $\{z, w_1, \dots, w_m, z\}$ is a uniform U-pseudo-orbit of g. Therefore, $CR(g)$ is closed in Z. □

Theorem 6.8. [7, Theorem 3.1.2] *Let Z be a compact Hausdorff space, and let $g\colon Z \to Z$ be a map. If g has uniform shadowing, then $\Omega(g) = CR(g)$.*

Proof. By Lemma 6.6, we have that $\Omega(g) \subset CR(g)$. Let $z \in CR(g)$ and let W be an open subset of Z containing z. Then there exists $U \in \mathfrak{U}_Z$ (Remark 2.2) such that $B(z, U) \subset W$. Since g has uniform shadowing, there exists $V \in \mathfrak{U}_Z$ such that every uniform V-pseudo-orbit is uniform U-shadowed by a point of Z. Since $z \in CR(g)$, there exists a uniform V-pseudo-orbit $\{z_0, \dots, z_m\}$ such that $z_0 = z = z_m$. Then, since g has uniform shadowing, there exists a point w in Z such that $\rho(g^j(w), z_j) < U$ for all $j \in \{0, \dots, m\}$. This implies that $g^m(B(z, U)) \cap B(z, U) \neq \emptyset$. Hence, $g^m(W) \cap W \neq \emptyset$. Therefore, $z \in \Omega(g)$, and $CR(g) \subset \Omega(g)$. □

7. **Uniform shadowing on hyperspaces.** We extend some results of [23] to the uniform case. In particular, we show that the induced map on the hyperspace of compact subsets of a compact Hausdorff space has uniform shadowing if and only if the map itself does.

Note that finite shadowing and shadowing are equivalent concepts for compact Hausdorff spaces (Theorem 5.5, compare with Theorem 5.4). Theorem 5.6 is useful in this section.

Let Z be a compact Hausdorff space. We consider the following *hyperspaces* of Z:

$$2^Z = \{A \subset Z \mid A \text{ is closed and nonempty}\};$$

$$\mathcal{C}_n(Z) = \{A \in 2^Z \mid A \text{ has at most } n \text{ components}\}, n \in \mathbb{N};$$
$$\mathcal{C}_\infty(Z) = \{A \in 2^Z \mid A \text{ has only finitely many components}\};$$
$$\mathcal{F}_n(Z) = \{A \in 2^Z \mid A \text{ has at most } n \text{ points}\}, n \in \mathbb{N};$$
$$\mathcal{F}_\infty(Z) = \{A \in 2^Z \mid A \text{ is finite}\}.$$

We define a uniformity on 2^Z as follows: If $U \in \mathfrak{U}_Z$ (Remark 2.2), then let $2^U = \{(A, A') \in 2^Z \times 2^Z \mid A \subset B(A', U) \text{ and } A' \subset B(A, U)\}$. Let $\mathfrak{B}_Z = \{2^U \mid U \in \mathfrak{U}_Z\}$. Then $\mathfrak{B}_Z$ is a base for a uniformity, denoted by $2^{\mathfrak{U}_Z}$ [22, 8.5.16]. Observe that the topology generated by $2^{\mathfrak{U}_Z}$ coincides with the Vietoris topology [31, 3.3]. Hence, 2^Z is compact and Hausdorff [31, 4.9]. Thus, $2^{\mathfrak{U}_Z}$ is unique (Remark 2.2), and $2^{\mathfrak{U}_Z} = \{\mathcal{U} \subset 2^Z \times 2^Z \mid \text{there exists } U \in \mathfrak{U}_Z \text{ such that } 2^U \subset \mathcal{U}\}$. For the other hyperspaces, we use the restriction of $2^{\mathfrak{U}_Z}$ to the corresponding hyperspace and we denote such restriction by: $\mathcal{C}_n(\mathfrak{U}_Z)$, $\mathcal{C}_\infty(\mathfrak{U}_Z)$, $\mathcal{F}_n(\mathfrak{U}_Z)$ and $\mathcal{F}_\infty(\mathfrak{U}_Z)$, respectively. In order to avoid confusion, we put a subindex to the expressions: $\rho_Z(z, z') < U$, $\rho_{2^Z}(A, A') < \mathcal{U}$, $\rho_{\mathcal{C}_n(Z)}(A, A') < \mathcal{U}$, $\rho_{\mathcal{C}_\infty(Z)}(A, A') < \mathcal{U}$, $\rho_{\mathcal{F}_n(Z)}(A, A') < \mathcal{U}$ and $\rho_{\mathcal{F}_\infty(Z)}(A, A') < \mathcal{U}$, respectively.

Let Z be a compact Hausdorff space and let $g\colon Z \to Z$ be a map. Then the functions: 2^g, $\mathcal{C}_n(g)$, $\mathcal{C}_\infty(g)$, $\mathcal{F}_n(g)$ and $\mathcal{F}_\infty(g)$, given by $2^g(A) = g(A)$ for all $A \in 2^Z$, $\mathcal{C}_n(g) = 2^g|_{\mathcal{C}_n(Z)}$, $\mathcal{C}_\infty(g) = 2^g|_{\mathcal{C}_\infty(Z)}$, $\mathcal{F}_n(g) = 2^g|_{\mathcal{F}_n(Z)}$ and $\mathcal{F}_\infty(g) = 2^g|_{\mathcal{F}_\infty(Z)}$ are the *induced maps* of g.

As a consequence of [31, 2.4.1], Theorem 5.6, we have:

Corollary 7.1. *Let Z be a compact Hausdorff space and let $g\colon Z \to Z$ be a map. Then 2^g has uniform finite shadowing if and only if $\mathcal{F}_\infty(g)$ has uniform finite shadowing.*

Theorem 7.2. [23, Theorem 3.2] *Let Z be a compact Hausdorff space and let $g\colon Z \to Z$ be a map. If either 2^g, $\mathcal{C}_n(g)$, $\mathcal{C}_\infty(g)$, $\mathcal{F}_n(g)$ or $\mathcal{F}_\infty(g)$ has (finite) uniform shadowing, then g has (finite) uniform shadowing.*

Proof. We give the proof for 2^g, the proofs for the other maps are similar.

Let $U \in \mathfrak{U}_Z$ (Remark 2.2). Then $2^U \in 2^{\mathfrak{U}_Z}$. Since 2^g has (finite) uniform shadowing, there exists $\mathcal{V} \in 2^{\mathfrak{U}_Z}$ such that each (finite) uniform $\mathcal{V}$-pseudo-orbit of 2^g is uniform 2^U-shadowed by an element of 2^Z. Let $V \in \mathfrak{U}_Z$ be such that $2^V \subset \mathcal{V}$. Let $\{z_0, z_1, \dots\}$ be a (finite) uniform V-pseudo-orbit of g. Then $\{\{z_0\}, \{z_1\}, \dots\}$ is a (finite) uniform 2^V-pseudo-orbit of 2^g. Since $2^V \subset \mathcal{V}$, $\{\{z_0\}, \{z_1\}, \dots\}$ is a (finite) uniform $\mathcal{V}$-pseudo-orbit of 2^g. Hence, there exists an element A of 2^Z such that for all $j \geq 0$, $\rho_{2^Z}((2^g)^j(A), \{z_j\}) < 2^U$. Thus, if $a \in A$, then for each $j \geq 0$, $\rho_{2^Z}((2^g)^j(\{a\}), \{z_j\}) < 2^U$. This implies that for every $j \geq 0$, $\rho_Z(g^j(a), z_j) < U$. Therefore, g has (finite) uniform shadowing. $\square$

Theorem 7.3. [23, Theorem 3.3] *Let Z be a compact Hausdorff space and let $g\colon Z \to Z$ be a map. If g has uniform finite shadowing, then $\mathcal{F}_\infty(g)$ has uniform finite shadowing.*

Proof. Suppose g has uniform finite shadowing. Let $\mathcal{U} \in \mathcal{F}_\infty(\mathfrak{U}_Z)$ (Remark 2.2). Then there exists $U \in \mathfrak{U}_Z$ such that $\mathcal{F}_\infty(U) \subset \mathcal{U}$. Since g has uniform finite shadowing, there exists $V \in \mathfrak{U}_Z$ such that each uniform finite V-pseudo-orbit of g is uniform U-shadowed by an element of Z. Let $\mathfrak{L} = \{A_0, \dots, A_r\}$ be a uniform finite $\mathcal{F}_\infty(V)$-pseudo-orbit of $\mathcal{F}_\infty(g)$. Assume that $A_r = \{a_{r,1}, \dots, a_{r,n_r}\}$. Since $\mathfrak{L}$ is a uniform $\mathcal{F}_\infty(V)$-pseudo-orbit of $\mathcal{F}_\infty(g)$, $\rho_{\mathcal{F}_\infty(Z)}(\mathcal{F}_\infty(g)(A_{r-1}), A_r) < \mathcal{F}_\infty(V)$. Hence, for each $j \in \{1, \dots, n_r\}$, there exists $a_{r-1,j} \in A_{r-1}$ such that

$\rho(g(a_{r-1,j}), a_{r,j}) < V$. Similarly, for each $j \in \{1, \ldots, n_r\}$, there exists $a_{r-2,j} \in A_{r_2}$ such that $\rho(g(a_{r-2,j}), a_{r-1,j}) < V$. Continuing with this process, we construct a uniform V-pseudo-orbit of g with n_r elements. If $j \in \{1, \ldots, n_r\}$, then let $\Lambda_j = \{a_{0,j}, \ldots, a_{r,j}\}$ be the jth uniform V-pseudo-orbit constructed. Suppose $\bigcup_{j=1}^{n_r} \Lambda_j \subsetneq \bigcup_{l=0}^{r} A_l$, and let $k = \max\{\ell \in \{0, \ldots, r\} \mid A_\ell \setminus \bigcup_{j=1}^{n_r} \Lambda_j \neq \emptyset\}$. Note the $k < r$. Let $a_{k,n_r+1} \in A_k \setminus \bigcup_{j=1}^{n_r} \Lambda_j$. Since $\rho_{\mathcal{F}_\infty(Z)}(\mathcal{F}_\infty(g)(A_{k-1}), A_k) < \mathcal{F}_\infty(V)$, there exists $a_{k-1,n_r+1} \in A_{k-1}$ such that $\rho(g(a_{k-1,n_r+1}), a_{k,n_r+1}) < V$. Continuing with this process, we construct a uniform V-pseudo-orbit $\Lambda'_{n_r+1} = \{a_{0,n_r+1}, \ldots, a_{k,n_r+1}\}$. Since $\rho_{\mathcal{F}_\infty(Z)}(\mathcal{F}_\infty(g)(A_k), A_{k+1}) < \mathcal{F}_\infty(V)$, there exists $a_{k+1} \in A_{k+1}$ such that $\rho(g(a_{k,n_r+1}), a_{k+1}) < V$. By the election of k, $a_{k+1} \in \bigcup_{j=1}^{n_r} \Lambda_j$. Suppose $a_{k+1} \in \Lambda_{j_0}$. Hence, $a_{k+1} = a_{k+1,j_0}$. Then $\Lambda_{n_r+1} = \Lambda'_{n_r+1} \cup \{a_{k+1,j_0}, \ldots, a_{r,j_0}\}$ is a uniform finite V-pseudo-orbit of g starting at a point of A_0 and ending at a point of A_r. Repeat this process for all points in $A_k \setminus \bigcup_{j=1}^{n_r} \Lambda_j$ to obtain more uniform finite V-pseudo-orbits starting at points of A_0, passing through the points of A_k and ending at points of A_r. Repeat the above process for all $\ell < k$ such that there exists a point of A_ℓ that is not in any of the uniform finite V-pseudo-orbits of g already constructed. Suppose that we have m uniform finite V-pseudo-orbits of g, say $\Lambda_1, \ldots, \Lambda_m$, each of which starts at a point of A_0, ends at a point A_r and each point of $\bigcup_{j=0}^{r} A_j$ belongs to at least one of the $\Lambda_1, \ldots, \Lambda_m$. Since g has uniform finite shadowing, for each $j \in \{1, \ldots, m\}$, there exists $z_j \in Z$ such that z_j uniform U-shadows Λ_j.

Now, we show that $\{z_1, \ldots, z_m\}$ uniform $\mathcal{F}_\infty(U)$-shadows $\mathfrak{L}$. Let $a_0 \in A_0$. Then there exists $j \in \{1, \ldots, m\}$ such that $a_0 \in \Lambda_j$. Since z_j uniform U-shadows Λ_j, we have that $\rho(z_j, a_0) < U$. Hence, $A_0 \subset B(\{z_1, \ldots, m\}, U)$. Let $z_j \in \{z_1, \ldots, z_m\}$. Since z_j uniform U-shadows Λ_j, there exists $a_0 \in A_0$ such that $\rho(a_0, z_j) < U$. Thus, $\{z_1, \ldots, z_m\} \subset B(A_0, U)$. As a consequence of this, $\rho_{\mathcal{F}_\infty(Z)}(A_0, \{z_1, \ldots, m\}) < \mathcal{F}_\infty(U)$. Let $k \in \{1, \ldots, r\}$ and let $a_k \in A_k$. Then, by the way we constructed the uniform V-pseudo-orbits of g, for every $l \in \{0, \ldots, k-1\}$, there exists $a_l \in A_l$ such that $\rho(g(a_l), a_{l+1}) < V$. Thus, there exists $j_k \in \{1, \ldots, m\}$ such that $\{a_0, \ldots, a_k\} \subset \Lambda_{j_k}$. Since z_{j_k} uniform U-shadows Λ_{j_k}, we have, in particular, $\rho(g^k(z_{j_k}), a_k) < U$. Hence, $A_k \subset B(\{z_1, \ldots, z_m\}, U)$. Let $z_j \in \{z_1, \ldots, z_m\}$. Since z_j uniform U-shadows Λ_j, there exists, for each $l \in \{0, \ldots, k\}$, $a_l \in A_l$ such that $\{a_0, \ldots, a_k\} \subset \Lambda_j$, and $\rho(g^l(z_j), a_l) < U$. In particular, $\rho(g^k(z_j), a_k) < U$. Thus, $\{z_1, \ldots, z_m\} \subset B(A_k, U)$. Hence, $\rho_{\mathcal{F}_\infty(Z)}(A_k, \{z_1, \ldots, z_m\}) < \mathcal{F}_\infty(U)$. Since $\mathcal{F}_\infty(U) \subset \mathcal{U}$, we have that for each $k \in \{0, \ldots, m\}$, $\rho_{\mathcal{F}_\infty(Z)}(A_k, \{z_1, \ldots, z_m\}) < \mathcal{U}$. Therefore, $\{z_1, \ldots, z_m\}$ uniform $\mathcal{U}$-shadows $\mathfrak{L}$, and $\mathcal{F}_\infty(g)$ has uniform finite shadowing. □

As a consequence of Theorems 7.3, 5.6, 5.5, and 7.2, we have:

Theorem 7.4. [23, Theorem 3.4] *Let Z be a compact Hausdorff space and let $g\colon Z \to Z$ be a map. Then g has uniform shadowing if and only if 2^g has uniform shadowing.*

As a consequence of Theorem 5.1 and 7.4, we obtain:

Theorem 7.5. *Let Z be a compact Hausdorff space and let $g\colon Z \to Z$ be a map. Then g has Hausdorff shadowing if and only if 2^g has Hausdorff shadowing.*

8. **Appendix.**

In this section we give the proofs of the equivalence of the definitions presented.

We begin with the various forms of sensitive dependence on initial conditions.

Proof of Theorem 3.2. Suppose g has Hausdorff sensitive dependence on initial conditions. Let $\mathcal{U}$ be a finite open cover of Z given by the Hausdorff sensitivity dependence on initial conditions of g. Since Z is compact, by Theorem 2.4, there exists $V \in \mathfrak{U}_Z$ (Remark 2.2) such that $\mathfrak{C}(V)$ refines $\mathcal{U}$. Let z be a point of Z and let $W \in \mathfrak{U}_Z$. Since $B(z,W)$ is a neighbourhood of z [22, 8.1.3], there exist $z' \in B(z,W)$ and $k \in \mathbb{N}$ such that $|\{g^k(z), g^k(z')\} \cap U| \leq 1$ for all $U \in \mathcal{U}$. This implies that $\rho(g^k(z), g^k(z')) \geq V$. Otherwise, $\rho(g^k(z), g^k(z')) < V$. Hence, $\{g^k(z), g^k(z')\} \subset B(g^k(z), V) \subset U$, for some $U \in \mathcal{U}$, a contradiction. Therefore, $\rho(g^k(z), g^k(z')) \geq V$ and g has uniform sensitive dependence on initial conditions.

Assume g has uniform sensitive dependence on initial conditions. Let $V \in \mathfrak{U}_Z$ be given by the uniform sensitivity dependence on initial conditions of g. Let $V' \in \mathfrak{U}_Z$ be such that $2V' \subset V$. Since $\mathfrak{C}(V')$ covers Z, we have that $\{Int_Z(B(z,V')) \mid z \in Z\}$ also covers Z. Since Z is compact, there exist $z_1, \dots, z_n$ in Z such that $\mathcal{U} = \{Int_Z(B(z_j, V'))\}_{j=1}^n$ is a finite subcover. We show that $\mathcal{U}$ satisfies the definition of Hausdorff sensitivity dependence on initial conditions of g. Let z be a point of Z and let A be an open subset of Z containing z. Then, since $\mathfrak{U}_Z$ induces the topology of Z, there exists $W \in \mathfrak{U}_Z$ such that $B(z,W) \subset A$. Since g has uniform sensitive dependence on initial conditions, there exist $z' \in B(z,W)$ and $k \in \mathbb{N}$ such that $\rho(g^k(z), g^k(z')) \geq V$. Hence, for each $j \in \{1, \dots, n\}$, $|\{g^k(z_1), g^k(z_2)\} \cap Int_Z(B(z_j, V'))| \leq 1$. Otherwise, there would exist $\ell \in \{1, \dots, n\}$ such that $\{g^k(z), g^k(z')\} \subset Int_Z(B(z_\ell, V'))$. Thus, $\rho(g^k(z), z_\ell) < V'$ and $\rho(z_\ell, g^k(z')) < V'$. Hence, $\rho(g^k(z), g^k(z')) < V' + V' = 2V'$. Since $2V' \subset V$, we obtain that $\rho(g^k(z), g^k(z')) < V$, a contradiction. Therefore, for each $j \in \{1, \dots, n\}$, $|\{g^k(z), g^k(z')\} \cap Int_Z(B(z_j, V'))| \leq 1$ and g has Hausdorff sensitive dependence on initial conditions.

For the rest of the proof, assume Z is metric. Suppose g has sensitive dependence on initial conditions. Let $\delta > 0$ be given by the sensitivity dependence on initial conditions of g. Note that $\left\{\mathcal{V}_{\frac{\delta}{2}}(z) \;\middle|\; z \in Z\right\}$ is an open cover of Z. Since Z is compact, there exist $z_1, \dots, z_n$ in Z such that $\mathcal{U} = \left\{\mathcal{V}_{\frac{\delta}{2}}(z_j)\right\}_{j=1}^n$ is a finite subcover. We show that $\mathcal{U}$ satisfies the definition of Hausdorff sensitivity dependence on initial conditions of g. Let w_1 be a point of Z and let V be an open subset of Z containing w_1. Since V is open, there exists $\varepsilon > 0$ such that $\mathcal{V}_\varepsilon(w_1) \subset V$. Since g has sensitive dependence on initial conditions, there exist $w_2 \in \mathcal{V}_\varepsilon(w_1)$ and $k \in \mathbb{N}$ such that $d(f^k(w_1), f^k(w_2)) \geq \delta$. Observe that for each $j \in \{1, \dots, n\}$, $|\{g^k(w_1), g^k(w_2)\} \cap \mathcal{V}_{\frac{\delta}{2}}(z_j)| \leq 1$; otherwise, there exists $\ell \in \{1, \dots, n\}$ such that $\{g^k(w_1), g^k(w_2)\} \subset \mathcal{V}_{\frac{\delta}{2}}(z_\ell)$. This implies that $d(g^k(w_1), g^k(w_2)) \leq d(g^k(w_1), z_\ell) + d(z_\ell, g^k(w_2)) < \delta$, a contradiction. Therefore, g has Hausdorff sensitive dependence on initial conditions.

Assume g has Hausdorff sensitive dependence on initial conditions. Let $\mathcal{U}$ be a finite open cover of Z given by the Hausdorff sensitivity dependence on initial conditions of g. Let $\delta > 0$ be a Lebesgue number for $\mathcal{U}$. Let z_1 be a point of Z and let $\varepsilon > 0$ be given. Since g has Hausdorff sensitive dependence on initial conditions, there exist $z_2 \in \mathcal{V}_\varepsilon(z_1)$ and $k \in \mathbb{N}$ such that $|\{g^k(z_1), g^k(z_2)\} \cap U| \leq 1$ for all $U \in \mathcal{U}$. Since δ is a Lebesgue number for $\mathcal{U}$, this implies that $d(g^k(z_1), g^k(z_2)) \geq \delta$. Therefore, g has sensitive dependence on initial conditions. $\square$

We continue with pseudo-orbits and shadowing.

Theorem 8.1. *Let X be a compactum, let $f\colon X \to X$ be a map and let $\delta > 0$. If $\{z_0, z_1, \ldots\}$ is a δ-pseudo-orbit, then there exists a finite open cover $\mathcal{V}$ of X such that $\{z_0, z_1, \ldots\}$ is a Hausdorff $\mathcal{V}$-pseudo orbit.*

Proof. Observe that $\left\{\mathcal{V}_{\frac{\delta}{2}}(x) \mid x \in X\right\}$ is an open cover of X. Since X is compact, there exist $x_1, \ldots, x_n$ in X such that $\mathcal{W} = \left\{\mathcal{V}_{\frac{\delta}{2}}(x_k)\right\}_{k=1}^{n}$ is a finite subcover. Let $\mathcal{V} = \{\mathcal{V}_{2\delta}(x_j)\}_{j=1}^{n}$. Then $\mathcal{V}$ is a finite open cover of X. Let $j \geq 0$. Since $\mathcal{W}$ is a cover of X, there exists $k \in \{1, \ldots, n\}$ such that $f(z_j) \in \mathcal{V}_{\frac{\delta}{2}}(x_k)$. Since $\{z_0, z_1, \ldots\}$ is a δ-pseudo-orbit, $d(f(z_j), z_{j+1}) < \delta$. Hence, $d(x_k, z_{j+1}) \leq d(x_k, f(z_j)) + d(f(z_j), z_{j+1}) \leq \frac{\delta}{2} + \delta = \frac{3}{2}\delta < 2\delta$. Therefore, $\{f(z_j), z_{j+1}\} \subset \mathcal{V}_{2\delta}(x_k)$ and $\{z_0, z_1, \ldots\}$ is a Hausdorff $\mathcal{V}$-pseudo orbit. $\square$

Theorem 8.2. *Let X be a compactum and let $f\colon X \to X$ be a map and let $\mathcal{V}$ be a finite open cover of X. If $\{z_0, z_1, \ldots\}$ is a Hausdorff $\mathcal{V}$-pseudo orbit, then there exists $\delta > 0$ such that $\{z_0, z_1, \ldots\}$ is a δ-pseudo-orbit.*

Proof. Let $\delta > \mathrm{mesh}(\mathcal{V})$. Let $j \geq 0$. Since $\{z_0, z_1, \ldots\}$ is a Hausdorff $\mathcal{V}$-pseudo-orbit, there exists $V_j \in \mathcal{V}$ such that $\{f(z_j), z_{j+1}\} \subset V_j$. This implies that $d(f(z_j), z_{j+1}) \leq \mathrm{diam}(V_j) \leq \mathrm{mesh}(\mathcal{V}) < \delta$. Therefore, $\{z_0, z_1, \ldots\}$ is a δ-pseudo-orbit. $\square$

Theorem 8.3. *Let X be a compactum, let $f\colon X \to X$ be a map and let $\varepsilon > 0$. If z is a point that ε-shadows the sequence $\{z_0, z_1, \ldots\}$, then there exists a finite open cover $\mathcal{U}$ of X such that z Hausdorff $\mathcal{U}$-shadows $\{z_0, z_1, \ldots\}$.*

Proof. Observe that $\{\mathcal{V}_{\frac{\varepsilon}{2}}(x) \mid x \in X\}$ is an open cover of X. Since X is compact, there exist $x_1, \ldots, x_n$ in X such that $\mathcal{W} = \{\mathcal{V}_{\frac{\varepsilon}{2}}(x_k)\}_{k=1}^{n}$ is a finite subcover. Let $\mathcal{U} = \{\mathcal{V}_{2\varepsilon}(x_j)\}_{j=1}^{n}$. Then $\mathcal{U}$ is a finite open cover of X. Let $j \geq 0$. Since $\mathcal{W}$ covers X, there exists $k \in \{1, \ldots, n\}$ such that $f^j(z) \in \mathcal{V}_{\frac{\varepsilon}{2}}(x_k)$. Since z ε-shadows $\{z_0, z_1, \ldots\}$, $d(x_k, z_j) \leq d(x_k, f^j(z)) + d(f^j(z), z_j)) < \frac{\varepsilon}{2} + \varepsilon = \frac{3}{2}\varepsilon < 2\varepsilon$. Therefore, $\{f^j(z), z_j\} \subset \mathcal{V}_{2\varepsilon}(x_k)$ and z Hausdorff $\mathcal{U}$-shadows $\{z_0, z_1, \ldots\}$. $\square$

Theorem 8.4. *Let X be a compactum, let $f\colon X \to X$ be a map and let $\mathcal{U}$ be a finite open cover of X. If z is a point that Hausdorff $\mathcal{U}$-shadows the sequence $\{z_0, z_1, \ldots\}$, then there exists $\varepsilon > 0$ such that z ε-shadows $\{z_0, z_1, \ldots\}$.*

Proof. Let $\varepsilon > \mathrm{mesh}(\mathcal{U})$. Let $j \geq 0$. Let z be a point of X that Hausdorff $\mathcal{U}$-shadows the sequence $\{z_0, z_1, \ldots\}$. Since z Hausdorff $\mathcal{U}$-shadows $\{z_0, z_1, \ldots\}$, there exists $U_j \in \mathcal{U}$ such that $\{f^j(z), z_j\} \subset U_j$. This implies that $d(f^j(z), z_j) \leq \mathrm{diam}(U_j) \leq \mathrm{mesh}(\mathcal{U}) < \varepsilon$. Therefore, z ε-shadows $\{z_0, z_1, \ldots\}$. $\square$

Theorem 8.5. *Let Z be a compact Hausdorff space, and let $g\colon Z \to Z$ be a map. Let $V \in \mathfrak{U}_Z$ (Remark 2.2). If $\{z_0, z_1, \ldots\}$ is a uniform V-pseudo-orbit, then there exists a finite open cover $\mathcal{V}$ of Z such that $\{z_0, z_1, \ldots\}$ is a Hausdorff $\mathcal{V}$-pseudo-orbit.*

Proof. Let $\{z_0, z_1, \ldots\}$ be a uniform V-pseudo-orbit. Note that $\{Int_Z(B(z, V)) \mid z \in Z\}$ is an open cover of Z. Since Z is compact, there exist $z'_1, \ldots, z'_n$ in Z such that $\mathcal{V}' = \{Int_Z(B(z'_j, V))\}_{j=1}^{n}$ is a finite subcover. Let $\mathcal{V} = \{Int_Z(B(z'_j, 3V))\}_{j=1}^{n}$. Then $\mathcal{V}$ is a finite cover of Z. Let $j \geq 0$. Since $\mathcal{V}'$ is a cover of Z, there exists $k \in \{1, \ldots, n\}$ such that $g(z_j) \in Int_Z(B(z'_k, V))$. Since $\{z_0, z_1, \ldots\}$ is a uniform V-pseudo-orbit, $\rho(g(z_j), z_{j+1}) < V$. Since $\rho(z'_k, g(z_j)) < V$ and $\rho(g(z_j), z_{j+1}) < V$, we have that $\rho(z'_k, z_{j+1}) < 2V$. Thus, $\{g(z_j), z_{j+1}\} \subset B(z'_k, 2V)$. Hence, by

Lemma 2.5, $\{g(z_j), z_{j+1}\} \subset Int_Z(B(z'_k, 3V))$. Therefore, $\{z_0, z_1, \ldots\}$ is a Hausdorff $\mathcal{V}$-pseudo-orbit. □

Theorem 8.6. *Let Z be a compact Hausdorff space, and let $g\colon Z \to Z$ be a map. Let $\mathcal{W}$ be a finite open cover of Z. If $\{z_0, z_1, \ldots\}$ is a Hausdorff $\mathcal{W}$-pseudo-orbit, then there exists $V \in \mathfrak{U}_Z$ (Remark 2.2) such that $\{z_0, z_1, \ldots\}$ is a uniform V-pseudo-orbit.*

Proof. Suppose $\{z_0, z_1, \ldots\}$ is a Hausdorff $\mathcal{W}$-pseudo-orbit. Assume that $\mathcal{W} = \{W_1, \ldots, W_n\}$, and let $V = \bigcup_{j=1}^{n} W_j \times W_j$. By Lemma 2.6, $V \in \mathfrak{U}_Z$. Let $j \geq 0$. Since $\{z_0, z_1, \ldots\}$ is a Hausdorff $\mathcal{W}$-pseudo orbit, there exists $k \in \{1, \ldots, n\}$ such that $\{g(z_j), z_{j+1}\} \subset W_k$. This implies that $\rho(g(z_j), z_{j+1}) < V$. Therefore, $\{z_0, z_1, \ldots\}$ is a uniform V-pseudo-orbit. □

Theorem 8.7. *Let Z be a compact Hausdorff space, and let $g\colon Z \to Z$ be a map. Let $U \in \mathfrak{U}_Z$ (Remark 2.2). If z uniform U-shadows the sequence $\{z_0, z_1, \ldots\}$, then there exists a finite open cover $\mathcal{U}$ of Z such that z Hausdorff $\mathcal{U}$-shadows $\{z_0, z_1, \ldots\}$.*

Proof. Suppose that z uniform U-shadows the set $\{z_0, z_1, \ldots\}$. Note that $\{Int_Z(B(z, U)) \mid z \in Z\}$ is an open cover of Z. Since Z is compact, there exist $z'_1, \ldots, z'_n$ in Z such that $\mathcal{U}' = \{Int_Z(B(z'_j, U))\}_{j=1}^{n}$ is a finite subcover. Let $\mathcal{U} = \{Int_Z(B(z'_j, 3V))\}_{j=1}^{n}$. Then $\mathcal{U}$ is a finite cover of Z. Let $j \geq 0$. Since $\mathcal{U}'$ covers Z, there exists $k \in \{1, \ldots, n\}$ such that $g^j(z) \in Int_Z(B(z'_k, U))$. Since z uniform U-shadows $\{z_0, z_1, \ldots\}$, $\rho(g^j(z), z_j) < U$. Also, since $\rho(z'_k, g^j(z)) < U$ and $\rho(g^j(z), z_j) < U$, we obtain that $\rho(z'_k, z_j) < 2U$. Thus, $\{g^j(z), z_j\} \subset B(z'_k, 2U)$. Hence, by Lemma 2.5, $\{g^j(z), z_j\} \subset Int_Z(B(z'_k, 3U))$. Therefore, z Hausdorff $\mathcal{U}$-shadows $\{z_0, z_1, \ldots\}$. □

Theorem 8.8. *Let Z be a compact Hausdorff space, and let $g\colon Z \to Z$ be a map. Let $\mathcal{U}$ be a finite open cover of Z. If z Hausdorff $\mathcal{U}$-shadows $\{z_0, z_1, \ldots\}$, then there exists $U \in \mathfrak{U}_Z$ (Remark 2.2) such that z uniform U-shadows $\{z_0, z_1, \ldots\}$.*

Proof. Suppose that z Hausdorff $\mathcal{U}$-shadows $\{z_0, z_1, \ldots\}$. Assume that $\mathcal{U} = \{U_1, \ldots, U_n\}$, and let $U = \bigcup_{j=1}^{n} U_j \times U_j$. By Lemma 2.6, $U \in \mathfrak{U}_Z$. Let $j \geq 0$. Since z Hausdorff $\mathcal{U}$-shadows $\{z_0, z_1, \ldots\}$, there exists $k \in \{1, \ldots, n\}$ such that $\{g^j(z), z_j\} \subset U_k$. Hence, $\rho(g^j(z), z_j) < U$. Therefore, z uniform U-shadows $\{z_0, z_1, \ldots\}$. □

Proof of Theorem 5.1. Assume g has uniform shadowing. Let $\mathcal{U}$ be a finite open cover of Z. Suppose $\mathcal{U} = \{U_1, \ldots, U_n\}$. Let $U = \bigcup_{j=1}^{n} U_j \times U_j$. By Lemma 2.6, we have that $U \in \mathfrak{U}_Z$ (Remark 2.2). Since g has uniform shadowing, there exists $V \in \mathfrak{U}_Z$ such that each uniform V-pseudo-orbit is uniform U-shadowed by a point of Z. Let $V' \in \mathfrak{U}_Z$ be such that $2V' \subset V$. Note that $\{Int_Z(B(z, V')) \mid z \in Z\}$ is an open cover of Z. Since Z is compact, there exist $z'_1, \ldots, z'_m$ in Z such that $\mathcal{V} = \{Int_Z(B(z'_j, V'))\}_{j=1}^{m}$ is a finite subcover. Let $\{z_0, z_1, \ldots\}$ be a Hausdorff $\mathcal{V}$-pseudo-orbit. Let $j \geq 0$. Since $\{z_0, z_1, \ldots\}$ is a Hausdorff $\mathcal{V}$-pseudo-orbit, there exists $k \in \{1, \ldots, m\}$ such that $\{g(z_j), z_{j+1}\} \subset Int_Z(B(z'_k, V'))$. This implies that $\rho(g(z_j), z'_k) < V'$ and $\rho(z'_k, z_{j+1}) < V'$. Hence, $\rho(g(z_j), z_{j+1}) < 2V'$. Since $2V' \subset V$, $\rho(g(z_j), z_{j+1}) < V$. Thus, $\{z_0, z_1, \ldots\}$ is uniform V-pseudo-orbit. Since g has uniform shadowing, there exists a point z in Z such that z uniform U-shadows $\{z_0, z_1, \ldots\}$; i.e., for every $j \geq 0$, $\rho(g^j(z), z_j) < U$. Let $j \geq 0$. Then $\rho(g^j(z), z_j) < U$. By the definition of U, there exists $l \in \{1, \ldots, n\}$ such that $(g^j(z), z_j) \in U_l \times U_l$. Thus, $\{g^j(z), z_j\} \subset U_l$. Therefore, g has Hausdorff shadowing.

Assume g has Hausdorff shadowing. Let $U \in \mathfrak{U}_Z$ and let $U' \in \mathfrak{U}_Z$ be such that $2U' \subset U$. Since Z is compact and $\{Int_Z(B(z,U')) \mid z \in Z\}$ is an open cover of Z, there exist $z'_1, \ldots, z'_n$ in Z such that $\mathcal{U} = \{Int_Z(B(z'_j,U'))\}_{j=1}^n$ is a finite subcover. Since g has Hausdorff shadowing, there exists a finite open cover $\mathcal{V}$ of Z such that each Hausdorff $\mathcal{V}$-pseudo-orbit is Hausdorff $\mathcal{U}$-shadowed by a point of Z. Since Z is compact, by Theorem 2.4, there exists $V \in \mathfrak{U}_Z$ such that $\mathfrak{C}(V)$ refines $\mathcal{V}$. Let $\{z_0, z_1, \ldots\}$ be a uniform V-pseudo-orbit. Let $j \geq 0$. Then $\rho(g(z_j), z_{j+1}) < V$. Since $\mathfrak{C}(V)$ refines $\mathcal{V}$, there exists $W_j \in \mathcal{V}$ such that $\{g(z_j), z_{j+1}\} \subset W_j$. Thus, $\{z_0, z_1, \ldots\}$ is a Hausdorff $\mathcal{V}$-pseudo-orbit. Since g has Hausdorff shadowing, there exists a point z in Z such that z Hausdorff $\mathcal{U}$-shadows $\{z_0, z_1, \ldots\}$; i.e., for every $j \geq 0$, there exists $k \in \{1, \ldots, n\}$ such that $\{g^j(z), z_j\} \subset Int_Z(B(z'_k, U'))$. Let $j \geq 0$. Then, since $\rho(g^j(z), z'_k) < U'$ and $\rho(z'_k, z_j) < U'$, we have that $\rho(g^j(z), z_j) < 2U'$. Since $2U' \subset U$, $\rho(g^j(z), z_j) < U$. Therefore, g has uniform shadowing.

For the rest of the proof, we assume Z is metric. Suppose g has shadowing. Let $\mathcal{U}$ be a finite open cover of Z and let $\varepsilon > 0$ be a Lebesgue number for $\mathcal{U}$. Since g has shadowing, there exists $\delta > 0$ such that every δ-pseudo-orbit is ε-shadowed by a point of Z. Note that $\left\{\mathcal{V}_{\frac{\delta}{2}}(z) \;\middle|\; z \in Z\right\}$ is an open cover of Z. Since Z is compact, there exist $z_1, \ldots, z_n$ in Z such that $\mathcal{V} = \left\{\mathcal{V}_{\frac{\delta}{2}}(z_k)\right\}_{k=1}^n$ is a finite subcover. Let $\{w_0, w_1, \ldots\}$ be a Hausdorff $\mathcal{V}$-pseudo-orbit. Let $j \geq 0$. Since $\{w_0, w_1, \ldots\}$ is a Hausdorff $\mathcal{V}$-pseudo-orbit, there exists $k \in \{1, \ldots, n\}$ such that $\{g(w_j), w_{j+1}\} \subset \mathcal{V}_{\frac{\delta}{2}}(z_k)$. This implies that $d(g(w_j), w_{j+1}) < \delta$. Hence, $\{w_0, w_1, \ldots\}$ is a δ-pseudo-orbit. Since g has shadowing, there exists a point w in Z such that w ε-shadows $\{w_0, w_1, \ldots\}$; i.e., for each $j \geq 0$, $d(g^j(w), w_j) < \varepsilon$. Let $j \geq 0$. Since ε is a Lebesgue number for $\mathcal{U}$ and $d(g^j(w), w_j) < \varepsilon$, there exists $U_j \in \mathcal{U}$ such that $\{g^j(w), w_j\} \subset U_j$. Therefore, g has Hausdorff shadowing.

Assume g has Hausdorff shadowing. Let $\varepsilon > 0$. Since $\left\{\mathcal{V}_{\frac{\varepsilon}{2}}(z) \;\middle|\; z \in Z\right\}$ is an open cover of Z and Z is compact, there exist $z_1, \ldots, z_n$ in Z such that $\mathcal{U} = \left\{\mathcal{V}_{\frac{\delta}{2}}(z_k)\right\}_{k=1}^n$ is a finite subcover. Since g has Hausdorff shadowing, there exists a finite open cover $\mathcal{V}$ of Z such that every Hausdorff $\mathcal{V}$-pseudo-orbit is Hausdorff $\mathcal{U}$-shadowed by point of Z. Let $\delta > 0$ be a Lebesgue number for $\mathcal{V}$. Let $\{w_0, w_1, \ldots\}$ be a δ-pseudo-orbit. Let $j \geq 0$. Since $d(g(w_j), w_{j+1}) < \delta$ and δ is a Lebesgue number for $\mathcal{V}$, there exists $V_j \in \mathcal{V}$ such that $\{g(w_j), w_{j+1}\} \subset V_j$. Thus, $\{w_0, w_1, \ldots\}$ is a Hausdorff $\mathcal{V}$-pseudo-orbit. Since g has Hausdorff shadowing, there exists a point w of Z such that w Hausdorff $\mathcal{U}$-shadows $\{w_0, w_1, \ldots\}$; i.e., for each $j \geq 0$, there exists $k_j \in \{1, \ldots, n\}$ such that $\{g^j(w), w_j\} \subset \mathcal{V}_{\frac{\varepsilon}{2}}(w_{k_j})$. Let $j \geq 0$. Since $\{g^j(w), w_j\} \subset \mathcal{V}_{\frac{\varepsilon}{2}}(w_{k_j})$, for some $k_j \in \{1, \ldots, n\}$, we have that $d(g^j(w), w_j) < \varepsilon$. Therefore, g has shadowing. □

REFERENCES

[1] E. Akin, *The General Topology of Dynamical Systems*, Graduate Studies in Mathematics (1), Amer. Math. Soc., Providence, RI, 1993.

[2] E. Akin, J. Auslander and K. Berg, When is a transitive map chaotic?, in *Convergence in Ergodic Theory and Probability*, (Columbus, OH, 1993), Ohio State Univ. Math. Res. Inst. Publ., de Gruyter, Berlin, **5** (1996), 25–40.

[3] E. Akin, J. Auslander and A. Nagar, Variations on the concept of topological transitivity, *Studia Math.*, **235** (2016), 225–249.

[4] E. Akin and J. D. Carlson, Conceptions of topological transitivity, *Topology Appl.*, **159** (2012), 2815–2830.

[5] E. Akin and J. Rautio, Chain transitive homeomorphisms on a space: All or none, *Pacific J. Math.*, **291** (2017), 1–49.

[6] N. Aoki, Topological dynamics, in *K. Morita and J. Nagata, Topics in General Topology,* North Holland, Amsterdam, New York, Oxford, Tokyo, **41** (1989), 625–740.

[7] N. Aoki and K. Hiraide, *Topological Theory of Dynamical Systems*, North Holland, Amsterdam, London, New York, Tokyo, 1994.

[8] J. Auslander, G. Greschonig and A. Nagar, Reflections on equicontinuity, *Proc. Amer. Math. Soc.*, **142** (2014), 3129–3137.

[9] M. Awartani and S. Elaydi, An extension of chaotic dynamics to general topological spaces, *Panamer. Math. J.,* **10** (2000), 61–71.

[10] J. Banks and S. Brett, A note on equivalent definitions of topological transitivity, *Discrete Contin. Dyn. Syst.*, **33** (2013), 1293–1296.

[11] J. Banks, J. Brooks, G. Cairns, G. Davis and P. Stace, On Devaney's definition of chaos, *Amer. Math. Month.*, **99** (1992), 332–334.

[12] A. D. Barwell, *ω-Limit Sets of Discrete Dynamical Systems*, Ph. D. Dissertation, The University of Birmingham, 2010.

[13] A. D. Barwell, C. Good and P. Oprocha, Shadowing and expansivity in subspaces, *Fund. Math.*, **219** (2012), 223–243.

[14] A. D. Barwell, C. Good, P. Oprocha and B. Raines, Characterizations of ω-limit sets in topologically hyperbolic systems, *Discrete Contin. Dyn. Syst.,* **33** (2013), 1819–1833.

[15] N. C. Bernardes and U. B. Darji, Graph theoretic structure of maps of the Cantor space, *Adv. Math.*, **231** (2012), 1655–1680.

[16] E. Bilokopytov and S. F. Kolyada, Transitive maps on topological spaces, *Ukrainian Math. J.*, **65** (2014), 1293–1318.

[17] W. R. Brian, Abstract omega-limit sets, preprint, Available from: `https://wrbrian.files.wordpress.com/2012/01/omegalimitsets.pdf`.

[18] T. A. Brown and W. W. Comfort, New method for expansion and contraction maps in uniform spaces, *Proc. Amer. Math. Soc.*, **11** (1960), 483–486.

[19] B. F. Bryant, On expansive homeomorphisms, *Pacific J. Math.*, **10** (1960), 1163–1167.

[20] T. Ceccherini-Silberstein and M. Coornaert, Sensitivity and Devaney's chaos in uniform spaces, *J. Dyn. Control Syst.*, **19** (2013), 349–357.

[21] M. Edelstein, On nonexpansive mappings of uniform spaces, *Indag. Math.,* **27** (1965), 47–51.

[22] R. Engelking, *General Topology*, Sigma series in pure mathematics, Vol. 6, Heldermann, Berlin, 1989.

[23] L. Fernández and C. Good, Shadowing for induced maps of hyperspaces, *Fund. Math.*, **235** (2016), 277–286.

[24] E. Glasner and B. Weiss, Sensitive dependence on initial conditions, *Nonlinearity*, **6** (1993), 1067–1075.

[25] B. M. Hood, Topological entropy and uniform spaces, *J. London Math. Soc.,* **8** (1974), 633–641.

[26] W. Huang, S. Kolyada and G. Zhang, *Analogues of Auslander-Yorke the- orems for multi-sensitivity*, Ergodic Theory Dynamical Systems, published online on 22 September 2016.

[27] X. Huang, F. Zeng and G. Zhang, Semi-openness and almost-openness of induced mappings, *Appl. Math. J. Chinese Univ. Ser. B*, **20** (2005), 21–26.

[28] W. J. Kammerer and R. M. Kasriel, On contractive mappings in uniform spaces, *Proc. Amer. Math. Soc.*, **15** (1964), 288–290.

[29] C. M. Lee, A development of contraction mapping principles on Hausdorff uniform spaces, *Trans. Amer. Math. Soc.*, **226** (1977), 147–159.

[30] E. Marczewski, Séparabilité et multiplication cartésienne des espaces topologiques, *Fund. Math.*, **34** (1947), 127–143.

[31] E. Michael, Topologies on spaces of subsets, *Trans. Amer. Math. Soc.*, **71** (1951), 152–182.

[32] C. A. Morales and V. Sirvent, Expansivity for measures on uniform spaces, *Trans. Amer. Math. Soc.,* **368** (2016), 5399–5414.

[33] W. L. Reddy, Expanding maps on compact metric spaces, *Topology Appl.*, **13** (1982), 327–334.

[34] F. Rhodes, A generalization of isometries to uniform spaces, *Proc. Cambridge Philos. Soc.*, **52** (1956), 399–405.

[35] S. Ruette, *Chaos on the Interval*, University Lecture Series, 67. American Mathematical Society, Providence, RI, 2017.

[36] S. Silverman, On maps with dense orbits and the definition of chaos, *Rocky Mountain J. Math.*, **22** (1992), 353–375.

[37] M. Vellekoop and R. Berglund, On intervals, transitivity = chaos, *Amer. Math. Monthly*, **101** (1994), 353–355.

[38] T. Wang, J. Yin and Q. Yan, Several transitive properties and Devaney's chaos, *Acta Math. Sin. (Engl. Ser.)*, **32** (2016), 373–383.

[39] Y. Wang, G. Wei, W. H. Campbell and S. Bourquin, A framework of induced hyperspace dynamical systems equipped with the hit-or-miss topology, *Chaos Solitons Fractals*, **41** (2009), 1708–1717.

[40] S. Willard, *General Topology*, Addison-Wesley, Reading, Massachusetts; London, 1970.

[41] K. Yan and F. Zend, Topological entropy, pseudo-orbits and uniform spaces, *Topology Appl.*, **210** (2016), 168–182.

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