PRINCIPLES OF MATHEMATICAL MODELLING

Problem

Initial growth of population is described by the model:

$$N(t) = 15.2\,\mathrm{e}^{1.1t}\,.$$

Determine a doubling time of the population.

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Solution.

$$T_d = \frac{\ln 2}{\alpha} = \frac{\ln 2}{1.1} = 0.63$$

Problem

In the modelling of a radioactive decay, exponential model easily follows from the assumption that each atom has the same probability of decay. In this case, instead of doubling time we are dealing with half life.
After 500 years, sample of radium-226 decreases to 80.4% of its initial mass. Determine the half life for radium-226.

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Solution.

$$N(t) = N_0\,\mathrm{e}^{-\alpha t}$$

$$N(t) = N_0\,\mathrm{e}^{-\alpha t} = 0.804 N_0 \quad \Rightarrow$$

$$\alpha = -\frac{\ln 0.804}{t} = -\frac{\ln 0.804}{500} = 0.000436312019606341$$

$$T_{1/2} = \frac{\ln 2}{\alpha} = \frac{\ln 2}{0.000436312019606341} = 1588,65020767782$$

Problem

Solve differential equations

1. $y' = y + 1$
2. $y' = y^2\,\mathrm{e}^t,\ \ y(0) = 1$
3. $y' = y^2 t$
4. $y' = 2y - 2y^2$
5. $y' = y - y \ln y$
6. $y' = y^{2/3} - y$

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$$y' = y + 1 \quad \Leftrightarrow \quad \frac{y'}{y+1} = 1 \quad \Leftrightarrow \quad \int \frac{dy}{y+1} = \int dt \quad \Leftrightarrow$$

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$$\ln(y+1) = t + C \quad \Leftrightarrow \quad y + 1 = e^{t+C} \quad \Leftrightarrow \quad y = Ce^t - 1$$

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$$y' = y^2\,\mathrm{e}^t \quad \Leftrightarrow \quad \frac{y'}{y^2} = \mathrm{e}^t \quad \Leftrightarrow \quad \int \frac{dy}{y^2} = \int \mathrm{e}^t\,dt \quad \Leftrightarrow$$

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$$-\frac{1}{y} = e^t + C \quad \Leftrightarrow \quad y = \frac{-1}{e^t + C}$$

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$$-\frac{1}{y} = \frac{t^2}{2} + C \quad \Leftrightarrow \quad y = \frac{-1}{\frac{t^2}{2} + C} = \frac{-2}{t^2 + C} = \frac{2}{C - t^2}$$

Problem

Solve the initial value problem

$$y' = 2y - y^2, \quad y(0) = 0.1$$

using Euler's method, on the interval $[0, 3]$.
Compare obtained approximation with the exact solution.

Plot approximation and exact solution.

More. Describe a behaviour of the error when the number of subintervals increases.