Structure of Inverse Limit Spaces of Tent Maps with Nonrecurrent Critical Points

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Abstract

In this paper we examine the structure of composants of inverse limit spaces generated by tent maps with a nonrecurrent critical point. We identify important structures and substructures of certain composants, and we prove the surprising result that, assuming the critical point is nonrecurrent, there are only finitely many "types" of structures in these composants. This is an important first step towards classifying this family of inverse limit spaces which would in turn lead us closer to a proof of the Ingram Conjecture.

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1 Introduction

The Ingram Conjecture states that if T_s and T_t are two different tent maps, then $\varprojlim\{[0,1],T_s\}$ is not homeomorphic to $\varprojlim\{[0,1],T_t\}$. There are many papers written on this topic, and, perhaps not surprisingly, the focus is usually on the structure of the postcritical orbit. If the critical point is periodic then the Ingram Conjecture is true, [K1], [K2], [S1] (for a particularly reader-friendly version of this proof see [Bl-J-K-Ke]) and recently the second author proved the Ingram Conjecture in the case that the critical point is pre-periodic, [S3] (see also [Bru] for a result in this direction). Thus in order to continue towards a proof of the Ingram Conjecture we must focus on the case when the critical point has an infinite (non-periodic and non-pre-periodic) orbit.

A natural subdivision of the family of tent maps with an infinite post-critical orbit is into the collection of tent maps with a recurrent critical point and the collection of tent maps without a recurrent critical point. The focus of this paper is the case that the critical point is nonrecurrent, but we should say something about the recurrent case. It is well-known that the set of parameters, $t \in [\sqrt{2}, 2]$, that correspond to tent maps T_t with a recurrent critical point is a set of full Lebesgue measure. Many of these tent maps generate horrendously complicated continua as inverse limit spaces. They frequently display the property of being *locally universal* in the sense that every open set contains a homeomorphic copy of every other tent map inverse limit, [Ba-B-D].

In this paper, though, we focus on the case that the critical point is nonrecurrent. We adopt the viewpoint of the second author in her recent work on the preperiodic case and consider the inverse limit as a quotient space of a certain set of bi-infinite sequences of 0's and 1's. We hope to make this viewpoint well-known and well-understood by demonstrating its utility towards proving the Ingram Conjecture.

Perhaps the most striking difference between the case we are considering and the previously solved cases (finite critical orbit) is that in the previous cases the inverse limits had only finitely many *inhomogeneities*, i.e. neighborhoods at which the continuum is not homeomorphic to the product of a Cantor set and an open arc. In the case under consideration there are always infinitely many such inhomogeneities (see [R] and [G-Kn-R] for a detailed discussion of these inhomogeneities), but we show in this paper that the amount of variation in the composant structure is still finite. In a forthcoming paper we will use the fact that there are only finitely many structures in a given composant to prove the Ingram Conjecture in the case that the critical point orbit is dense in a countable set.

It is natural in thinking about Ingram's Conjecture to attempt to describe the structure of the composants of these inverse limit spaces. All of these spaces are indecomposable metric continua, and as such they have uncountably many composants. Every homeomorphism will preserve the composants, and it will send a composant containing an inhomogeneity to another composant containing an inhomogeneity. In this paper we describe many properties of the composants of $\varprojlim\{[0,1], T_s\}$ with an aim towards using these properties in a proof of the Ingram Conjecture.

We begin the paper with a detailed description of the symbolic representation of the inverse limit space. We state some of the properties of that description paying particular attention to composant properties. We then list many of the lemmas and theorems from [S2] which, even though they were originally proved in the case of a finite critical orbit, are true in this more general setting. We occasionally give some indication as to how the proofs would need to be altered to fit this case. We end the paper with our main theorem which states that even though the critical point has an infinite orbit, if the critical point is nonrecurrent then there are only finitely many different "types" of structures in the composants of the inverse limit space.

2 Preliminaries

Let $s \in (\sqrt{2}, 2]$ be such that the tent map $T_s : [0,1] \to [0,1]$ has critical point which is not recurrent. Let $f_s : [0,1] \to [0,1]$ be the rescaled core of the tent map T_s, i.e.

$$f_s(\xi) = \begin{cases} s\xi + 2 - s, & \text{if } 0 \le \xi \le c_s\,, \\ s(1-\xi), & \text{if } c_s \le \xi \le 1\,, \end{cases}$$

where $c_s = \frac{s-1}{s}$ is the critical point. Let C_s denote the limit of the inverse sequence consisting of copies of $[0,1]$, where the bonding map is the rescaled core f_s,

$$C_s = \varprojlim\{[0,1], f_s\} = \{(\ldots, \xi_{-3}, \xi_{-2}, \xi_{-1}) \in [0,1]^{\mathbb{N}} : \xi_{-i} = f_s(\xi_{-i-1}),\, i \in \mathbb{N}\}.$$

C_s is a continuum (compact connected metric space). It is well known that to describe the structure of continua $\varprojlim\{[0,1], T_s\}$, $s \in (1,2]$, it is sufficient to describe the structure of continua C_s, $s \in (\sqrt{2}, 2]$.

Now we recall a symbolic representation of the inverse limit spaces C_s provided by Brucks and Diamond in [B-D].

For every point $\xi \in [0,1]$ an *itinerary of* ξ *under the map* f_s is a right-infinite sequences of zeros and ones $\overrightarrow{x}(\xi) = (x_i)_{i\in\mathbb{Z}_+} = x_0x_1x_2\cdots \in \{0,1\}^{\mathbb{Z}_+}$, where

$$x_i = \begin{cases} 0\,, & f_s^i(\xi) \le c_s\,, \\ 1\,, & f_s^i(\xi) \ge c_s\,. \end{cases}$$

Note that every point $\xi \in [0,1]$ has at most two itineraries and the points which have two itineraries are the preimages of the critical point. The *kneading sequence* of the map f_s, denoted by $\overrightarrow{c_1} = (c_i)_{i\in\mathbb{N}}$, is the itinerary of $f_s(c_s) = 1$. Note that $\overrightarrow{c_2} = c_2c_3c_4\ldots$ is the itinerary of $f_s^2(c_s) = 0$. A sequence $\overrightarrow{x} \in \{0,1\}^{\mathbb{Z}_+}$ is called *allowed* (with respect to f_s) if there is $\xi \in [0,1]$ such that $\overrightarrow{x}$ is the itinerary of ξ under the map f_s. By Theorem II.3.8 in [C-E], $\overrightarrow{x}$ is allowed if and only if $\overrightarrow{c_2} \preceq \overrightarrow{x}$ and $\sigma^k\overrightarrow{x} \preceq \overrightarrow{c_1}$, for every $k \in \mathbb{Z}_+$. Let us denote by X_s^+ the set of all allowed sequences $\overrightarrow{x} \in \{0,1\}^{\mathbb{Z}_+}$. The metric d on the space X_s^+ is given as follows: For two sequences $\overrightarrow{x} = (x_i)_{i\in\mathbb{Z}_+}$ and

$\overrightarrow{y} = (y_i)_{i\in\mathbb{Z}_+}$, let $d(\overrightarrow{x}, \overrightarrow{y}) = 0$ if $\overrightarrow{x} = \overrightarrow{y}$, and let $d(\overrightarrow{x}, \overrightarrow{y}) = 2^{-k}$ if $\overrightarrow{x} \neq \overrightarrow{y}$, where $k = \min\{j \in \mathbb{Z}_+ : x_j \neq y_j\}$. The one-sided shift $\sigma : X_s^+ \to X_s^+$, given by $\sigma((x_i)_{i\in\mathbb{Z}_+}) = (x_{i+1})_{i\in\mathbb{Z}_+}$, is continuous. Let us define an equivalence relation $\sim$ on X_s^+ as follows: $\overrightarrow{x} \sim \overrightarrow{y}$ if either $\overrightarrow{x} = \overrightarrow{y}$, or there exists $m \in \mathbb{Z}_+$, such that $x_0x_1\dots x_{m-1} = y_0y_1\dots y_{m-1}$, $x_m \neq y_m$ and $\overrightarrow{x}_{m+1} = \overrightarrow{y}_{m+1} = \overrightarrow{c_1}$. If $[\overrightarrow{x}] \in X_s^+/_\sim$ and there exists $\overrightarrow{y} \in [\overrightarrow{x}]$ with $\overrightarrow{y} \neq \overrightarrow{x}$, we will write, for simplicity, $[\overrightarrow{x}] = x_0x_1\dots x_{m-1}{}^0_1\overrightarrow{c_1}$. The mapping $\pi : X_s^+/_\sim \to [0,1]$, given by $\pi[\overrightarrow{x}] = \xi$ if $\overrightarrow{x}$ is an itinerary of the point ξ, is a homeomorphism, and $\pi(\tilde{\sigma}([\overrightarrow{x}])) = f_s(\pi([\overrightarrow{x}]))$, for every $[\overrightarrow{x}] \in X_s^+/_\sim$, where $\tilde{\sigma} : X_s^+/_\sim \to X_s^+/_\sim$ is given by $\tilde{\sigma}([\overrightarrow{x}]) = [\sigma\overrightarrow{x}]$. For this reason, we will often identify $[0,1]$ and $X_s^+/_\sim$.

For a bi-infinite sequence $\bar{x} = (x_i)_{i\in\mathbb{Z}}$, we denote the right-infinite sequence $x_jx_{j+1}x_{j+2}\dots$, also called a *right tail* of $\bar{x}$, by $\overrightarrow{x_j} = x_jx_{j+1}x_{j+2}\dots$. A bi-infinite sequence $\bar{x} \in \{0,1\}^{\mathbb{Z}}$ is called *allowed* (with respect to f_s), if all of its right tails $\overrightarrow{x_j}$ are itineraries (with respect to f_s), i.e., if for every right tail $\overrightarrow{x_j}$, $j \in \mathbb{Z}$, one has $\overrightarrow{c_2} \preceq \overrightarrow{x_j}$ and $\sigma^k\overrightarrow{x_j} \preceq \overrightarrow{c_1}$, for every $k \in \mathbb{Z}_+$. Let $X_s = \{\bar{x} \in \{0,1\}^{\mathbb{Z}} : \bar{x}$ is allowed with respect to $f_s\}$ denote the space of all bi-infinite allowed sequences with respect to f_s. The metric d on the space X_s is given as follows: For two sequences $\bar{x}, \bar{y} \in X_s$, $\bar{x} = (x_i)_{i\in\mathbb{Z}}$, $\bar{y} = (y_i)_{i\in\mathbb{Z}}$, if $\bar{x} \neq \bar{y}$, let $k = \min\{|j| : j \in \mathbb{Z},\ x_j \neq y_j\}$. Then $d(\bar{x}, \bar{y}) = 2^{-k}$ if $\bar{x} \neq \bar{y}$, and $d(\bar{x}, \bar{y}) = 0$ if $\bar{x} = \bar{y}$. The shift map $\sigma : X_s \to X_s$ given by $(\sigma\bar{x})_i = x_{i+1}$, for every $i \in \mathbb{Z}$, is a homeomorphism. Let us define an equivalence relation $\approx$ on the space X_s as follows: Two sequences $\bar{x}, \bar{y} \in X_s$, $\bar{x} = (x_i)_{i\in\mathbb{Z}}$, $\bar{y} = (y_i)_{i\in\mathbb{Z}}$, are equivalent, $\bar{x} \approx \bar{y}$, if either $\bar{x} = \bar{y}$, or if there is $k \in \mathbb{Z}$ with $x_i = y_i$, for $i < k$, $x_k \neq y_k$ and $\overrightarrow{x_{k+1}} = \overrightarrow{y_{k+1}} = \overrightarrow{c_1}$. By Theorem 2.5 in [B-D] there is a homeomorphism $h : X_s/_\approx \to C_s$ such that $h(\tilde{\sigma}([\bar{x}])) = \hat{f}_s(h([\bar{x}]))$, for every $[\bar{x}] \in X_s/_\approx$, where $\tilde{\sigma} : X_s/_\approx \to X_s/_\approx$ is given by $\tilde{\sigma}([\bar{x}]) = [\sigma\bar{x}]$, and $\hat{f}_s : C_s \to C_s$ is given by $\hat{f}_s(\dots, \xi_{-3}, \xi_{-2}, \xi_{-1}) = (\dots, \xi_{-2}, \xi_{-1}, f_s(\xi_{-1}))$, i.e., the maps $\tilde{\sigma}$ and $\hat{f}_s$ are conjugate. Note that the maps $\tilde{\sigma}$ and $\hat{f}_s$ are homeomorphisms. We will often identify C_s and $X_s/_\approx$. If there is a sequence $\bar{y} \in [\bar{x}]$

with $\bar{y} \neq \bar{x}$, it is unique, and we denote it by $\bar{x}^* = (x_i^*)_{i\in\mathbb{Z}}$. If there is no such $\bar{y} \in [\bar{x}]$ with $\bar{y} \neq \bar{x}$, we put $\bar{x}^* = \bar{x}$. Let $\pi_j : X_s/_\approx \to [0,1]$, $j \in \mathbb{Z}_+$, be the projection on the j-th coordinate, i.e., $\pi_j[\bar{x}] = \pi(\overrightarrow{x}_{-j})$, where $\pi(\overrightarrow{x}_{-j}) = \xi$ if $\overrightarrow{x}_{-j}$ is an itinerary of the point ξ.

For a bi-infinite sequence $\bar{x} = (x_i)_{i\in\mathbb{Z}}$, we denote the left-infinite sequence $\dots x_{j-2}x_{j-1}x_j$, also called *left tail* of $\bar{x}$, by $\overleftarrow{x_j} = \dots x_{j-2}x_{j-1}x_j$. A left-infinite sequence $\overleftarrow{x} = (x_{-i})_{i\in\mathbb{N}}$ is allowed if for every $k \in \mathbb{N}$, there exists an itinerary, such that its initial part of length k is the finite sequence $x_{-k}\dots x_{-1}$. Note that if $\bar{x}$ is allowed, then all of its left tails $\overleftarrow{x_j}$ are allowed. Each left-infinite sequence $\overleftarrow{x} = \dots x_{-3}x_{-2}x_{-1}$ describes one composant in C_s which is the set of bi-infinite sequences having a left tail common to $\overleftarrow{x}$. Two sequences $\overleftarrow{x}$ and $\overleftarrow{y}$ describe the same composant if and only if they have a common left tail (Corollary 2.10 in [B-D]).

Next we introduce some of our own definitions and results which appeared in [S2] for inverse limit spaces of tent maps that have a preperiodic critical point. These are also valid in this more general setting of inverse limit spaces of tent maps with nonrecurrent critical points.

Every composant of C_s is arcwise connected. Let $\overleftarrow{a} = \dots a_{-3}a_{-2}a_{-1}$ and let $n \in \mathbb{Z}_+$. The set $A^n_{\overleftarrow{a}} = \{[\bar{x}] \in C_s : \exists \bar{x} \in [\bar{x}],\ \overleftarrow{x}_{-n} = \overleftarrow{a}\}$ is an arc and we call it a *basic arc*. For a fixed left-infinite sequence $\overleftarrow{y} = \dots y_{-3}y_{-2}y_{-1}$, let C be the corresponding composant of C_s. If $A^n_{\overleftarrow{v}}$ is a basic arc contained in the composant C, then either $\overleftarrow{v}_{-1} = \overleftarrow{y}_{-n}$, or there is $k \in \mathbb{N}$ with $v_{-k} \neq y_{-n-k+1}$ and $\overleftarrow{v}_{-k-1} = \overleftarrow{y}_{-n-k}$. In the first case we put $k = 0$. When $k = 0$, and whenever it is clear which sequence $\overleftarrow{y}$ represents the composant containing the basic arc $A^n_{\overleftarrow{v}}$, we write only A^n instead of $A^n_{\overleftarrow{y_{-n}}}$. When $k > 0$, we write only A^n_v instead of $A^n_{\overleftarrow{v}}$, where $v = v_{-k}\dots v_{-1}$, and we understand that $\overleftarrow{v}_{-k-1} = \overleftarrow{y}_{-n-k}$.

For $n \in \mathbb{N}$, let $P(n) = \mathrm{card}\,\{\, i :\ y_{-i} = 1, 1 \leq i \leq n\}$. If $n = 0$, let $P(0) = 0$. An arc A^n is called *even* (respectively *odd*), if $P(n)$ is even (respectively odd). An arc A^n_v, $v = v_{-k}\dots v_{-1}$, $v_{-k} \neq y_{-n-k}$, is called *even* (respectively *odd*) if $(-1)^{P(n+k)} = \prod_{i=1}^{k}(-1)^{v_{-i}}$ (respectively $(-1)^{P(n+k)} \neq \prod_{i=1}^{k}(-1)^{v_{-i}}$).

We introduce an ordering on the composant C denoted by $\preceq$ and called *generalized parity-lexicographical ordering*, as follows: For $[\bar{x}], [\bar{z}] \in C$, let $k = k([\bar{x}], [\bar{z}]) = \max\{i \in \mathbb{N} : x_{-i} \neq y_{-i}$ or $z_{-i} \neq y_{-i}$, $\bar{x} = (x_i)_{i\in\mathbb{Z}} \in [\bar{x}]$, $\bar{z} = (z_i)_{i\in\mathbb{Z}} \in [\bar{z}]\}$. If $x_{-i} = y_{-i}$ and $z_{-i} = y_{-i}$, for all $i \in \mathbb{N}$, $\bar{x} \in [\bar{x}]$, $\bar{z} \in [\bar{z}]$, let $k = 0$. We say that $\bar{x} \prec \bar{z}$ if either $(-1)^{P(k)}x_{-k} < (-1)^{P(k)}z_{-k}$, or there exists $l \in \mathbb{Z}$, $l > -k$, such that $x_i = z_i$, for $-k \leq i < l$, and $(-1)^{P(k)}\varepsilon x_l < (-1)^{P(k)}\varepsilon z_l$, where $\varepsilon = \prod_{i=-k}^{l-1}(-1)^{x_i} = \prod_{i=-k}^{l-1}(-1)^{z_i} \in \{-1, 1\}$. We say that $[\bar{x}] \preceq [\bar{z}]$ if $\bar{x} \prec \bar{z}$ or $\bar{x} = \bar{z}$.

Note that the ordering depends on the chosen left-infinite sequence $\overleftarrow{y}$. The choice of another representative of this particular composant would lead either to the same, or to the opposite ordering. There exists an order-preserving bijection ϕ between the real line, endowed with its natural order, and C, endowed with the ordering $\preceq$. Therefore, the ordering $\preceq$ on the composant C is natural. Note that ϕ is continuous, but its inverse is not.

We define some special points as follows: A point $[\bar{x}] \in C_s$ is called an *identification point* or shorter an *i-point* if there is $m \in \mathbb{Z}_+$ with $\overrightarrow{x}_{-m+1} = \overrightarrow{c}_1$.

Let $[\bar{x}] \in C_s$ be an i-point with $\bar{x} \neq \bar{x}^*$. The *level* of $[\bar{x}]$ is defined by $L[\bar{x}] = m$ if $|x_{-m} - x^*_{-m}| = 1$. If $\bar{x} = \bar{x}^*$, let $L[\bar{x}] = \infty$.

The importance of the i-points and their levels is visible from the following: Let $\overleftarrow{a} = (a_{-i})_{i\in\mathbb{N}}$ and $\overleftarrow{b} = (b_{-i})_{i\in\mathbb{N}}$, $\overleftarrow{a} \neq \overleftarrow{b}$, be allowed sequences. For $n \in \mathbb{N}$, let $A^n_{\overleftarrow{a}}$ and $A^n_{\overleftarrow{b}}$ be the basic arcs. If there is $[\bar{x}] \in A^n_{\overleftarrow{a}} \cap A^n_{\overleftarrow{b}}$, then $\overleftarrow{x}_{-n} = \overleftarrow{a}$ and $\overleftarrow{x}^*_{-n} = \overleftarrow{b}$. Hence, $[\bar{x}]$ is an i-point, and there is $m \geq n$ with $x_{-i} = x^*_{-i} = a_{n-i-1}$, for $i > m$, $|x_{-m} - x^*_{-m}| = 1$ and $\overrightarrow{x}_{-m+1} = \overrightarrow{x}^*_{-m+1} = \overrightarrow{c}_1$, implying that $L[\bar{x}] = m$. Also, if $[\bar{y}] \in A^n_{\overleftarrow{a}}$ is an i-point with $L[\bar{y}] > n$, then $[\bar{y}] \in \partial A^n_{\overleftarrow{a}}$.

Note that $\pi_{n-1}|_{A^n_{\overleftarrow{a}}}$ is an injection and if $A^n_{\overleftarrow{a}}$ has boundary points $[\bar{x}]$ and $[\bar{y}]$ with $L[\bar{x}] = l$ and $L[\bar{y}] = k$, then $\pi_{n-1}(A^n_{\overleftarrow{a}}) = \{\pi_{n-1}[\bar{x}] : [\bar{x}] \in A^n_{\overleftarrow{a}}\}$ is a closed interval with boundary points $f_s^{l-n+1}(c_s)$ and $f_s^{k-n+1}(c_s)$. Let $A^n_{\overleftarrow{b}}$ be another basic arc. Let $\{[\bar{x}^0] \prec \cdots \prec [\bar{x}^i]\}$ be the ordered set of all i-points of $A^n_{\overleftarrow{a}}$, and $\{[\bar{u}^0] \prec \cdots \prec [\bar{u}^j]\}$ be the ordered set of all i-points of $A^n_{\overleftarrow{b}}$. If $\pi_{n-1}(\partial A^n_{\overleftarrow{a}}) = \pi_{n-1}(\partial A^n_{\overleftarrow{b}})$, then $i = j$ and either $L[\bar{x}^m] = L[\bar{u}^m]$, for every $m \in \{1, \dots, j-1\}$, if $A^n_{\overleftarrow{a}}$ and $A^n_{\overleftarrow{b}}$ have the same parity, or $L[\bar{x}^m] = L[\bar{u}^{j-m}]$, for every $m \in \{1, \dots, j-1\}$, if they have different parity. For every $k \in \{0, \dots, n-1\}$, the arc $A^n_{\overleftarrow{a}}$ is a union of arcs A^k_w, i.e., $A^n_{\overleftarrow{a}} = \cup_w A^k_w$, where the union is computed over all finite sequences w of length $n-k$ such that $\overleftarrow{a}\, w$ is allowed. Since f_s is l.e.o. and $\pi \circ \sigma = f_s \circ \pi$, for every arc A, there is $m \in \mathbb{Z}_+$ such that $\tilde{\sigma}^m(A) = \{\tilde{\sigma}^m[\bar{x}] : [\bar{x}] \in A\}$ contains at least one i-point.

In [S2] (Proposition 2.10) we proved the following properties for basic arcs:

Proposition 2.1 *Let $\overleftarrow{a} = (a_{-i})_{i \in \mathbb{N}}$ be an allowed sequence, $n \in \mathbb{N}$, and let $A^n_{\overleftarrow{a}}$ be the associated basic arc. Then, for every i-point $[\bar{y}] \in int A^n_{\overleftarrow{a}}$, there are points $[\bar{x}], [\bar{z}] \in A^n_{\overleftarrow{a}}$, $[\bar{x}] \prec [\bar{y}] \prec [\bar{z}]$, such that, for every point $[\bar{u}] \in A^n_{\overleftarrow{a}}$, $[\bar{x}] \preceq [\bar{u}] \prec [\bar{y}]$, there is a point $[\bar{v}] \in A^n_{\overleftarrow{a}}$, $[\bar{y}] \prec [\bar{v}] \preceq [\bar{z}]$, such that $[\overrightarrow{u}_{-l+1}] = [\overrightarrow{v}_{-l+1}]$, where $l = L[\bar{y}]$.*

The proof for nonrecurrent case is the same as the proof for preperiodic case.

We say that the arc $A^n_{\overleftarrow{a}}$ is $[\bar{y}]$*-symmetric between* $[\bar{x}]$ *and* $[\bar{z}]$. If either $[\bar{x}] \in \partial A^n_{\overleftarrow{a}}$, or $[\bar{z}] \in \partial A^n_{\overleftarrow{a}}$, we say that the arc $A^n_{\overleftarrow{a}}$ is $[\bar{y}]$*-symmetric.*

Note that, if the basic arc $A^n_{\overleftarrow{a}}$ contains an i-point $[\bar{y}]$ such that $L[\bar{y}] = n-1$, then $A^n_{\overleftarrow{a}}$ is $[\bar{y}]$-symmetric. If $A^n_{\overleftarrow{a}}$ is $[\bar{y}]$-symmetric and $[\bar{x}] \in \partial A^n_{\overleftarrow{a}}$ then, in the nonrecurrent case (as in the strictly preperiodic case) the corresponding point $[\bar{z}]$ is not an i-point.

Since every basic arc contains finitely many i-points, the following corollary (Corollary 2.12 in [S2]) is a direct consequence of the previous proposition:

Corollary 2.2 *Let $A^n_{\overleftarrow{a}}$ and $A^n_{\overleftarrow{b}}$ be two neighboring arcs, let $\{[\bar{x}^0] \prec [\bar{x}^1] \prec \cdots \prec [\bar{x}^m]\}$ be their i-points and let $k \in \{1, \dots, m-1\}$ be such that $[\bar{x}^k] = A^n_{\overleftarrow{a}} \cap A^n_{\overleftarrow{b}}$. Let $j = \min\{k, m-k\}$. Then for every $[\bar{u}]$, $[\bar{x}^{k-j+1}] \preceq [\bar{u}] \prec [\bar{x}^k]$, there is $[\bar{v}]$, $[\bar{x}^k] \prec [\bar{v}] \preceq [\bar{x}^{k+j-1}]$, such that $[\overrightarrow{u}_{-n+1}] = [\overrightarrow{v}_{-n+1}]$. In particular, $L[\bar{x}^{k-i}] = L[\bar{x}^{k+i}]$, for every $i \in \mathbb{N}$, $i \leq j-1$.*

3 The Structure of a Composant Containing a Periodic Point

Let $[\bar{a}]$ be any periodic point of continuum C_s. Let the period of $[\bar{a}]$ be $N \in \mathbb{N}$. Then $\bar{a} = \bar{a}^* = \overline{w^\infty . w^\infty}$ for some finite word w of the length $|w| = N$. Let us denote by C the composant represented by the sequence $\overleftarrow{w^\infty}$. Let $K = 2kN$, for some $k \in \mathbb{N}$. For every $i \in \mathbb{Z}$, the mapping $\tilde{\sigma}^{iK}$ is an order-preserving homeomorphism on C having $[\bar{a}]$ as a fixed point. We will study the structure of composant C. In what follows we 'adjust' most of our definitions, lemmas, and theorems to the $\tilde{\sigma}^{pK}$th image of the space. Since $\tilde{\sigma}$ is a homeomorphism this is obviously the same space, but the indices will have changed. The reason for doing this is two-fold: (1) Since K is an even multiple of the period of $[\bar{a}]$, $\tilde{\sigma}^{pK}$ will send the composant we are studying back to itself in an order-preserving way and (2) in a forthcoming paper we will prove that a homeomorphism of these spaces must 'almost commute' with some power of the shift, so in that context we will need to be working in this adjusted space.

We sort the i-points of C in the following way: For every $p \in \mathbb{Z}_+$ a point $[\bar{x}] \in C$ is called *p-point*, if either there is $m \in \mathbb{Z}_+$ with $[\overrightarrow{x}_{-pK-m+1}] = [\overrightarrow{c_1}]$, or if $[\bar{x}] = [\bar{a}]$. A p-point $[\bar{x}]$ has *p-level* $L_p[\bar{x}] = m$ if $|x_{-pK-m} - x^*_{-pK-m}| = 1$. Let us define $L_p[\bar{a}] = \infty$, for every $p \in \mathbb{Z}_+$. For every $p, m \in \mathbb{Z}_+$, the set $E_{p,m} = \{[\bar{x}] \in C^i : \exists \bar{x} \in [\bar{x}], \overrightarrow{x}_{-pK-m+1} = \overrightarrow{c_1}\}$ is the set of all p-points of level m and $E_p = \cup_{m=0}^{\infty} E_{p,m} \cup \{[\bar{a}]\}$ is the set of all p-points of the composant C. Note that $E_{p+1} \subset E_p$, for every $p \in \mathbb{Z}_+$. Since there is an order-preserving bijection from $(\mathbb{Z}, \leq)$ to $(E_p, \preceq)$, such that $0 \in \mathbb{Z}$ is mapped to $[\bar{a}] \in E_p$, from now on, the points of E_p will be indexed by $\mathbb{Z}$. So, $E_p = \{\dots, [\bar{x}^{-1}], [\bar{x}^0], [\bar{x}^1], \dots\}$ and $[\bar{x}^0] = [\bar{a}]$.

The sequence $L_p[\bar{x}^0], L_p[\bar{x}^1], L_p[\bar{x}^2], \dots$ is called the *folding pattern* of the composant C. Note that $[\bar{x}^0] \prec [\bar{x}^1] \prec [\bar{x}^2] \prec \dots$. Let $q \in \mathbb{Z}_+$, $q > p$, and $E_q = \{\dots, [\bar{y}^{-1}], [\bar{y}^0], [\bar{y}^1], \dots\}$. Since $\tilde{\sigma}^{(q-p)K}$ is an order-preserving homeomorphism on C, it is easy to see that, for every $i \in \mathbb{Z}_+$, one has $\tilde{\sigma}^{(q-p)K}([\bar{x}^i]) = [\bar{y}^i]$ and $L_p[\bar{x}^i] = L_q[\bar{y}^i]$. Therefore, the folding pattern of the composant C does not depend on p.

Now, we give some basic properties of the folding pattern of the composant C. These properties were first proved in [S2] for the preperiodic case. In that paper they were stated and proved for a composant containing a point in the inverse limit generated by the particular periodic orbit to which the critical point was mapped. However, the results are true in a more general setting. As long as the critical point is nonrecurrent and the composant contains a point periodic under the shift-map we have the following results. The proofs require no change and so we simply list the relevant results here:

1. Let $p \in \mathbb{Z}_+$. Let $[\bar{x}^n] \in E_p$ and $L_p[\bar{x}^n] = iK + k$, for some $i \in \mathbb{N}$ and $k \in \mathbb{Z}_+$, $k < K$. Then, for every $j \in \mathbb{Z}_+$, $j < i$, there is $[\bar{x}^m] \in E_p$, $[\bar{x}^m] \prec [\bar{x}^n]$, such that $L_p[\bar{x}^m] = jK + k$ (The paragraph preceding Lemma 3.2).

2. Let $p, q, k \in \mathbb{Z}_+$ and let arcs $A, B \subset C$ be such that there are no i-points $[\bar{x}] \in \operatorname{int} A$ and $[\bar{y}] \in \operatorname{int} B$ with $L_p[\bar{x}] > k$ and $L_q[\bar{y}] > k$. If $\pi_{pK+k}(A) = \pi_{qK+k}(B)$, then for $E_p \cap \operatorname{int} A = \{[\bar{x}^0] \prec \cdots \prec [\bar{x}^n]\}$ and for $E_q \cap \operatorname{int} B = \{[\bar{y}^0] \prec \cdots \prec [\bar{y}^m]\}$, one has $m = n$ and either $L_p[\bar{x}^i] = L_q[\bar{y}^i]$, for every $0 \leq i \leq n$, or $L_p[\bar{x}^i] = L_q[\bar{y}^{n-i}]$, for every $0 \leq i \leq n$ (The paragraph preceding Lemma 3.2).

3. Let $p \in \mathbb{Z}_+$. Let $[\bar{x}^n] \in E_p$ be such that $[\bar{x}^n] \neq [\bar{a}]$ and $L_p[\bar{x}^n] \neq 0$. Let $i, j \in \mathbb{N}$ be the smallest numbers with $L_p[\bar{x}^{n+i}] > L_p[\bar{x}^n]$ and $L_p[\bar{x}^{n-j}] > L_p[\bar{x}^n]$. Then the arc between the points $[\bar{x}^{n-j}]$ and $[\bar{x}^{n+i}]$ is $[\bar{x}^n]$-symmetric and $L_p[\bar{x}^{n-k}] = L_p[\bar{x}^{n+k}]$, for every k, $0 < k < \min\{i, j\}$ (Lemma 3.2).

4. Let $p \in \mathbb{Z}_+$ and $[\bar{x}], [\bar{y}] \in E_p$, $[\bar{x}] \neq [\bar{y}]$. If $L_p[\bar{x}] = L_p[\bar{y}]$, then there is $[\bar{z}] \in E_p$ between $[\bar{x}]$ and $[\bar{y}]$ such that $L_p[\bar{z}] > L_p[\bar{x}]$ (Lemma 3.4 and Remark 3.5).

5. Let $p \in \mathbb{Z}_+$ and $[\bar{x}^n], [\bar{x}^m] \in E_p$, $|m-n| \geq 2$. If there is $k \in \mathbb{Z}_+$ such that $L_p[\bar{x}^m] = L_p[\bar{x}^n]$, and $L_p[\bar{x}^j] \neq L_p[\bar{x}^n]$, for every $n < j < m$, then $n+m$ is even and, for $l = \max\{L_p[\bar{x}^j] : n < j < m\}$, one has $L_p[\bar{x}^n] < l = L_p[\bar{x}^{\frac{n+m}{2}}]$ (Corollary 3.7 and Remark 3.8).

6. Let $p \in \mathbb{Z}_+$. Let $[\bar{x}], [\bar{y}] \in E_p$ be such that $L_p[\bar{x}] = k$, $L_p[\bar{y}] = k+1$, $k \in \mathbb{Z}_+$, and there is no $[\bar{w}] \in E_p$ between $[\bar{x}]$ and $[\bar{y}]$, satisfying $L_p[\bar{w}] \geq k$. Then, for every $n < k$, there is $[\bar{z}] \in E_p$ between $[\bar{x}]$ and $[\bar{y}]$, such that $L_p[\bar{z}] = n$ (Lemma 3.9 and Remark 3.10).

An arc A of the composant C such that $\partial A = \{[\bar{u}], [\bar{v}]\}$ and $A \cap E_p = \{[\bar{y}^0], \ldots, [\bar{y}^n]\}$ is called *p-symmetric* if $[\overrightarrow{u}_{-pK}] = [\overrightarrow{v}_{-pK}]$ and $L_p[\bar{y}^i] = L_p[\bar{y}^{n-i}]$, for every $0 \leq i \leq n$. By statement 2. above, every p-symmetric arc is also q-symmetric, for every $0 \leq q \leq p$. Note that if A is a p-symmetric arc of the composant C and $A \cap E_p = \{[\bar{x}^0], \ldots, [\bar{x}^n]\}$, then by statement 5. above, n is even. The p-point $[\bar{x}^{\frac{n}{2}}]$ is called the *center* of A, it is denoted by $[\bar{\chi}^A]$, and also by statement 5. above, $L_p[\bar{\chi}^A] = \max\{L_p[\bar{x}] : [\bar{x}] \in E_p \cap \mathrm{int} A\}$. Therefore, the centers of the p-symmetric arcs of the composant C are the "turning points" of the composant C.

In order to describe the folding pattern of the composant C, we study some special arcs. For $p \in \mathbb{Z}_+$ an arc B of the composant C is called a *p-bridge* if $\partial B \subset E_p$, $L_p[\bar{x}] = 0$, for every $[\bar{x}] \in \partial B$, and $L_p[\bar{x}] \neq 0$, for every $[\bar{x}] \in \mathrm{int}\, B \cap E_p$. Note that for every $[\bar{x}] \in \mathrm{int}\, B$ one has either $x_{-pK} = 0$, or $x_{-pK} = 1$. If for every $[\bar{x}] \in \mathrm{int}\, B$ one has $x_{-pK} = 0$ (respectively $x_{-pK} = 1$), we will say that B is a p-bridge of *sign* 0 (respectively of *sign* 1).

For $q \leq p$, let $B \cap E_q = \{[\bar{z}^0], \ldots, [\bar{z}^m]\}$. We will call the finite sequence $FP_q(B) = L_q[\bar{z}^0], \ldots, L_q[\bar{z}^m]$ the *q-folding pattern* of the p-bridge B. If $q = p$, we will write, for simplicity, $FP(B)$ instead of $FP_p(B)$. It is easy to see that p-bridges are p-symmetric, and that $L_p[\bar{\chi}^B]$ determines the q-folding pattern of the p-bridge B, for all $q \leq p$. Therefore, it is natural to ask which kind of p-bridges with respect to the p-levels of their centers exist? The answer is the same as in the preperiodic case:

Lemma 3.1 *Let $p \in \mathbb{Z}_+$ and $n \in \mathbb{N}$. There is a p-bridge $B \subset C$ such that $L_p[\bar{\chi}^B] = n$ if and only if $c_s \in f_s^n([0, c_s])$.*

The proof is the same as the proof of Lemma 3.12 in [S2]. The following corollary is a direct consequence of the previous lemma:

Corollary 3.2 *Let $p \in \mathbb{Z}_+$. If $c_3 = 0$, then for every $n \in \mathbb{N}$, there is a p-bridge $B \subset C$ such that $L_p[\bar{\chi}^B] = n$.*

Lemma 3.3 *Let $p \in \mathbb{Z}_+$. For every $n \in \mathbb{N}$, there is a p-bridge $B \subset C$ such that $L_p[\bar{\chi}^B] = 2n$.*

The proof is the same as the proof of Lemma 3.14 in [S2].

Lemma 3.4 *Let $p \in \mathbb{Z}_+$ and $m = \min\{i \in \mathbb{N} : c_{2i+1} = 0\}$. There is a p-bridge $B \subset C$ such that $L_p[\bar{\chi}^B] = 2n - 1$ if and only if $n \geq m$.*

The proof is the same as the proof of Lemma 3.15 in [S2].

Corollary 3.5 *Let $p \in \mathbb{Z}_+$ and let $[\bar{x}], [\bar{y}] \in E_p$ be such that $L_p[\bar{x}] = k$, $L_p[\bar{y}] = k + 1$, $k \in \mathbb{Z}_+$. Then for every $n < k$, there is $[\bar{z}] \in E_p$ between $[\bar{x}]$ and $[\bar{y}]$, such that $L_p[\bar{z}] = n$. Furthermore, either there is a p-bridge B between $[\bar{x}]$ and $[\bar{y}]$ such that $L_p[\bar{\chi}^B] = n$, or there is no p-bridge whose p-level of the center equals n.*

The proof is similar to the proof of Corollary 3.17 in [S2] for the case $i = j = 0$ (see Remark 3.18 in [S2]).

Let B be a p-bridge with $B \cap E_p = \{[\bar{x}^0], \ldots, [\bar{x}^n]\}$. Let

$$T(B) = \min\{L_p[\bar{\chi}^A] : A \text{ is a } p\text{-bridge of the same sign as } B \text{ such that for } A \cap E_p = \{[\bar{u}^0], \ldots, [\bar{u}^n]\} \text{ one has } L_p[\bar{u}^i] = L_p[\bar{x}^i],\ 0 \leq i < n/2\}.$$

For $q \in \mathbb{Z}_+$, let $D \subset C$ be a q-bridge and $D \cap E_q = \{[\bar{y}^0], \ldots, [\bar{y}^m]\}$. If $T(B) = T(D)$, than there are a p-bridge $B_1 \subset C$ and a q-bridge $D_1 \subset C$ with $L_p[\bar{\chi}^{B_1}] = L_q[\bar{\chi}^{D_1}]$. Hence, $m = n$ and $L_p[\bar{x}^i] = L_q[\bar{y}^i]$, for every $0 \leq i \leq n$, $i \neq n/2$. Therefore, we will call the number $T(B)$ the *type* of the p-bridge B.

Theorem 3.6 *There are finitely many bridge types.*

Proof: Let $p \in \mathbb{Z}_+$. It is sufficient to prove that there exists $M \in \mathbb{N}$ such that for every p-bridge $B \subset C$ and for every $[\bar{x}] \in B \cap E_p$, $[\bar{x}] \neq [\bar{\chi}^B]$, one has $L_p[\bar{x}] < M$.
The kneading sequence $\overrightarrow{c_1}$ satisfies the following properties:

(a) There is the smallest $k_1 \in \mathbb{N}$ such that for every $i \in \mathbb{N}$ one has $c_{i+1} \dots c_{i+k_1} \neq c_1 \dots c_{k_1}$ (otherwise $\overrightarrow{c_1}$ is recurrent),

(b) There is the smallest odd k_2 such that $c_{k_2} = 0$ (if $c_{2i-1} = 1$ for every $i \in \mathbb{N}$, then $\overrightarrow{c_1} = 10111c_6 1 c_8 1 \dots = 1 * 10(1-c_6)(1-c_8)\dots$ which contradicts the assumption that $s > \sqrt{2}$).

Let $A \subset C$ be an arc and let $A \cap E_p = \{[\bar{x}^0], \dots, [\bar{x}^m]\}$. Let us denote by $FP(A)$ the folding pattern of the arc A, i.e. $FP(A) = (L_p[\bar{x}^i])_{i=0}^m = L_p[\bar{x}^0], \dots, L_p[\bar{x}^m]$.
Let us suppose, on the contrary, that for every $i \in \mathbb{N}$, there exists a p-bridge B^i which contains a p-point $[\bar{x}^{m_i}]$, $[\bar{x}^{m_i}] \neq [\bar{\chi}^{B^i}]$ such that $L_p[\bar{x}^{m_i}] = n_i = \max\{L_p[\bar{x}] : [\bar{x}] \in B^i \cap E_p, [\bar{x}] \neq [\bar{\chi}^{B^i}]\}$ and the sequence $(n_i)_{i \in \mathbb{N}}$ is strictly increasing. Then for every p-bridge $B^l \in (B^i)_{i \in \mathbb{N}}$ there exists a p-bridge A with $FP(A) = 0\ n\ 0$, for some $n \in \mathbb{N}$, with the following properties:

(i) $\tilde{\sigma}^{n_l}(A) = A^{n_l} \subset B^l$,

(ii) $[\bar{x}^{m_l}] \in \partial A^{n_l}$.

Note that $L_p[\bar{x}^{m_l}] = n_l$ and $n = L_p[\bar{\chi}^{B^l}] - n_l$. Now we fix some p-bridge $B^l \in (B^i)_{i \in \mathbb{N}}$ and the corresponding p-bridge A. We will study arcs $A^i = \tilde{\sigma}^i(A)$, $i \leq n_l$, and their folding patterns $FP(A^i)$.

(1) Since $FP(A) = 0\ n\ 0$, one has $c_{n+1} = c_1$ and $FP(A^1) = 1\ n+1\ 1$ (if $c_{n+1} \neq c_1$ then $FP(A^1) = 1\ 0\ n+1\ 0\ 1$ and $A^{n_l} \not\subseteq B^l$).

(2) Suppose that $c_{n+2} \neq c_2$. Then $FP(A^2) = 2\ 0\ n+2\ 0\ 2$. Therefore, $c_{n+3} = c_3 = c_1$ and $FP(A^3) = 3\ 1\ n+3\ 1\ 3$. Since $c_3 = c_1$ implies $c_4 = c_1$, one has that $FP(A^4)$ contains the pattern $2\ 0\ 4$. Thus, $c_5 = c_1$ and $FP(A^5)$ contains the pattern $3\ 1\ 5$. Continuing we get that $FP(A^{2i})$ contains 0 for every $2i \leq n_l$, and $FP(A^{2i+1})$ contains $1, 3, \ldots, 2i+1$ and does not contain 0 for every $2i+1 \leq n_l$. Since the sequence $(n_i)_{i\in\mathbb{N}}$ is strictly increasing, one has $c_1 = c_{2i+1}$ for every $i \in \mathbb{N}$ which contradicts (b). Therefore, $c_{n+2} = c_2$ and $FP(A^2) = 2\ n+2\ 2$.

(3) Suppose that $c_{n+3} \neq c_3$. Then $FP(A^3) = 3\ 0\ n+3\ 0\ 3$ and hence $c_{n+4} = c_4 = c_1$ and $FP(A^4) = 4\ 1\ n+4\ 1\ 4$. If $c_5 \neq c_2$, then $FP(A^5)$ contains the pattern $2\ 0\ 5$. This implies that $c_1 = c_3 = c_6$ and that $FP(A^6)$ contains the pattern $3\ 1\ 6$. Going on we get that $FP(A^{2i+1})$ contains 0 for every $2i+1 \leq n_l$, and $FP(A^{2i+2})$ contains $1, 3, \ldots, 2i+1$ and does not contain 0 for every $2i+2 \leq n_l$. But $c_1 = c_{2i+1}$ for every $i \in \mathbb{N}$ contradicts (b). Therefore, $c_5 = c_2$.

Suppose that $c_i = c_{i+3}$ for every $i < j$ and $c_j \neq c_{j+3}$. Then $FP(A^{j+3})$ contains the pattern $j+3\ 0\ j$.

If $j = 3i+1$ for some i, then

$$\begin{array}{ccccccccc} c_1 & = & c_4 & = & \cdots & = & c_j & \neq & c_{j+3} \\ c_2 & = & c_5 & = & \cdots & = & c_{j+1} & & \\ c_3 & = & c_6 & = & \cdots & = & c_{j+2}. & & \end{array}$$

Since $c_2 = c_{j+1}$, then $FP(A^{j+4})$ contains the pattern $1\ 0\ j+1$ which contradicts the assumption that $A^{n_l} \subset B^l$.

If $j = 3i+2$ for some i, then

$$\begin{array}{ccccccccc} c_1 & = & c_4 & = & \cdots & = & c_{j+2} & & \\ c_2 & = & c_5 & = & \cdots & = & c_j & \neq & c_{j+3} \\ c_3 & = & c_6 & = & \cdots & = & c_{j+1}. & & \end{array}$$

Since $FP(A^{j+3})$ contains the pattern $j+3\ 0\ j$, then $c_{j+1} = c_1$ and $c_3 = c_1$. Since $c_{j+2} = c_1$, then $FP(A^{j+5})$ contains the pattern $2\ 0\ j+2$

and $FP(A^{j+6})$ contains the pattern $3\ 1\ j+3$. Since $c_4 = c_1 \neq c_2$, then $FP(A^{j+7})$ contains the pattern $4\ 0\ 2$. Since $c_5 = c_2 \neq c_1$, then $FP(A^{j+8})$ contains the pattern $5\ 0\ 1$ which contradicts the assumption that $A^{n_l} \subset B^l$.

If $j = 3i$ for some i, then

$$\begin{array}{ccccccccc} c_1 & = & c_4 & = & \cdots & = & c_{j+1} & & \\ c_2 & = & c_5 & = & \cdots & = & c_{j+2} & & \\ c_3 & = & c_6 & = & \cdots & = & c_j & \neq & c_{j+3}. \end{array}$$

Therefore, $c_3 = 0$ (otherwise $\overrightarrow{c_1}$ is not σ-maximal). Since $FP(A^{j+3})$ contains the pattern $3\ 0\ j+3$, then $c_{j+4} = c_1$. If $c_{j+5} \neq c_2$, then $FP(A^{j+5})$ contains the pattern $2\ 0\ j+5$ and $FP(A^{j+6})$ contains the pattern $3\ 0\ 1$ which contradicts the assumption that $A^{n_l} \subset B^l$. Therefore, $c_{j+5} = c_2$. If $c_{j+7} \neq c_4$, then $FP(A^{j+7})$ contains the pattern $4\ 0\ j+7$ and $FP(A^{j+8})$ contains the pattern $5\ 0\ 1$ which contradicts the assumption that $A^{n_l} \subset B^l$. If $c_{j+8} \neq c_5$, then $FP(A^{j+8})$ contains the pattern $5\ 0\ j+8$ and $FP(A^{j+9})$ contains the pattern $6\ 0\ 1$ which again contradicts the assumption that $A^{n_l} \subset B^l$. Continuing we get that $c_1 = c_{j+1} = c_{j+4} = \cdots = c_{3i+1}$ and $c_2 = c_{j+2} = c_{j+5} = \cdots = c_{3i+2}$ for every i with $3i+2 \leq n_l$. Since the sequence $(n_i)_{i\in\mathbb{N}}$ is strictly increasing, this implies that $\overrightarrow{c_1} = 10 * (1-c_3)(1-c_6)\dots(1-c_{3i})$ which contradicts the assumption that $s > \sqrt{2}$. Therefore, $c_{n+3} = c_3$ and $FP(A^3) = 3\ n+3\ 3$.

Note that the only assumptions on n are the following:

(0) $FP(A) = 0\ n\ 0$, i.e. n is the p-level of p-point whose both neighboring p-points have p-levels 0,

and the assumptions (i) and (ii). In (1), (2) and (3) we have proved that for every n which satisfies (0), (i) and (ii), one has $c_i = c_{n+i}$, for every $i < 4$, i.e. the proofs in (1), (2) and (3) do not depend on n.

(4) Suppose that we have proved that there exists j such that for every n which satisfies (0), (i) and (ii), one has $c_i = c_{n+i}$, for every $i < j$, and suppose that $c_j \neq c_{n+j}$. Then $FP(A^j) = j\ 0\ n+j\ 0\ j$. Therefore, similarly to (1), one has $c_{n+j+1} = c_{j+1} = c_1$ and $FP(A^{j+j}) = j+1\ 1\ n+j+1\ 1\ j+1$. Similarly to (2), one has $c_{j+2} = c_2$, and so on.

Since j satisfies (0), (i) and (ii), one has $c_{j+i} = c_i$ for every $i < j$.

$$
\begin{array}{ccccccccc}
0 & & & & n & & & & 0 \\
1 & & & & n+1 & & & & 1 \\
2 & & & & n+2 & & & & 2 \\
\vdots & & & & \vdots & & & & \vdots \\
j-1 & & & & n+j-1 & & & & j-1 \\
j & & 0 & & n+j & & 0 & & j \\
j+1 & & 1 & & n+j+1 & & 1 & & j+1 \\
j+2 & & 2 & & n+j+2 & & 2 & & j+2 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
2j-1 & & j-1 & & n+2j-1 & & j-1 & & 2j-1 \\
2j & (0) & j & (0) & n+2j & (0) & j & (0) & 2j \\
2j+1 & & j+1=1 & & n+2j+1 & & j+1=1 & & 2j+1 \\
2j+2 & & j+2=2 & & n+2j+2 & & j+2=2 & & 2j+2 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
3j-1 & & 2j-1=j-1 & & n+3j-1 & & 2j-1=j-1 & & 3j-1 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}
$$

Therefore, $c_{dj+i} = c_i$ for every $i < j$ and for every d such that $(d+1)j < n_l$. Since the sequence $(n_i)_{i \in \mathbb{N}}$ is strictly increasing, this implies that $\overrightarrow{c_1} = 10c_3 \ldots c_{j-1} * \overrightarrow{y}$ for some $\overrightarrow{y}$, which contradicts the assumption that $s > \sqrt{2}$. Therefore, $c_i = c_{n+i}$ for every i, which contradicts (a).

Therefore, there exists $M \in \mathbb{N}$ such that for every p-bridge $B \subset C$ and for every $[\bar{x}] \in B \cap E_p$, $[\bar{x}] \neq [\bar{\chi}^B]$, one has $L_p[\bar{x}] < M$. ■

We should point out that we do not know which bridge types are allowed for a given tent map with a nonrecurrent critical point, only that there are finitely many types.

Next, we consider relations between different bridges of the composant C. For two p-bridges $B^1, B^2 \subset C$, we say that $B^1 \prec B^2$, if for every $[\bar{x}] \in B^1$, and for every $[\bar{y}] \in B^2$, one has $[\bar{x}] \preceq [\bar{y}]$. Let $B \subset C$ be a p-bridge and let $B \cap E_{p-1} = \{[\bar{x}^0], \ldots, [\bar{x}^n]\}$. The arc between the points $[\bar{x}^0]$ and $[\bar{\chi}^B]$ we will denote by A^2, and the arc between the point $[\bar{\chi}^B]$ and $[\bar{x}^n]$ we will denote by A^1. The arcs A^1 and A^2 we will call the $(p-1)$-*semibridges*. Note that $L_{p-1}[\bar{x}^i] = L_{p-1}[\bar{x}^{n-i}]$, for every $i \in \{0, \ldots, n/2\}$. We say that the $(p-1)$-semibridges A^1 and A^2 have the *semitype* $sT(A^1) = sT(A^2) = T(B)$. Let A be an arc such that $\partial A \subset E_{p-1}$ and let $A \cap E_{p-1} = \{[\bar{y}^0], \ldots, [\bar{y}^m]\}$. If $m = n/2$ and either $L_{p-1}[\bar{y}^i] = L_{p-1}[\bar{x}^i]$ for every $i \in \{0, \ldots, n/2-1\}$, and $[\overrightarrow{y}^{\,m}_{-(p-1)K}] = [\overrightarrow{\chi}^{\,B}_{-(p-1)K}]$, or $L_{p-1}[\bar{y}^i] = L_{p-1}[\bar{x}^{n/2+i}]$, for every $i \in \{1, \ldots, n/2\}$, and $[\overrightarrow{y}^{\,0}_{-(p-1)K}] = [\overrightarrow{\chi}^{\,B}_{-(p-1)K}]$, then the arc A is a $(p-1)$-semibridge with the semitype $sT(A) = T(B)$.

If $K > M$, then the only p-bridges which contain p-points of p-level K are p-bridges whose centers are p-points of p-level K. Therefore, from now on we assume that $K > M$.

Let D be a p-bridge and $D \cap E_{p-1} = \{[\bar{x}^0], \ldots, [\bar{x}^n]\}$. Then $L_p[\bar{x}^0] = 0$, $L_{p-1}[\bar{x}^0] = K$, and $[\bar{x}^0]$ is the center of $(p-1)$-bridge of type K. Let $i \in \mathbb{N}$ be the smallest number with $L_{p-1}[\bar{x}^i] = 0$, and let $j < n$ be the largest number with $L_{p-1}[\bar{x}^j] = 0$. Let A^1_D be the arc between the points $[\bar{x}^0]$ and $[\bar{x}^i]$, and let A^2_D be the arc between the points $[\bar{x}^j]$ and $[\bar{x}^n]$. Then $sT(A^1_D) = sT(A^2_D) = K$. The arc A^1_D we will call the *first* $(p-1)$-semibridge of the p-bridge D, and the arc A^2_D we will call the *last* $(p-1)$-semibridge of the p-bridge D. Between the points $[\bar{x}^i]$ and $[\bar{x}^j]$ there is one or more $(p-1)$-bridges. The ordered set of the first and the last $(p-1)$-semibridges and all $(p-1)$-bridges contained in the p-bridge B is called the *structure* of the p-bridge B, and it is denoted by $S(B)$.

Lemma 3.7 *Let $p \in \mathbb{Z}_+$. Let $B \subset C$ be a p-bridge, $B \cap E_p = \{[\bar{x}^0], \ldots, [\bar{x}^n]\}$ and $S(B) = (A^1_B, B_1, \ldots, B_m, A^2_B)$. Let A be the arc between the points $[\bar{x}^0]$*

and $[\bar{x}^1]$. *Then* $\{[\overrightarrow{x}_{-pK}] : [\bar{x}] \in A\} = \{[\overrightarrow{x}_{-pK}] : [\bar{x}] \in B\}$ *and* $A^1_B \subset A$.

The proof is the same as the proof of Lemma 3.21 of [S2].

References

[Ba-B-D] M. Barge, K. Brucks, B. Diamond, Self-similarity in inverse limit spaces of tent family, Proc. Amer. Math. Soc. 124 (1996) 3563–3570.

[Bl-J-K-Ke] L. Block, S. Jakimovik, L. Kailhofer, J. Keesling, On the classification of inverse limits of tent maps, Fund. Math. **187** (2005), 171–192.

[B-D] Brucks, K. M., Diamond, B., *A symbolic representation of inverse limit space for a class of unimodal maps*, Continua with the Houston problem book, Eds.: Cook, H. et al., Lecture notes in pure and appl. math. **170**, Dekker (1995), 207–226.

[Bru] Bruin, H., *Inverse limit spaces of post-critically finite tent maps*, Fund. Math. **165** (2000), 125–138.

[C-E] Collet, P., Eckmann, J.-P., *Iterated maps on the interval as dynamical systems*, Progress in Physics, Birkhäuser, Basel (1980).

[G-Kn-R] Good, C., Knight, R., Raines, B. E., *Non-hyperbolic one-dimensional invariant sets with a countably infinite collection of inhomogeneities*, Preprint (2006)

[K1] Kailhofer, L., *A partial classification of inverse limit spaces of tent maps with periodic critical points*, Topology and its Applications **123** (2002), 235–265.

[K2] Kailhofer, L., *A classification of inverse limit spaces of tent maps with periodic critical points*, Fund. Math. **177** (2003), 95–120.

[R] Raines, B. E., *Inhomogeneities in non-hyperbolic one-dimensional invariant sets*, Fund. Math. **182** (2004), 241–268.

[S1] S. Štimac, Topological classification of Knaster continua with finitely many endpoints, Ph.D. thesis (in Croatian), University of Zagreb, Zagreb, 2002.

[S2] S. Štimac, Structure of inverse limit spaces of tent maps with finite critical orbit, Fund. Math. **191** (2006), 125–150.

[S3] Štimac, S., *A Classification of inverse limit spaces of tent maps with finite critical orbit*, Preprint (2005).