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On Intervals, Transitivity = Chaos.

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In an earlier article in the American Mathematical Monthly a redundancy was found in the definition of chaos by Devaney [1]:

Let V be a set. A continuous map $f\colon V \to V$ is said to be chaotic on V if

(1) f is *topologically transitive*: for any pair of open non-empty sets $U, W \subset V$ there exists a $k > 0$ such that $f^k(U) \cap W \neq \varnothing$.
(2) the *periodic points* of f are *dense* in V.
(3) f has *sensitive dependence on initial conditions*: there exists a $\delta > 0$ such that, for any $x \in V$ and any neighbourhood N of x, there exists a $y \in N$ and an $n \geq 0$ such that $|f^n(x) - f^n(y)| > \delta$.

In [2], Banks et al. prove that (1) and (2) imply (3) in any metric space V, and in [3] Assaf IV and Gadbois show that for *general* maps this is the only redundancy: (1) and (3) do not imply (2), and (2) and (3) do not imply (1). But if we restrict our attention to maps on an interval a stronger result can be obtained:

Proposition. *Let I be a, not necessarily finite, interval and $f\colon I \to I$ a continuous and topologically transitive map. Then* (1) *the periodic points of f are dense in I and* (2) *f has sensitive dependence on initial conditions.*

The first result (1) can be found in [4] (Chapter IV.5, Lemma 41) but the proof uses a lot of other highly non-trivial results. Since Devaney's text is being used by so many students, we think that it is interesting to give a very short, intuitive proof of this proposition.

We will need the following lemma, which can be found in [4] (Chapter IV.1, Corollary 10) in a more general form:

Lemma. *Suppose that I is a, not necessarily finite, interval and $f\colon I \to I$ is a continuous map. If $J \subset I$ is an interval which contains no periodic points of f and z, $f^m(z)$ and $f^n(z) \in J$ with $0 < m < n$, then either $z < f^m(z) < f^n(z)$ or $z > f^m(z) > f^n(z)$.*

Proof of the lemma: Suppose we can find such a $z \in J$ with $z < f^m(z)$ and $f^m(z) > f^n(z)$. Define the function $g(x) = f^m(x)$. Then we know that $z < g(z)$ and this implies $z < g(z) < g^{k+1}(z)$ for all natural numbers $k \geq 1$ by induction. Because, if $g^{k+1}(z) < g(z)$ for a certain k then the function $g^k(x) - x$ has a positive value in z and a negative value in $g(z)$ and this would mean, by the Intermediate Value Theorem, that there exists a point $c \in]z, g(z)[\subset J$ with $g^k(c) - c = 0$, giving a km-periodic point of f in J. Thus $z < g^k(z)$ for all positive k so in particular for $k = n - m > 0$, giving $z < f^{(n-m)m}(z)$. Since we assumed that $f^{n-m}(f^m(z)) < f^m(z)$ we could prove analogously, taking $g = f^{n-m}$ that $f^{(n-m)m}(f^m(z)) < f^m(z)$. But then we have that the function $f^{(n-m)m}(x) - x$ has a positive value in z and a negative value in $f^m(z)$, giving an $(n - m)m$-periodic point in J and thus a contradiction. The other case can be proven analogously. □

Proof of the Proposition: Suppose that f is continuous and topologically transitive. Because of the result in **[2]** we only need to prove that the periodic points are dense in I. Suppose that this is not the case, then there exists an interval $J \subset I$ containing no periodic points. Take an $x \in J$ which is not an endpoint of J, an open neighbourhood $N \subsetneq J$ of x and an open interval $E \subset J \setminus N$. Since f is topologically transitive on I there exists a natural number $m > 0$ with $f^m(N) \cap E \neq \emptyset$ and thus a $y \in J$ with $f^m(y) \in E \subset J$. Since J contains no periodic points we know that $y \neq f^m(y)$ and since f is continuous this implies that we can find a neighbourhood U of y with $f^m(U) \cap U = \emptyset$. Since U is an open set we can use the topological transitivity again and find an $n > m$ and a $z \in U$ with $f^n(z) \in U$. But then we have $0 < m < n$ and $z, f^n(z) \in U$ while $f^m(z) \notin U$ and this violates our earlier lemma. □

We know now that for maps *on an interval* the only condition that has to be checked for Devaney's definition of chaos is the first one, topological transitivity. Note that the proof cannot be generalized for higher dimensions or the unit circle S^1 because our lemma uses the ordering on $\mathbb{R}$ in an essential way.

For completeness we note that there are no other trivialities in Devaney's definition when restricted to intervals:

A continuous function on an interval whose periodic points are dense doesn't need to have sensitive dependence on initial conditions.

The identity function on any interval trivially proves this.

A continuous function on an interval which has sensitive dependence on initial conditions and whose periodic points are dense does not have to be transitive.

Define on $I = \mathbb{R}_+$ the function

$$f(x) = \begin{cases} 3x & 0 \le x < \frac{1}{3} \\ -3x + 2 & \frac{1}{3} \le x < \frac{2}{3} \\ 3x - 2 & \frac{2}{3} \le x < 1 \\ f(x-1) + 1 & x \ge 1 \end{cases}$$

It is sensitive on initial conditions since $|df/dx(x)| = 3$ for all points on I, so every neighbourhood around a point will expand under iteration. It is easy to establish that f^n has $3^n - 2$ fixed points between any two integer values with distances between these points smaller than $(\frac{1}{3})^{n-1}$, so the periodic points are dense. But since $f([0,1]) = [0,1]$ the function is not topologically transitive. When one restricts this function to the interval $I = [0, 2]$ one sees that it is a counterexample for finite I as well.

A continuous function on an interval which has sensitive dependence on initial conditions doesn't need to have periodic points which are dense.

As counterexample take the interval $I = [0, \frac{3}{4}]$ and the function

$$f(x) = \begin{cases} \frac{3}{2}x & 0 \le x < \frac{1}{2} \\ \frac{3}{2}(1-x) & \frac{1}{2} \le x \le \frac{3}{4} \end{cases}$$

Sensitive dependence is clear again since the function is expanding, but there can be no periodic points in $]0, 3/8[$ since it is easy to establish that any trajectory with initial value in this subinterval, will not return there. For a counterexample in the infinite case, take $I = \mathbb{R}_+$ and $f(x) = 2x$.

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Proof of a Mixed Arithmetic-Mean, Geometric-Mean Inequality

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The following conjecture was made by F. Holland in [**3**].

Conjecture. *Let $x_1, x_2, \ldots, x_n$ be positive real numbers. The arithmetic mean of the numbers*

$$x_1, \sqrt{x_1 x_2}, \sqrt[3]{x_1 x_2 x_3}, \ldots, \sqrt[n]{x_1 x_2 \cdots x_n}$$

does not exceed the geometric mean of the numbers

$$x_1, \frac{x_1 + x_2}{2}, \frac{x_1 + x_2 + x_3}{3}, \ldots, \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

There is equality if and only if $x_1 = x_2 = \cdots = x_n$.

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