Workshop on Proof Theory, Modal Logic and Reflection Principles

Wormshop 2023

Bern

Booklet of abstracts

ALBERT VISSER

1. R meets Effective Essential Hereditary Creativity

This talk is about research in collaboration with Taishi Kurahashi.

The Tarski-Mostowski-Robinson theory R has all kinds of remarkable properties. I zoom in on *effective essential hereditary creativity* and sketch a new proof that R has this property. This result extends a theorem of Alan Cobham.

I discuss the equivalence of effective essential hereditary creativity and strong effective inseparability. This insight is an analogue of Marian Boykan Pour-El's theorem that effective essential incompleteness is equivalent to effective inseparability.

Finally, I show how desirable properties of R can be transferred to other theories like Vaught's Set Theory and restricted sequential theories using the notion of *sourcedness*.

My worms, my friends

A biased overview of Π^0_1-ordinal analysis through provability algebras

Joost J. Joosten*

August 27, 2023

Abstract

In this talk I will provide an overview of Π^0_1-ordinal analysis through provability algebras. I guess we should start at Turing's PhD thesis and from there I will follow an itinerary of results and developments that I have mainly chosen on the basis of its importance but also on the basis of personal preference and closeness to the results. In the event, I shall mention various of my own results, old and recent.

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The limits of determinacy in third-order arithmetic

J.P.Aguilera, T. Kouptchinsky

Abstract

This talk is about the foundations of mathematics, studying determinacy axioms derived from game theory, with a reverse mathematics point of view. It exposes their relationship with second-order and third-order arithmetic, examining a paper by Montalbán and Shore. These results are a refinement of the proof from Martin of Borel determinacy, which shows along the way that nth-order arithmetic (Z_n) proves the determinacy of Π^0_{n+1} Gale-Stewart games ($n > 1$).

Based on the equivalence between some of the subsystems of second-order arithmetic and extensions of the Kripke-Platek set theory, we present a generalisation of the results of Montalbán and Shore, in some natural interpretation of third-order arithmetic about the difference hierarchy of Π^0_4 sets. Along the way, we underline a shift compared to the analogous situation in the countable case (about differences of Π^0_3 sets), which is the object of the paper of Montalbán and Shore. Namely, we need less of the separation scheme than in the countable case.

Finally, we use these generalisations following the results of Yokoyama and Pachecho to show that, while

$$\forall m \in \omega \ \mathsf{Z}_n \vdash (\Pi^0_{n+1})_m\text{-}\mathsf{Det} \qquad \text{but} \qquad \mathsf{Z}_n \nvdash \forall m \ (\Pi^0_{n+1})_m\text{-}\mathsf{Det},$$

the last theorem is equivalent to a form of reflection principle.

Van Benthem's characterisation theorem for interpretability logic IL with respect to Verbrugge semantics

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Keywords:
Interpretability logic, Verbrugge semantics, Weak bisimulations, Van Benthem's theorem

Van Benthem's characterisation theorem belongs to the field of correspondence theory which systematically investigates the relationship between modal and classical logic. It is the paradigmatic result of this kind which shows that modal languages correspond to the bisimulation invariant fragment of first–order languages (see [6]). A. Dawar and M. Otto [1] proved a characterization theorem over the provability logic GL. M. Vuković and T. Perkov extended this result to Veltman models for the interpretability logic IL. The key tools in these proofs are bisimulations, saturated unraveling and standard translation, in appropriate form.

Generalised Veltman semantics for interpretability logic, or nowadays called Verbrugge semantics, was developed to obtain certain non-derivability results since Veltman semantics for interpretability logic is not fine-grained enough for certain applications. It has been proven that this semantics has various good properties (see e.g. [3] and [4]). However, it turns out that the notion of bisimulation of Verbrugge models, which has been used so far (see e.g. [7] and [8]), does not satisfy an important property: if there are only finitely many propositional variables, n-modal equivalence of two worlds in a Verbrugge model doesn't imply their n-bisimilarity [2]. So, we have defined a new notion of weak bisimulation (or short, w-bisimulation) and its finite approximation called n-w-bisimulation [2]. In this talk, that new notion will be presented along with the appropriate definitions of a standard translation and q-saturated unravelling. Finally, using previously introduced concepts and proven results, we will show the key steps in proving an analogue of van Benthem's characterization theorem for the logic of interpretability with Verbrugge's semantics: a two-sorted first-order formula is invariant under w-bisimulation if, and only if, it is equivalent to standard translation of an IL-formula with respect to Verbrugge models.

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On Provability Logic of Heyting Arithmetic

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1 Abstract

In this talk we provide a complete axiomatization of the Provability Logic for Heyting Arithmetic HA (this is the first-order intuitionistic fragment of the Peano Arithmetic PA). It turns out that the provability logic of HA is equal to iGL (intuitionistic fragment of the Gödel-Löb logic GL) plus $\Box A \to \Box B$ for *some* admissible rules A/B of iGL.

Then we describe a crucial tools which were helpful during the completeness proof: a new Kripke style semantic for intuitionistic modal logics, called *mixed semantics*, which is a combination of derivability and usual validity in Kripke models.

This talk is based on the following two manuscripts: [1, 2].

References

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On a Problem Posed by Montagna

Reinhard Kahle, Isabel Oitavem, Paulo Guilherme Santos

Submission for the
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Peter Clote and Jan Krajíček complied in 1993 a list of *Open Problems* in Arithmetic, Proof Theory, and Computational Complexity. In one of the problems, number 25, Montagna asked for the validity of Löb's Theorem in terms of k-provability in Peano Arithmetic:

> [D]oes $\mathsf{PA} \vdash_{k\text{-steps}} \Box\varphi \rightarrow \varphi$ implies $\mathsf{PA} \vdash_{k\text{-steps}} \varphi$, where $\Box\varphi$ is the formalized statement that φ is provable?

In this talk, a negative answer to this question is given, for its formulation in provability logic as well as in Peano Arithmetic. We have, however, to use a specific axiomatization of the propositional part. Instead of the standard axioms A1 – A3:

A1. $\varphi \rightarrow (\psi \rightarrow \varphi)$;

A2. $(\varphi \rightarrow (\psi \rightarrow \mu)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \mu))$;

A3. $(\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi)$.

we use the following ones, A'1 – A'4:

A'1. $\varphi \rightarrow (\psi \rightarrow \psi)$;

A'2. $(\varphi \rightarrow (\psi \rightarrow \mu)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \mu))$;

A'3. $(\neg\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \varphi)$;

A'4. $(\varphi \rightarrow (\psi \rightarrow \mu)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \mu))$.

We will discuss, in particular, the dependency of Montagna's Problem on the specific axiomatization of propositional logic.

A realization theorem for the modal logic of transitive closure K^+

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Justification logics are an interesting group of epistemic logics whose language is obtained from the language of modal logic by replacing assertions $\Box A$ with expressions $[h]\, A$, where $[h]\, A$ is interpreted as 'h is a justification for A'. A variety of studies on justification logics has appeared since Artemov introduced the logic of proofs LP [2], where the expression $[h]A$ is understood as 'h is a proof for A'. Artemov proved, among other things, a realization theorem connecting LP and the standard modal logic $\mathsf{S4}$. This theorem involves the following notion of forgetful projection. For any formula B of LP, the forgetful projection of B is obtained from the given formula by replacing all subformulas of the form $[h]\, C$ with $\Box C$. Trivially, the forgetful projection of each formula provable in LP is provable in $\mathsf{S4}$. The realization theorem states the converse: any formula A provable in $\mathsf{S4}$ turns out to be the forgetful projection of a formula B provable in LP.

To date, the justification counterparts of many modal logics have been presented and the corresponding realization theorems have been obtained. However, the epistemically important case of the modal logic of common knowledge still needs to be investigated. The concept of common knowledge is captured in the given logic according to the so-called fixed-point account, i.e. common knowledge of A is defined as the greatest fixed-point of a mapping $X \mapsto$ (everybody knows A and everybody knows X). Therefore, the logic of common knowledge belongs to the family of modal fixed-point logics and, like other logics from this family, is in many respects difficult to study. Although Bucheli, Kuznets and Studer introduced a justification logic analogous to the modal logic of common knowledge (see [5, 4]), whether one can obtain the realization theorem remains to be an open question. At the same time, as Antonakos [1] showed, the modal logic of generic common knowledge is realizable in a justification logic proposed by Artemov in [3].

In this talk, we will focus on the case of the modal logic of transitive closure K^+, which is very similar to the case of the modal logic of common knowledge. We present a justification counterpart of K^+ and syntactically establish the corresponding realization theorem by applying a sequent calculus, the essential feature of which is that it allows non-well-founded proof trees.

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On the Proof Theory and Correspondence Theory of Atomic and Molecular Logics[1]

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Atomic and molecular logics, introduced in [4], are based on Dunn's Gaggle theory and are a generalization of modal logics. Molecular logics are logics whose primitive connectives are compositions of connectives of atomic logics. So, atomic logics are also molecular logics. We showed in [4] that every non-classical logic such that the truth conditions of its connectives can be expressed in first-order logic is as expressive as an atomic or a molecular logic. We also proved that first-order logic is as expressive as a specific atomic logic. Moreover, from a model-theoretic point of view, invariance notions for atomic and molecular logics can be defined systematically from the truth conditions of their connectives and when those are uniform we obtain automatically a van Benthem characterization theorem for the logic considered [5]. These results support formally our claim that atomic and molecular logics are somehow 'universal'.

In this talk, sound and strongly complete display calculi and Hilbert calculi for basic atomic and molecular logics are introduced with a Kripke-style relational semantics. All these calculi can be automatically computed from the definition of the connectives constituting a basic atomic or molecular logic, yet with some restrictions on the class of molecular logics. We develop the correspondence theory for these logics on the basis of the work of Goranko & Vakarelov. We reformulate the notion of inductive formulas introduced by Goranko & Vakarelov into our framework. This allows us to prove correspondence theorems for atomic logics by adapting their results. Then, we apply the general results of Ciabattoni & Ramanayake about display calculi to atomic and molecular logics. We obtain an instantiation of their "I2 acyclic" formulas in our framework. We show that our versions of I2 acyclic formulas are inductive formulas in the sense of Goranko & Vakarelov, again adapted to our framework. This allows us to provide a characterization of displayable calculi extending basic atomic and molecular logics in terms of these formulas: a calculus is properly displayable iff it is sound and complete w.r.t. a class of models defined by I2 acyclic formulas. Afterwards, we apply our characterization theorem and we obtain proper display calculi for propositional logic. In doing so, we show how classical and standard inference rules reappear as simplifications of more general and abstract rules that stem from the display calculi of our atomic logics.

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[1]This talk is based on [1, 2, 3] which are submissions currently under revision.

Universes in Explicit Mathematics – Some Irritations ?

Gerhard Jäger

After a very short introduction into the main ideas of Explicit Mathematics EM we turn to the notion of *universe* and look at universes from different perspectives. In doing that some emphasis is placed on ontological properties of universes, pointing out some features of universes that look a bit surprising at first glance.

We also turn to the question of the predictivity of universes and what it means to claim the minimality of universes. Time permitting, we will discuss a few open problems in connection with Mahloness in EM.

Reflection and Induction for subsystems of HA

Philipp Provenzano[1], Mojtaba Mojtahedi[1], Fedor Pakhomov[1], and Albert Visser[2]

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By a classical result of Leivant and Ono [1, 2], the subsystem $I\Pi_n$ of PA is equivalent to the scheme of uniform reflection $\mathrm{RFN}_{\Pi_{n+2}}(\mathrm{EA})$ over elementary arithmetic EA. In the present paper, we study the correspondence between the schemes of induction and reflection for subsystems of Heyting arithmetic HA.
In an intuitionistic setting, complexity classes of formulas behave quite differently than over classical logic. Underpinning this, we show by an application of realizability that reflection over prenex formulas $\mathrm{RFN}_{\Pi_\infty}(i\mathrm{EA})$ is equivalent over intuitionistic elementary arithmetic iEA to just $\mathrm{RFN}_{\Sigma_1}(i\mathrm{EA})$ or the totality of hyperexponentiation. More generally, for any class $\Gamma \supseteq \Sigma_1$ of formulas, we have an equivalence between $\mathrm{RFN}_{\Pi_\infty\Gamma}(i\mathrm{EA})$ and $\mathrm{RFN}_{\Gamma}(i\mathrm{EA})$. This phenomenon does not have any counterpart in classical logic where Π_∞ exhausts all arithmetical formulas.
As our main result, we show that a suitable generalization of the result by Leivant and Ono holds true intuitionistically. We show for some natural classes Γ of formulas that the principle of induction $I\Gamma$ for Γ is equivalent over iEA to the reflection principle $\mathrm{RFN}_{\forall(\Gamma\to\Gamma)\to\Gamma}(i\mathrm{EA})$. Here $\forall(\Gamma \to \Gamma) \to \Gamma$ denotes the class of formulas of type $\forall x.\,(\phi(x) \to \psi(x)) \to \theta$ with $\phi, \psi, \theta \in \Gamma$. This appears as the natural class containing the induction axioms for Γ. Note that classically, for $\Gamma = \Pi_n$, (the universal closure of) this class is just equivalent to Π_{n+2}, in harmony with the classical result.

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▸ ANDREAS WEIERMANN, *The fine structure of certain classes of monotone primitive recursive functions.*

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Motivated by a problem of Reshetnikov (shown to us by Harry Altman) we consider the class F of unary number-theoretic functions containing the successor and which contains with two functions f and g the function h which is the function f iterated $g(x)$ many times applied to x. Let $f < g$ if f is eventually dominated by g. We are interested in the maximal order type of this relation. This order type is bounded from below by ω^{ω^ω} and from above by ε_0. We expect that ω^{ω^ω} will be the sharp bound and that F is actually well ordered under $<$. We are also interested in the case that the function class in question is closed under composition. Here $\varphi_\omega 0$ is a lower bound for the resulting maximal order type and we believe that this is a sharp bound. Many variations of this problems are possible (even for functionals of finite type). (Joint work with Zongshu Wu and others.)

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The Provability of Consistency

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For Hilbert, the consistency of a formal theory T was a **serial property**, i.e., an infinite collection $\mathcal{C}$ of claims, here "*D does not contain a contradiction*" for each derivation D.

$$\textit{What is a proof of such a serial property } \mathcal{C} \textit{ in a given theory } S? \qquad (1)$$

Provability of each instance of $\mathcal{C}$ in S is not sufficient, since it does not take into account **why** they are provable whereas justification of that can lay beyond S.

An approach, *replace $\mathcal{C}$ with a single formula $\widehat{\mathcal{C}}$, yielding all instances of $\mathcal{C}$, and reduce* (1) *to the provability of $\widehat{\mathcal{C}}$ in S*, fails. In the case of Peano Arithmetic PA, the formula $\mathsf{Con}_{\mathsf{PA}}$ yields Hilbert's consistency $\mathcal{C}$ but not *visa versa*. So, Gödel's Theorem statement that $\mathsf{PA} \not\vdash \mathsf{Con}_{\mathsf{PA}}$ does not shed light to the provability of $\mathcal{C}$ in PA.

Another reason to avoid using provability of a **consistency formula** as the provability of consistency test: by compactness, it does not distinguish between, e.g., "provability in PA" and "provability in some finite fragment of PA" which turns out to be critical here.

The answer to (1) can be mined from Hilbert's own works on consistency. Hilbert scholars characterize his view of a consistency proof for $\mathcal{C}$ as a pair of

(i) an operation (we call it **selector**) that given D produces a proof that D does not contain a contradiction, e.g., by reducing D to a special contradiction-free form,

(ii) a proof (**verifier**) that the selector works for all inputs D.

This agrees with Brouwer's approach to proving universal sentences since, according to the BHK semantics with Kreisel's adjustments, a proof of $\forall x f(x)$ is a pair $\langle s, v\rangle$ such that

$$v \textit{ is a proof that for all } x,\ s(x) \textit{ is a proof of } f(x).$$

Hilbert-style proofs of serial properties are *de facto* adopted by mathematicians and even became indispensable in metamathematics.

We provide a proof of the consistency of PA in the original Hilbert's format and formalize this proof in PA. This undermines the Unprovability of Consistency paradigm that "*there exists no consistency proof of a system that can be formalized in the system itself*" *(Encyclopædia Britannica)* and suggests reopening studies of Hilbert's consistency program.

Simplicial Approaches to Crashing Agents

Roman Kuznets*

(joint work with
Hans van Ditmarsch and Rojo Randrianomentsoa)

Abstract

Kripke models have long been a preferred semantics for modeling knowledge and belief. For distributed systems, this is commonly done via the runs-and-systems framework, where the Kripke model is induced from a given interpreted system by abstracting away most low-level details, including agents' local states. Simplicial complexes provide an alternative algebraic-topological semantics that is categorically dual to Kripke models and treats agents' local states as primary objects. This is more in line with how agents' knowledge is determined in interpreted systems. One of the benefits of the tangible presence of agents and their local states is the ability to model the absence of some of the agents, which is necessary to faithfully represent distributed systems with crash failures. Modeling knowledge of crashed agents presents interesting dilemmas on the purely logical level. We outline the available choices, discuss the difficulties involved, e.g., the failure of modus ponens in one, and outline the benefits and pitfalls of the existing approaches.

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The Compactness of Gödel Logic

Juan Aguilera

The compactness cardinal of first-order Gödel logic with identity is the least ω_1-strongly compact cardinal.

Intuitionistic Gödel-Löb logic, à la Simpson: labelled systems and birelational semantics

Anupam Das

We derive an intuitionistic version of Gödel-Löb modal logic (GL) in the style of Simpson, via proof theoretic techniques. We recover a labelled system, lIGL, by restricting a non-wellfounded labelled system for GL to have only one formula on the right. The latter is obtained using techniques from cyclic proof theory, sidestepping the barrier that GL's usual frame condition (converse well-foundedness) is not first-order definable. While existing intuitionistic versions of GL are typically defined over only the box (and not the diamond), our presentation includes both modalities.

Our main result is that lIGL coincides with a corresponding semantic condition in birelational semantics: the composition of the modal relation and the intuitionistic relation is conversely well-founded. We call the resulting logic IGL. While the soundness direction is proved using standard ideas, the completeness direction is more complex and necessitates a detour through several intermediate characterisations of IGL.

This talk is based on joint work with Iris van der Giessen and Sonia Marin: https://arxiv.org/abs/2309.00532 .

A decision procedure for IS4

Marianna Girlando, Roman Kuznets, Sonia Marin,
Marianela Morales, Lutz Straßburger

In these two talks we demonstrate decidability for the intuitionistic modal logic S4 first formulated by Fischer Servi. This solves a problem that has been open for almost thirty years since it had been posed in Simpson's PhD thesis in 1994. We obtain this result by performing proof search in a labelled deductive system that, instead of using only one binary relation on the labels, employs two: one corresponding to the accessibility relation of modal logic and the other corresponding to the order relation of intuitionistic Kripke frames. Our search algorithm outputs either a proof or a finite counter-model, thus, additionally establishing the finite model property for intuitionistic S4, which has been another long-standing open problem in the area. In Part 1 we will introduce intuitionistic modal logics, we will present our labelled proof system, and we show how it can be employed for decision procedures in general. In Part 2 we will go into the specifics of the IS4 decision algorithm.

Dilators, Reflection, and Forcing: A Proof-Theoretic Analysis of $\Pi^1_1{-}\mathsf{CA}_0$.

Fedor Pakhomov

In this talk I will sketch a new approach to ordinal analysis of $\Pi^1_1{-}\mathsf{CA}_0$ that uses the interplay of the methods of functorial proof theory and iterated reflection. Usual idea of ordinal analysis via refleciton principles is to use reflection iterated along countable well-orders. However, since the notion of well-order is just Π^1_1-complete this kinds of iterations have significant limitations for the case of applications to proof-theoretic analysis of second-order arithmetic. Dilators form a Π^1_2-complete class of set-coded functors on the category of well-orders and strictly monotone functions. The central idea of the method are iteration of reflection along dilators D with the intended meaning that the iteration as iterations along $D(\omega_1)$. In our analysis it is instrumental to calculate the poof-theoretic dilators $Dil(T)$ of theories T ($Dil(T)$ is embeddability smallest dilators, where all provable dilators of the theory are embeddable). Note that the proof-theoretic ordinal of T is $Dil(T)(0)$. The key technical result here is that the proof-theoretic dilator of the Π^1_2 uniform reflection iterated along D could be described as a dilator performing collapsing of $D(\Omega)$. For a theory T let $T' = \{\phi \mid T$ proves that ϕ holds in all β-models of $\mathsf{ACA}_0\}$. Final ingredient in our analysis is the result obtained using forcing in second order arithmetic that T' could be characterized as the theory of reflection iterated along the proof-theoretic dilator of T. Since, Π^1_2-consequences of $\Pi^1_1{-}\mathsf{CA}_0$ are precisely $\bigcup\{T^{(n)} \mid n < \omega\}$, we see that the proof theoretic dilator of $\Pi^1_1{-}\mathsf{CA}_0$ could be described in terms all finite iterations of collapsing procedure or in other words to recover the standard notation system that uses ω-many collapsing functions.

▶ LEV D. BEKLEMISHEV and YUNSONG WANG, *General topological frames for polymodal provability logic based on periodic sets of ordinals*.
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Topological semantics of provability logic is well-known since the work of Harold Simmons and Leo Esakia in the 1970s. The diamond modality can be interpreted as a topological derivative operator acting on a scattered topological space. This type of semantics is especially interesting in the case of polymodal provability logic GLP (and even its bimodal fragment GLB), because these logics are incomplete w.r.t. any class of Kripke frames. Although GLP is known to be complete w.r.t. topological semantics, the topologies needed for the completeness proof are highly non-constructive [1]. On the other hand, the question of completeness of GLP w.r.t. *natural* ordinal GLP-spaces turns out to be dependent on large cardinal axioms of set theory (cf [2]).

In this paper we define a natural class of *countable* general topological frames on ordinals for which GLB is sound and complete. The associated topologies happen to be the same as the ordinal topologies introduced by T. Icard [3]. However, the key point is to consider a suitable algebra of subsets of an ordinal closed under the boolean and topological derivative operations. The algebras we define are based on the notion of a periodic set of ordinals generalizing that of a periodic set of natural numbers.

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Provability Logic in Fragments of HA
A First Look

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Keywords:
Intuitionistic Logic, Heyting Arithmetic, Provability Logic.

In [1], Ardeshir and Mojtahedi, calculated the Σ_1-provability logic of Heyting arithmetic (HA), the intuitionistic version of PA. In [2], Visser and Zoethout have given an alternative way of characterizing the Σ_1-provability of HA. This alternative method resembles Solovay's original proof for PA.

We try to apply the method of [2] to subtheories of HA. The method can be divided in two parts: one related to Solovay's construction in the intuitionistic setting and one related to the NNIL algorithm. We found that one of the relevant properties needed for the construction is that the theory proves some degree of sentential reflection with provability predicates of its finite subtheories. This makes the tools hard to apply to fragments of HA. However, we are able to provide a natural fragment for which the whole method is applicable: $\mathsf{iI}\Sigma_1+$ sentential reflection for the provability predicate of $\mathsf{iI}\Sigma_1$.

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References

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[2] Albert Visser and Jetze Zoethout, *Provability logic and the completeness principle*, Annals of Pure and Applied Logic, 170(6):718-753, 2019.