ON ISOTOPY AND UNIMODAL INVERSE LIMIT SPACES

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ABSTRACT. We prove that every self-homeomorphism $h : K_s \to K_s$ on the inverse limit space K_s of tent map T_s with slope $s \in (\sqrt{2}, 2]$ is isotopic to a power of the shift-homeomorphism $\sigma^R : K_s \to K_s$.

1. Introduction

The solution of Ingram's Conjecture constitutes a major advancement in the classification of unimodal inverse limit spaces and the group of self-homeomorphisms on them. This conjecture was posed by Tom Ingram in 1992 for tent maps $T_s : [0,1] \to [0,1]$ with slope $\pm s$, $s \in [1,2]$, defined as $T_s(x) = \min\{sx, s(1-x)\}$. The turning point is $c = \frac{1}{2}$ and we denote its iterates by $c_n = T_s^n(c)$. The inverse limit space $K_s = \varprojlim([0, s/2], T_s)$ consists of the *core* $\varprojlim([c_2, c_1], T_s)$ and the 0-composant $\mathfrak{C}_0$, *i.e.,* the composant of the point $\bar{0} := (\dots, 0, 0, 0)$, which compactifies on the core of the inverse limit space. Ingram's Conjecture reads:

> If $1 \le s < s' \le 2$, then the corresponding inverse limit spaces $\varprojlim([0, s/2], T_s)$ and $\varprojlim([0, s'/2], T_{s'})$ are non-homeomorphic.

The first results towards solving this conjecture were obtained for tent maps with a finite critical orbit [9, 12, 3]. Raines and Štimac [11] extended these results to tent maps with a possibly infinite, but non-recurrent critical orbit. Recently Ingram's Conjecture was solved completely (in the affirmative) in [2], but we still know very little of the structure of inverse limit spaces (and their subcontinua) for the case that orb(c) is infinite and recurrent, see [1, 5, 8].

Given a continuum K and $x \in K$, the *composant A of x* is the union of the proper subcontinua of K containing x. For slopes $s \in (\sqrt{2}, 2]$, the core is indecomposable (*i.e.,* it cannot be written as the union of two proper subcontinua), and in this case we also proved [2] that any self-homeomorphism $h : K_s \to K_s$ is pseudo-isotopic to a power σ^R of the shift-homeomorphism

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σ on the core. This means that h permutes the composants of the core of K_s in the same way as σ^R does, and it is a priori a weaker property than isotopy. This is for instance illustrated by the $\sin \frac{1}{x}$-continuum, defined as the graph $\{(x, \sin \frac{1}{x}) : x \in (0,1]\}$ compactified with a *bar* $\{0\} \times [-1,1]$. There are homeomorphisms that reverse the orientation of the bar, and these are always pseudo-isotopic, but never isotopic, to the identity. Since such $\sin \frac{1}{x}$-continua are precisely the non-trivial subcontinua of Fibonacci-like inverse limit spaces [8], this example is very relevant to our paper.

In this paper we make the step from pseudo-isotopy to isotopy. To this end, we exploit so-called *folding points*, *i.e.,* points in the core of K_s where the local structure of the core of K_s is not that of a Cantor set cross an arc. In the next section we prove the following results:

Theorem 1.1. If $s \in (\sqrt{2}, 2]$, and $h : K_s \to K_s$ is a homeomorphism, then there is $R \in \mathbb{Z}$ such that $h(x) = \sigma^R(x)$ for every folding point x in K_s.

Folding points $x = (\dots, x_{-2}, x_{-1}, x_0)$ are characterized by the fact that each entry x_{-k} belongs to the omega-limit set $\omega(c)$ of the turning point $c = \frac{1}{2}$, see [10]. This gives the immediate corollary for those slopes such that the critical orbit $\mathrm{orb}(c)$ is dense in $[c_2, c_1]$, which according to [7] holds for Lebesgue a.e. $s \in [\sqrt{2}, 2]$.

Corollary 1.2. If $\mathrm{orb}(c)$ is dense in $[c_2, c_1]$, then for every homeomorphism $h : K_s \to K_s$ there is $R \in \mathbb{Z}$ such that $h = \sigma^R$ on the core of K_s.

The more difficult case, however, is when $\mathrm{orb}(c)$ is not dense in $[c_2, c_1]$. In this case, h can be at best isotopic to a power of the shift, because at non-folding points, where the core of K_s is a Cantor set cross an arc, h can easily act as a local translation. It is shown in [4] that for tent maps with non-recurrent critical point (or in fact, more generally long-branched tent maps), every homeomorphism $h : K_s \to K_s$ is indeed isotopic to a power of the shift. The proof exploits the fact that in this case, so-called p-points (indicating folds in the arc-components of K_s) are separated from each other, at least in arc-length semi-metric. Here we prove the general result.

Theorem 1.3. If $s \in (\sqrt{2}, 2]$, and $h : K_s \to K_s$ is a homeomorphism, then there exists $R \in \mathbb{Z}$ such that h is isotopic to σ^R.

The paper is organized as follows. In Section 2 we give basic definitions and prove results on how homeomorphisms act on folding points, *i.e.,* Theorem 1.1 and Corollary 1.2. These proofs depend largely on the results obtained in [2]. In Section 3 we present the additional arguments needed for the isotopy result and finally prove Theorem 1.3.

2. Inverse limit spaces of tent maps and folding points

Let $\mathbb{N} = \{1, 2, 3, \dots\}$ be the set of natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The tent map $T_s : [0,1] \to [0,1]$ with slope $\pm s$ is defined as $T_s(x) = \min\{sx, s(1-x)\}$. The critical or turning point is $c = 1/2$ and we write $c_k = T_s^k(c)$, so in particular $c_1 = s/2$ and $c_2 = s(1-s/2)$. Also let $\operatorname{orb}(c)$ and $\omega(c)$ be the orbit and the omega-limit set of c. We will restrict T_s to the interval $I = [0, s/2]$; this is larger than the *core* $[c_2, c_1] = [s - s^2/2, s/2]$, but it contains the fixed point 0 on which the 0-composant $\mathfrak{C}_0$ is based.

The inverse limit space $K_s = \varprojlim([0, s/2], T_s)$ is

$$\{x = (\dots, x_{-2}, x_{-1}, x_0) : T_s(x_{i-1}) = x_i \in [0, s/2] \text{ for all } i \leq 0\},$$

equipped with metric $d(x,y) = \sum_{n\leqslant 0} 2^n |x_n - y_n|$ and *induced* (or *shift*) *homeomorphism*

$$\sigma(\dots, x_{-2}, x_{-1}, x_0) = (\dots, x_{-2}, x_{-1}, x_0, T_s(x_0)).$$

Let $\pi_k : \varprojlim([0, s/2], T_s) \to I$, $\pi_k(x) = x_{-k}$ be the k-th projection map. Since $0 \in I$, the endpoint $\bar{0} := (\dots, 0, 0, 0)$ is contained in $\varprojlim([0, s/2], T_s)$. The composant of $\varprojlim([0, s/2], T_s)$ of $\bar{0}$ will be denoted as $\mathfrak{C}_0$; it is a ray converging to, but disjoint from the core $\varprojlim([c_2, c_1], T_s)$ of the inverse limit space. We fix $s \in (\sqrt{2}, 2]$; for these parameters T_s is not renormalizable and $\varprojlim([c_2, c_1], T_s)$ is indecomposable. Moreover, the arc-component of $\bar{0}$ coincides with the composant of $\bar{0}$, but for points in the core of K_s, we have to make the distinction between arc-component and composant more carefully.

A point $x = (\dots, x_{-2}, x_{-1}, x_0) \in K_s$ is called a *p-point* if $x_{-p-l} = c$ for some $l \in \mathbb{N}_0$. The number $L_p(x) := l$ is the *p-level* of x. In particular, $x_0 = T_s^{p+l}(c)$. By convention, the endpoint $\bar{0}$ of $\mathfrak{C}_0$ is also a p-point and $L_p(\bar{0}) := \infty$, for every p. The ordered set of all p-points of the composant $\mathfrak{C}_0$ is denoted by E_p, and the ordered set of all p-points of p-level l by $E_{p,l}$. Given an arc $A \subset K_s$ with successive p-points $x^0, \dots, x^n$, the sequence of their p-level is denoted as

$$FP_p(A) := L_p(x^0), \dots, L_p(x^n),$$

where FP stands for folding pattern. Note that every arc of $\mathfrak{C}_0$ has only finitely many p-points, but an arc A of the core of K_s can have infinitely many p-points. In this case, if $(u^i)_{i\in\mathcal{I}}$ is the set of p-points of A, then $FP_p(A) = (L_p(u^i))_{i\in\mathcal{I}}$, for some countable index set $\mathcal{I}$ (not necessarily of the same ordinal type as $\mathbb{N}$ or $\mathbb{Z}$). The *folding pattern of the composant* $\mathfrak{C}_0$, denoted by $FP(\mathfrak{C}_0)$, is the sequence $L_p(z^1), L_p(z^2), \dots,\ L_p(z^n), \dots$, where $E_p = \{z^1, z^2, \dots, z^n, \dots\}$ and p is any nonnegative integer. Let $q \in \mathbb{N}$, $q > p$, and $E_q = \{y^0, y^1, y^2, \dots\}$. Since σ^{q-p} is an order-preserving homeomorphism of $\mathfrak{C}_0$, it is easy to see that $\sigma^{q-p}(z^i) = y^i$ for every $i \in \mathbb{N}$, and $L_p(z^i) = L_q(y^i)$. Therefore, the folding pattern of $\mathfrak{C}_0$ does not depend on p.

Definition 2.1. We call a p-point $s \in \mathfrak{C}_0$ *salient* if $0 \le L_p(x) < L_p(s)$ for every p-point $x \in (\bar{0}, s)$. Let $(s_i)_{i \in \mathbb{N}}$ be the sequence of all salient p-points of $\mathfrak{C}_0$ ordered such that $s_i \in (\bar{0}, s_{i+1})$ for all $i \ge 1$.

Since for every slope $s > 1$ and $p \in \mathbb{N}_0$, the folding pattern of the 0-composant $\mathfrak{C}_0$ starts as ∞ 0 1 0 2 0 1 ..., and since by definition $L_p(s_i) > 0$, for all $i \ge 1$, we have $L_p(s_1) = 1$. Also, since $s_i = \sigma^{i-1}(s_1)$, we have $L_p(s_i) = i$, for every $i \in \mathbb{N}$. Therefore, for every p-point x of K_s with $L_p(x) \ne 0$, there exists a unique salient p-point s_l such that $L_p(x) = L_p(s_l)$ and $l = L_p(x)$. Also, for every $l \in \mathbb{N}$, amongst all p-points $E_{p,l}$ of $\mathfrak{C}_0$ with p-level l there exists precisely one p-point s_l which is salient and has p-level l. Note that the salient p-points depend on p: if $p \ge q$, then the salient p-point s_i equals the salient q-point s_{i+p-q}.

A *folding point* is any point x in the core of K_s such that no neighborhood of x in core of K_s is homeomorphic to the product of a Cantor set and an arc. In [10] it was shown that $x = (\dots, x_{-2}, x_{-1}, x_0)$ is a folding point if and only if $x_{-k} \in \omega(c)$ for all $k \ge 0$. We can characterize folding points in terms of p-points as follows:

Lemma 2.2. Let p be arbitrary. A point $x \in K_s$ is a folding point if and only if there is a sequence of p-points $(x^k)_{k \in \mathbb{N}}$ such that $x^k \to x$ and $L_p(x^k) \to \infty$.

Proof. $\Rightarrow$ Take $m \ge p$ arbitrary. Since $\pi_m(x) \in \omega(c)$ there is a sequence of post-critical points $c_{n_i} \to \pi_m(x)$. This means that any point $y^i = (\dots, c_{n_i}, c_{n_i+1}, \dots, c_{n_i+m})$ is a p-point with p-level $L_p(y^i) = n_i + m - p$. Furthermore, for each $0 \le j \le m$, $|\pi_j(y^i) - \pi_j(x)| \to 0$ as $i \to \infty$. Since m is arbitrary, we can construct a diagonal sequence $(x^k)_{k \in \mathbb{N}}$ of p-points, by taking a single element from $(y^i)_{i \in \mathbb{N}}$ for each m, such that $\sup_{j \le k} |\pi_j(x^k) - \pi_j(x)| \to 0$ as $k \to \infty$. This proves that $x^k \to x$ and $L_p(x^k) \to \infty$.
$\Leftarrow$ Take m arbitrary. Since $x^k \to x$, also $|\pi_m(x^k) - \pi_m(x)| \to 0$ and $\pi_m(x^k) = c_n$ for $n = L_p(x^k) + p - m$. But $L_p(x^k) \to \infty$, so $\pi_m(x) \in \omega(c)$. □

A continuum is *chainable* if for every $\varepsilon > 0$, there is a cover $\{\ell^1, \dots, \ell^n\}$ of open sets (called *links*) of diameter $< \varepsilon$ such that $\ell^i \cap \ell^j \ne \emptyset$ if and only if $|i - j| \le 1$. Such a cover is called a *chain*. Clearly the interval $[0, s/2]$ is chainable. Throughout, we will use sequence of chains $\mathcal{C}_p$ of $\varprojlim([0, s/2], T_s)$ satisfying the following properties:

(1) there is a chain $\{I_p^1, I_p^2, \dots, I_p^n\}$ of $[0, s/2]$ such that $\ell_p^j := \pi_p^{-1}(I_p^j)$ are the links of $\mathcal{C}_p$;
(2) each point $x \in \cup_{i=0}^p T_s^{-i}(c)$ is the boundary point of some link I_p^j;
(3) for each i there is j such that $T_s(I_{p+1}^i) \subset I_p^j$.

If $\max_j |I_p^j| < \varepsilon s^{-p}/2$ then $\mathrm{mesh}(\mathcal{C}_p) := \max\{\mathrm{diam}\,(\ell) : \ell \in \mathcal{C}_p\} < \varepsilon$, which shows that $\varprojlim([0,s/2],T_s)$ is indeed chainable. Condition (3) ensures that $\mathcal{C}_{p+1}$ *refines* $\mathcal{C}_p$ (written $\mathcal{C}_{p+1} \preceq \mathcal{C}_p$).

Note that all p-point $E_{p,l}$ of p-level l belong to the same link of $\mathcal{C}_p$. (This follows by property (1) of $\mathcal{C}_p$, because $L_p(x) = L_p(y)$ implies $\pi_p(x) = \pi_p(y)$.) Therefore, every link of $\mathcal{C}_p$ which contains a p-point of p-level l contains also the salient p-point s_l.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $h : K_s \to K_s$ be a homeomorphism. Let $x, y \in K_s$ be folding points with $h(x) = y$. For $i \in \mathbb{N}_0$ let $q_i, p_i \in \mathbb{N}$ be such that for sequences of chains $(\mathcal{C}_{q_i})_{i\in\mathbb{N}_0}$ and $(\mathcal{C}_{p_i})_{i\in\mathbb{N}_0}$ of K_s we have

$$\cdots \prec h(\mathcal{C}_{q_{i+1}}) \prec \mathcal{C}_{p_{i+1}} \prec h(\mathcal{C}_{q_i}) \prec \mathcal{C}_{p_i} \prec \cdots \prec h(\mathcal{C}_{q_1}) \prec \mathcal{C}_{p_1} \prec h(\mathcal{C}_q) \prec \mathcal{C}_p,$$

where $q_0 = q$ and $p_0 = p$. Let $(\ell^x_{q_i})_{i\in\mathbb{N}_0}$ be a sequence of links such that $x \in \ell^x_{q_i} \in \mathcal{C}_{q_i}$, and similarly for $(\ell^y_{p_i})_{i\in\mathbb{N}_0}$. Then $\ell^x_{q_{i+1}} \subset \ell^x_{q_i}$, $\ell^y_{p_{i+1}} \subset \ell^y_{p_i}$ and $h(\ell^x_{q_i}) \subset \ell^y_{p_i}$. By Lemma 2.2 and by a remark above this proof, there exists a sequence $(s'_{d_i})_{i\in\mathbb{N}}$ of salient q-points with $s'_{d_i} \to x$ as $i \to \infty$. Then for every i there exist j_i such that $s'_{d_{j_i}} \in \ell^x_{q_i}$, $h(s'_{d_{j_i}}) \in \ell^y_{p_i}$ and $h(s'_{d_{j_i}}) \to y$ as $i \to \infty$. By [2, Theorem 4.1] the midpoint of the arc component A_i of $\ell^y_{p_i}$ which contains $h(s'_{d_{j_i}})$ is a salient p_i-point s''_{m_i}. Since $s''_{m_i}, y \in \ell^y_{p_i}$, for every i and $\mathrm{diam}\ \ell^y_{p_i} \to 0$ as $i \to \infty$, we have $s''_{m_i} \to y$. Since s'_{d_i} is a salient q-point and $s'_{d_i} \in \ell^x_q$, s''_{m_i} can be also considered as a salient p-point and is also the midpoint of the arc component $B_i \supset A_i$ of ℓ^y_p which contains $h(s'_{d_{j_i}})$. Therefore, $s''_{m_i} = s_{d_{j_i}+M}$, where M is as in [2, Theorem 4.1].

Let $R = M - q + p$. By [2, Corollary 5.3], R does not depend on q, p and M. Since $\sigma^R : K_s \to K_s$ is a homeomorphism, and since $s'_{d_i} \to x$ as $i \to \infty$, we have $\sigma^R(s'_{d_i}) \to \sigma^R(x)$ as $i \to \infty$. Note that $\sigma^R(s'_{d_{j_i}}) = s_{d_{j_i}+M}$ and $s_{d_{j_i}+M} \to y$. Therefore $\sigma^R(x) = y$, *i.e.*, $\sigma^R(x) = h(x)$. □

Proof of Corollary 1.2. If $\mathrm{orb}(c)$ is dense in $[c_2, c_1]$, every point x in the core of K_s satisfies $\pi_k(x) \in \omega(c)$ for all $k \in \mathbb{N}$. By [10], this means that every point is a folding point, and hence the previous theorem implies that $h \equiv \sigma^R$ on the core of K_s. □

Remark 2.3. A point $x \in K_s$ is an *endpoint* of an atriodic continuum, if for every pair of subcontinua A and B containing x, either $A \subseteq B$ or $B \subseteq A$. The notion of folding point is more general than that of endpoint. For example, if the critical point of a tent map is preperiodic, then the folding points of the inverse limit space of this tent map are not endpoints.

It is natural to classify arc-components $\mathfrak{A}$ according to the folding points they may contain. For arc-components $\mathfrak{A}$, we have the following possibilities:

- $\mathfrak{A}$ contains no folding point.
- $\mathfrak{A}$ contains one folding point x, *e.g.* if x is an endpoint of $\mathfrak{A}$.
- $\mathfrak{A}$ contains two folding points, *e.g.* if $\mathfrak{A}$ is the bar of a $\sin \frac{1}{x}$-continuum.
- $\mathfrak{A}$ contains countably many folding points. One can construct a tent map such that the folding points of its inverse limit space belong to finitely many arc-components that are periodic under σ, but where there are still countably folding points.[1]
- $\mathfrak{A}$ contains uncountably many folding points, *e.g.* if $\omega(c) = [c_2, c_1]$, because then every point in the core is a folding point.

This is clearly only a first step towards a complete classification.

Definition 2.4. Let $\ell^0, \ell^1, \dots, \ell^k$ be those links in $\mathcal{C}_p$ that are successively visited by an arc $A \subset \mathfrak{C}_0$ (hence $\ell^i \neq \ell^{i+1}$, $\ell^i \cap \ell^{i+1} \neq \emptyset$ and $\ell^i = \ell^{i+2}$ is possible if A turns in ℓ^{i+1}). Let $A^i \subset \ell^i$ be the corresponding arc components such that $\operatorname{Cl} A^i$ are subarcs of A. We call the arc A

- *p-link symmetric* if $\ell^i = \ell^{k-i}$ for $i = 0, \dots, k$;
- *maximal p-link symmetric* if it is p-link symmetric and there is no p-link symmetric arc $B \supset A$ and passing through more links than A.

The p-point of $A^{k/2}$ with the highest p-level is called the *center* of A, and the link $\ell^{k/2}$ is called the *central link* of A.

3. Isotopic Homeomorphisms of Unimodal Inverse Limits

It is shown in [2] that every salient p-point $s_l \in \mathfrak{C}_0$ is the center of the maximal p-link symmetric arc A_l. We denote the central link that s_l belongs to by $\ell_p^{s_l}$. For a better understanding of this section, let us mention that a key idea in [2] is that under a homeomorphism h such that $h(\mathcal{C}_q) \prec \mathcal{C}_p$, (maximal) q-link symmetric arcs have to map to (maximal) p-link symmetric arcs, and for this reason $h(s_m) \in \ell_p^{s_l}$ for some appropriate $m \in \mathbb{N}$ (see [2, Theorem 4.1]).

Lemma 3.1. Let $h : K_s \to K_s$ be a homeomorphism pseudo-isotopic to σ^R, and let $q, p \in \mathbb{N}_0$ be such that $h(\mathcal{C}_q) \preceq \mathcal{C}_p$. Let x be a q-point in the core of K_s and let $\ell_p^{s_l} \in \mathcal{C}_p$ be the link

[1] An example is the tent-map where c_1 has symbolic itinerary (kneading sequence) $\nu = 100101^2 01^3 01^4 01^5 \dots$. Then the two-sided itineraries of folding points are limits of $\{\sigma^j(\nu)\}_{j \geq 0}$. The only such two-sided limit sequences are $1^\infty.1^\infty$ and $\{\sigma^j(1^\infty.01^\infty) : j \in \mathbb{Z}\}$. Since they all have left tail $\dots 1111$, these folding points belong to the arc-component of the point $(\dots, p, p, p)$ for the fixed point $p = \frac{s}{1+s}$. This use of two-sided symbolic itineraries was introduced for inverse limit spaces in [6].

containing both $\sigma^R(x)$ and salient p-point s_l, where $l = L_p(\sigma^R(x))$. Suppose that the arc-component W_x of $\ell_p^{s_l}$ containing $\sigma^R(x)$ does not contain any folding point. Then $h(x) \in W_x$.

Proof. Since W_x does not contain any folding point, it contains finitely many p-points. Note that W_x contains at least one p-point since $\sigma^R(x) \in W_x$ is a p-point. Since $\mathfrak{C}_0$ is dense in K_s, there exists a sequence $(W_i)_{i\in\mathbb{N}}$ of arc-components of $\ell_p^{s_l}$ such that $W_i \subset \mathfrak{C}_0$, $FP_p(W_i) = FP_p(W_x)$ for every $i \in \mathbb{N}$, and $W_i \to W_x$ in the Hausdorff metric. Let $(x_i)_{i\in\mathbb{N}}$ be a sequence of q-points such that for every $i \in \mathbb{N}$, $L_q(x_i) = L_q(x)$, $x_i \to x$ and $\sigma^R(x_i) \in W_i$. Obviously $(x_i)_{i\in\mathbb{N}} \subset \mathfrak{C}_0$, $L_p(\sigma^R(x_i)) = L_p(\sigma^R(x))$ and $\sigma^R(x_i) \to \sigma^R(x)$. Since h is a homeomorphisms, $h(x_i) \to h(x)$. It follows by the construction in the proof of [2, Proposition 4.2] that $h(x_i) \in W_i$ for every $i \in \mathbb{N}$. Therefore $h(x) \in W_x$. $\square$

Corollary 3.2. Let $h : K_s \to K_s$ be a homeomorphism pseudo-isotopic to σ^R. Then h permutes arc-components of K_s in the same way as σ^R.

Proof. Since h is a homeomorphism, h maps arc-components to arc-components. Let $\mathfrak{A}$ be an arc-component of K_s. Let us suppose that $\mathfrak{A}$ contains a folding point, say x. Then $h(x) = \sigma^R(x)$ implies $h(\mathfrak{A}) = \sigma^R(\mathfrak{A})$.

Let us assume now that $\mathfrak{A}$ does not contain any folding point; in particular this means that $\mathfrak{A}$ has no endpoint. There exist $q, p \in \mathbb{N}_0$ such that $h(\mathcal{C}_q) \preceq \mathcal{C}_p$ and that $h(\mathfrak{A})$ is not contained in a single link of $\mathcal{C}_p$. Then $\mathfrak{A}$ is not contained in a single link of $\mathcal{C}_q$. Since $\mathfrak{A}$ cannot go straight through all the links of $\mathcal{C}_q$, we can find a link $\ell_q \in \mathcal{C}_q$ and arc-component $V \in \ell_q \cap \mathfrak{A}$ of ℓ_q such that V contains at least one q-point, say x. Let $\ell_p^{s_l} \in \mathcal{C}_p$ be such that $l = L_p(\sigma^R(x))$. Let $W \subset \ell_p^{s_l}$ be arc-component containing $\sigma^R(x)$. Since $\mathfrak{A}$ does not contain any folding point, $h(\mathfrak{A})$ does not contain any folding point implying W does not contain any folding point. Then, by Lemma 3.1, $h(x) \in W$ implying $h(\mathfrak{A}) = \sigma^R(\mathfrak{A})$. $\square$

Lemma 3.3. Let $h : K_s \to K_s$ be a homeomorphism that is pseudo-isotopic to the identity. Then h preserves orientation of every arc-component $\mathfrak{A}$, *i.e.,* given a parametrization $\varphi : \mathbb{R} \to \mathfrak{A}$ (or $\varphi : [0,1] \to \mathfrak{A}$ or $\varphi : [0,\infty) \to \mathfrak{A}$) that induces an order $\prec$ on $\mathfrak{A}$, then $x \prec y$ implies $h(x) \prec h(y)$.

Proof. Let us first suppose that $h : K_s \to K_s$ is any homeomorphism. Then, by [2, Theorem 1.2] there is an $R \in \mathbb{Z}$ such that h, restricted to the core, is pseudo-isotopic to σ^R, *i.e.,* h permutes the composants of the core of the inverse limit in the same way as σ^R. Therefore, by Corollary 3.2, it permutes the arc-components of the inverse limit in the same way as σ^R.

Let $\mathfrak{A}, \mathfrak{A}'$ be arc-components of the core such that $h, \sigma^R : \mathfrak{A} \to \mathfrak{A}'$, and let $x, y \in \mathfrak{A}$, $x \prec y$. We want to prove that $h(x) \prec h(y)$ if and only if $\sigma^R(x) \prec \sigma^R(y)$. Since h and σ^R are homeomorphisms on arc-components, each of them could be either order preserving or order reversing. Therefore, to prove the claim we only need to pick two convenient points $u, v \in \mathfrak{A}$, $u \prec v$, and check if we have either $h(u) \prec h(v)$ and $\sigma^R(u) \prec \sigma^R(v)$, or $h(v) \prec h(u)$ and $\sigma^R(v) \prec \sigma^R(u)$. If $\mathfrak{A}$ contains at least two folding points, we can choose u, v to be folding points. Then $h(u) = \sigma^R(u)$ and $h(v) = \sigma^R(v)$ and the claim follows.

Let us suppose now that $\mathfrak{A}$ contains at most one folding point. Then there exist $q, p \in \mathbb{N}_0$ such that $h(\mathcal{C}_q) \preceq \mathcal{C}_p$ and q-points $u, v \in \mathfrak{A}$, $u \prec v$ (on the same side of the folding point if there exists one) such that $\sigma^R(u)$ and $\sigma^R(v)$ are contained in disjoint links of $\mathcal{C}_p$ each of which does not contain the folding point of $\mathfrak{A}$, if there exists one.

Let $\ell_p^{s_j}, \ell_p^{s_k} \in \mathcal{C}_p$ with $j = L_p(\sigma^R(u))$ and $k = L_p(\sigma^R(v))$ be links containing $\sigma^R(u)$ and $\sigma^R(v)$ respectively. Let $W_u \subset \ell_p^{s_j}$ and $W_v \subset \ell_p^{s_k}$ be arc-components containing $\sigma^R(u)$ and $\sigma^R(v)$ respectively. Then W_u and W_v do not contain any folding point and by Lemma 3.1 $h(u) \in W_u$ and $h(v) \in W_v$. Therefore obviously $h(u) \prec h(v)$ if and only if $\sigma^R(u) \prec \sigma^R(v)$.

If h is a homeomorphism that is pseudo-isotopic to the identity, then $R = 0$ and the claim of lemma follows. □

Corollary 3.4. If h is pseudo-isotopic to the identity, then the arc A connecting x and $h(x)$ is a single point, or A contains no folding point.

Proof. Since h is pseudo-isotopic to the identity, x and $h(x)$ belong to the same composant, and in fact the same arc-component. So let A be the arc connecting x and $h(x)$. If $x = h(x)$, then there is nothing to prove. If $h(x) \neq x$, say $x \prec h(x)$, and A contains a folding point y, then $x \prec y = h(y) \prec h(x)$, contradicting Lemma 3.3. □

In particular, any homeomorphism h that is pseudo-isotopic to the identity cannot reverse the bar of a $\sin \frac{1}{x}$-continuum. The next lemma strengthens Lemma 3.1 to the case that W_x is allowed to contain folding points.

Lemma 3.5. Let $h : K_s \to K_s$ be a homeomorphism that is pseudo-isotopic to the identity. Let $q, p \in \mathbb{N}_0$ be such that $h(\mathcal{C}_q) \preceq \mathcal{C}_p$. Let x be a q-point in the core of K_s and let $\ell_p^{s_l} \in \mathcal{C}_p$ be such that $l = L_p(x)$. Let $W_x \subset \ell_p^{s_l}$ be an arc-component of $\ell_p^{s_l}$ containing x. Then $h(x) \in W_x$.

Proof. If W_x does not contain any folding point the proof follows by Lemma 3.1 for $R = 0$.

Let W_x contain at least one folding point. If x is a folding point, then $h(x) = x \in W_x$ by Theorem 1.1. If W_x contains at least two folding points, say y and z, such that $x \in [y, z] \subset W_x$, then $h(x) \in [y, z] \subset W_x$ by Corollary 3.4.

The last possibility is that $x \in (y, z) \subset W_x$, where $z \in W_x$ is a folding point, $y \notin W_x$, *i.e.*, y is a boundary point of W_x, and (y, z) does not contain any folding point. Since $\mathfrak{C}_0$ is dense in K_s, there exists a sequence $(W_i)_{i\in\mathbb{N}}$ of arc-components of $\ell_p^{s_l}$ such that $W_i \subset \mathfrak{C}_0$ and $W_i \to (y, z]$ in the Hausdorff metric. Note that in this case there exists the sequence of p-points $(m_i)_{i\in\mathbb{N}}$, where m_i is the midpoint of W_i, and for this sequence we have $m_i \to z$ and $L_p(m_i) \to \infty$. Also, for every i large enough, every W_i contains a q-point x_i with $L_q(x_i) = L_q(x)$, and for the sequence of q-points $(x_i)_{i\in\mathbb{N}}$ we have $x_i \to x$. Obviously $(x_i)_{i\in\mathbb{N}} \subset \mathfrak{C}_0$ and $L_p(x_i) = L_p(x)$. By the proof of [2, Proposition 4.2] applied for $R = 0$ we have $h(x_i) \in W_i$ for every i. Since h is a homeomorphisms, $h(x_i) \to h(x)$. Therefore, $h(x) \in (y, z) \subset W_x$. □

Proposition 3.6. Let $h : K_s \to K_s$ be a homeomorphism. If $z^n \to z$ and $A^n = [z^n, h(z^n)]$, then $A^n \to A := [z, h(z)]$ in Hausdorff metric.

Proof. We know that h is pseudo-isotopic to σ^R for some $R \in \mathbb{Z}$; by composing h with σ^{-R} we can assume that $R = 0$. By Corollary 3.2, h preserves the arc-components, and by Lemma 3.3, preserves the orientation of each arc-component as well.

Take a subsequence such that A^{n_k} converges in Hausdorff metric, say to B. Since $x, h(x) \in B$, we have $B \supseteq A$. Assume by contradiction that $B \neq A$. Fix q, p arbitrary such that $h(\mathcal{C}_q)$ refines $\mathcal{C}_p$, and such that $\pi_p(B) \neq \pi_p(A)$ and a fortiori, that there is a link $\ell \in \mathcal{C}_p$ such that $\ell \cap A = \emptyset$ and $\pi_p(\ell)$ contains a boundary point of $\pi_p(B)$.

Let $d_n = \max\{L_p(y) : y \text{ is } p\text{-point in } A^n\}$. If $D := \sup d_n < \infty$ (if A^n contains no p-point, then we set $D = 0$ be default), then we can pass to the chain $\mathcal{C}_{p+D}$ and find that all A^{n_k}'s go straight through $\mathcal{C}_{p+D}$, hence the limit is a straight arc as well, stretching from x to $h(x)$, so $B = A$. Therefore $D = \infty$, and we can assume without loss of generality that $d_{n_k} \to \infty$.

Since the link in ℓ is disjoint from A but $\pi_p(\ell)$ contains a boundary point of $\pi_p(B)$, the arcs A^{n_k} intersects ℓ for all k sufficiently large. Therefore $A^{n_k} \cap \ell$ separates x^{n_k} from $h(x^{n_k})$; let W^{n_k} be a component of $A^{n_k} \cap \ell$ between x^{n_k} and $h(x^{n_k})$. Since $\pi_p(\ell)$ contains a boundary point of $\pi_p(B)$, W^{n_k} contains at least one p-point for each k. Lemma 3.5 states that there is $y^{n_k} \in W^{n_k}$ such that $h(y^{n_k}) \in W^{n_k}$ as well, and therefore $x^{n_k} \prec y^{n_k}, h(y^{n_k}) \prec h(x^{n_k})$ (or $y^{n_k} \prec x^{n_k}, h(x^{n_k}) \prec h(y^{n_k})$), contradicting that h preserves orientation. □

Let us finally prove Theorem 1.3:

Proof of Theorem 1.3. Fix R such that h is pseudo-isotopic to σ^R. Then $\sigma^{-R} \circ h$ is pseudo-isotopic to the identity. So renaming $\sigma^{-R} \circ h$ to h again, we need to show that h is isotopic to the identity.

If x is a folding point of K_s, then $h(x) = x$ by Theorem 1.1. In this case, and in fact for any point such that $h(x) = x$, we let $H(x,t) = x$ for all $t \in [0,1]$. If $h(x) \neq x$, then x and $h(x)$ belong to the same arc-component, and the arc $A = [x, h(x)]$ contains no folding point by Corollary 3.4. By Lemma 2.2, A contains only finitely many p-points, so there is m such that $\pi_m : A \to \pi_m(A)$ is one-to-one. In this case,

$$H(x,t) = \pi_m^{-1}|_A[(1-t)\pi_m(x) + t\pi_m(h(x))].$$

Clearly $t \mapsto H(\cdot,t)$ is a family of maps connecting h to the identity in a single path as $t \in [0,1]$. We need to show that H is continuous both in x and t, and that $H(\cdot,t)$ is a bijection for all $t \in [0,1]$.

Let $z \in K_s$ and $(z^n, t^n) \to (z,t)$. If $h(z) = z$, then $H(z,t) \equiv z$, and Proposition 3.6 implies that $H(z^n,t^n) \to z = H(z,t)$. So let us assume that $h(z) \neq z$. The arc $A = [z, h(z)]$ contains no folding point, so by Lemma 2.2, for all $x \in A$, there is $\varepsilon(x) > 0$ and $W(x) \in \mathbb{N}$ such that $B_{\varepsilon(x)}(x)$ contains no p-point of p-level $\geq W(x)$. By compactness of A, $\varepsilon := \inf_{x \in A} \varepsilon(x) > 0$ and $\sup_{x\in A} W(x) < \infty$, whence there is $m > p + W$ such that $V := \pi_m^{-1} \circ \pi_m(A)$ is contained in an ε-neighborhood of A that contains no p-point.

By Proposition 3.6, there is N such that $A^n \subset V$ for all $n \geq N$, and in fact $\pi_m(A^n) \to \pi_m(A)$. It follows that $H(z^n,t^n) \to H(z,t)$.

To see that $x \mapsto H(\cdot,t)$ is injective for all $t \in [0,1]$, assume by contradiction that there is $t_0 \in [0,1]$ and $x \neq y$ such that $H(x,t_0) = H(y,t_0)$. Then x and y belong to the same arc-component $\mathfrak{A}$, which is the same as the arc-component containing $h(x)$ and $h(y)$. The smallest arc J containing all four point contains no folding point by Corollary 3.4. Therefore there is m such that $\pi_m : J \to \pi_m(J)$ is injective, and we can choose an orientation on $\mathfrak{A}$ such that $x < y$ on J, and $\pi_m(x) < \pi_m(y)$. Since $t \mapsto \pi_m \circ H(x,t)$ is monotone with constant speed depending only on x, we find

$$\pi_m(x) < \pi_m(y) < \pi_m \circ H(x,t_0) = \pi_m \circ H(y,t_0) < \pi_m \circ h(y) < \pi_m \circ h(x)$$

This contradicts that h preserves orientation on arc-components, see Lemma 3.3.

To prove surjectivity, choose $x \in K_s$ arbitrary. If $h(x) = x$, then $H(x,t) = x$ for all $t \in [0,1]$. Otherwise, say if $h(x) > x$, there is $y < x$ in the same arc-component as x such that $h(y) = x$. The map $t \mapsto H(\cdot,t)$ moves the arc $[y,x]$ continuously and monotonically to $[h(y), h(x)] =$

$[x, h(x)]$. Therefore, for every $t \in [0,1]$, there is $y_t \in [y, x]$ such that $H(y_t, t) = x$. This proves surjectivity.

We conclude that $H(x,t)$ is the required isotopy between h and the identity. □

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