PRINCIPLES OF MATHEMATICAL MODELLING

4. ANALYSIS OF SYSTEMS OF DIFFERENTIAL EQUATIONS

4.1. System of differential equations

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Chemostat model is an example for system of differential equations:

$$\begin{aligned} S' &= -V\frac{S}{K+S}\frac{P}{Y} + \omega\, S_0 - \omega\, S, \qquad & S(0) = s_0 \\ P' &= V\frac{S}{K+S}P - \omega\, P, & P(0) = p_0 \end{aligned}$$

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$$X(t) = \begin{bmatrix} S(t) \\ P(t) \end{bmatrix}, \quad X : \mathbb{R} \to \mathbb{R}^2$$

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X - vector function

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For

$$F(X) = \begin{bmatrix} -V\dfrac{S}{K+S}\dfrac{P}{Y} + \omega\, S_0 - \omega\, S \\ V\dfrac{S}{K+S}P - \omega\, P \end{bmatrix} \quad \text{and} \quad X_0 = \begin{bmatrix} s_0 \\ p_0 \end{bmatrix},$$

vector function

$$X(t) = \begin{bmatrix} S(t) \\ P(t) \end{bmatrix}$$

is a solution of the differential equation

$$X'(t) = F(X(t)), \quad X(0) = X_0.$$

Generally, system of differential equations

$$
\begin{aligned}
y_1' &= f_1(y_1,\dots,y_n), \quad y_1(0) = y_1^0\\
y_2' &= f_2(y_1,\dots,y_n), \quad y_2(0) = y_2^0\\
&\vdots\\
y_n' &= f_n(y_1,\dots,y_n), \quad y_n(0) = y_n^0
\end{aligned}
$$

may be written in a vector form.

$$Y'(t) = F(Y(t)), \quad Y(0) = Y_0,$$

where

$$Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, \quad F(Y) = \begin{bmatrix} f_1(y_1,\dots,y_n) \\ \vdots \\ f_n(y_1,\dots,y_n) \end{bmatrix} \quad \text{i} \quad Y_0 = \begin{bmatrix} y_1^0 \\ \vdots \\ y_n^0 \end{bmatrix},$$

4.3. Linear system of differential equations

Diiferential equation

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for $A \in M_n(\mathbb{R})$ is called a linear system of differential equations.

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-Otherwise, nonlinear system of differential equations.

$$\begin{aligned}
x_1'(t) &= a_{11}x_1(t) + a_{12}x_2(t) + \ldots + a_{1n}x_n(t) \\
x_2'(t) &= a_{21}x_1(t) + a_{22}x_2(t) + \ldots + a_{2n}x_n(t) \\
\vdots \quad & \quad \vdots \\
x_n'(t) &= a_{n1}x_1(t) + a_{n2}x_2(t) + \ldots + a_{nn}x_n(t)
\end{aligned}$$

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Scalar λ is an eigenvalue of matrix $A \in M_n(\mathbb{R})$ if there exists $x \neq 0$ such that

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$\rightarrow \quad \lambda$ is zero (root) of characteristic polynomial (characteristic root).

Example

Find eigenvalues and eigenvectors of matrix

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}.$$

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$$p(\lambda) = \begin{vmatrix} 3-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix} = (3-\lambda)(4-\lambda) - 1 = \lambda^2 - 7\,\lambda + 11$$

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$$\lambda_{1,2} = \frac{7 \pm \sqrt{49 - 4 \cdot 11}}{2} = \frac{7 \pm \sqrt{5}}{2}$$

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$$\lambda_1 = \frac{7+\sqrt{5}}{2}, \quad \lambda_2 = \frac{7-\sqrt{5}}{2}$$

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Solve the system:

$$Ax = \lambda_1 x \quad \Leftrightarrow \quad (A - \lambda_1 I)x = 0$$

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Augmented matrix (last column i a zero-vector and we omitted it):

$$\begin{bmatrix} 3-\frac{7+\sqrt{5}}{2} & 1 \\ 1 & 4-\frac{7+\sqrt{5}}{2} \end{bmatrix} \sim$$

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$$(3-\lambda_2)x_1 + x_2 = 0 \quad \Rightarrow \quad x_2 = -(3-\lambda_2)x_1 = -\left(3 - \frac{7-\sqrt{5}}{2}\right)x_1$$

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$$A\,X_1 \quad = \quad \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix}$$

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$$\Rightarrow \quad A\, X_1 = \lambda_1\, X_1$$

Solution of differemtial equation $x' = A\,x$

Example

Solve differential equation $x' = A\,x$, $x(0) = x_0$ where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{i} \quad x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

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Each equation can be solved separatelly.

$$x_1' = x_1 \quad \Rightarrow \quad x_1(t) = c_1 \,\mathrm{e}^t$$

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Constants c_1 i c_2 are determined from the initial condition

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Theorem

Let matrix $A \in M_n(\mathbb{R})$ is similar to diagonal matrix. then a general solution of differential equation $x'(t) = A\,x$ is given by

$$x(t) = \sum_{i=1}^{n} c_i\, \mathrm{e}^{\lambda_i t}\, v_i$$

where λ_i are eigenvalues and v_i corresponding eigenvectors of matrix A ($A\,v_i = \lambda_i v_i$). Constants c_i are determined from initial conditions.

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e_i - vector of canonical basis

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Each equation may be solved separately (first solve second equation and after that solve first equation).

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Note. In the case of multiple eigenvalues,

we obtain terms $\mathrm{e}^{\lambda_i t}, t\,\mathrm{e}^{\lambda_i t}, t^2\,\mathrm{e}^{\lambda_i t}, \ldots$ in the solution.

Stability of the linear system of differential equations

Definition

Linear system of differential equations

$$X' = AX$$

where $A \in M_n(\mathbb{R})$, is said to be stable if any solution $X(t)$ satisfies

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Theorem

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Q.E.D.

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Viete's formulae $\Rightarrow$

$$\begin{aligned} \lambda_1 + \lambda_2 &= b = \operatorname{tr} A \\ \lambda_1\,\lambda_2 &= c = \det A \end{aligned}$$

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$$\lambda_1 < 0, \lambda_2 < 0 \quad \Rightarrow \quad \lambda_1 + \lambda_2 < 0 \quad \text{i} \quad \lambda_1\,\lambda_2 > 0$$

$\Leftarrow$. Let

$$\lambda_1 + \lambda_2 < 0 \quad \text{i} \quad \lambda_1\,\lambda_2 > 0.$$

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Since $\lambda_1 + \lambda_2 < 0 \Rightarrow \quad \lambda_1 < 0$ i $\lambda_2 < 0$.

2. $\lambda_1, \lambda_2 \in \mathbb{C}\backslash\mathbb{R}. \quad \Rightarrow \quad \lambda_1 = a + i\,c, \lambda_2 = a - i\,c \quad \Rightarrow$

$$\begin{aligned} \lambda_1\,\lambda_2 &= a^2 + b^2 > 0 \\ \lambda_1 + \lambda_2 &= 2a = 2\mathrm{Re}\lambda_i \end{aligned}$$

Theorem

For $A \in M_2(\mathbb{R})$, system of differential equations $x' = A\,x$ is stable $\Leftrightarrow$ $tr\,A < 0$ i $\det A > 0$

Proof.

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$$\lambda_1 + \lambda_2 < 0 \quad \Leftrightarrow \quad \mathrm{Re}\lambda_1 < 0 \quad \text{and} \quad \mathrm{Re}\lambda_2 < 0$$

4.3. Phase portrait

Consider differential equation

$$X(t)' = F(X(t)), \quad X : \mathbb{R} \to \mathbb{R}^2$$

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Phase portrait is obtained by displaying trajectories for several initial conditions.

Cartesian plane containing phase portrait is sometimes named phase plane.

Example

Sketch phase portrait of differential equation

$$x' = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x$$

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We have to plot several solutions (with different initial conditions).

Note that

$$x(t) = c_k \, e^{\lambda_k t} \, v_k, \quad k = 1, 2$$

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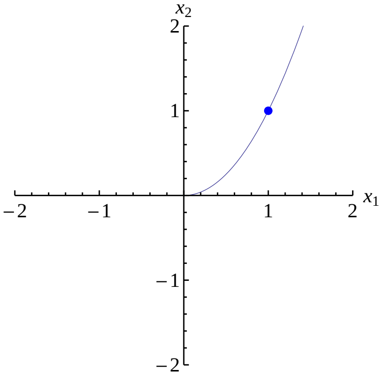
Trajectory for $x_0 = [1, 1]^T$:



In what direction solution goes?

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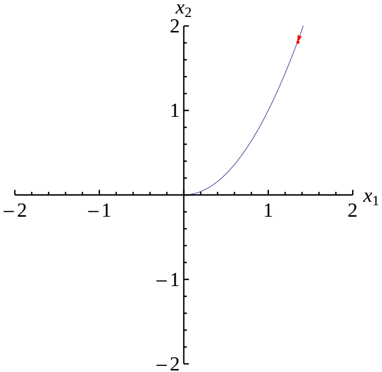
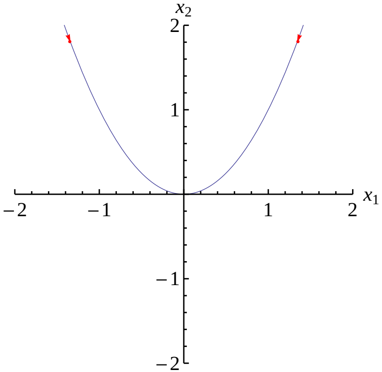
Direction in $[1, 1]$ is

$$\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

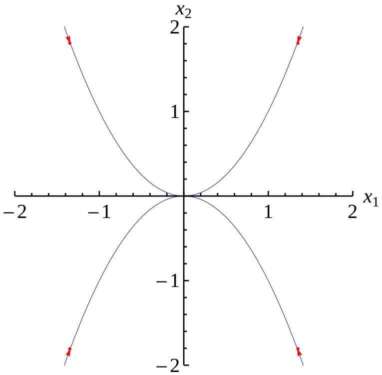
Trajectory for $x_0 = [1, 1]^T$:



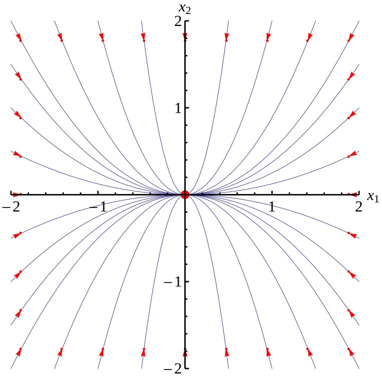
Immediately, we have another trajectory



and another two



Phase portrait:



Phase portrait for

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -5 \end{bmatrix}?$$

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We obtain solution of differential equation $x' = A\,x$ as before:

$$x(t) = \begin{bmatrix} c_1\,\mathrm{e}^{-t} \\ c_2\,\mathrm{e}^{-5t} \end{bmatrix}$$

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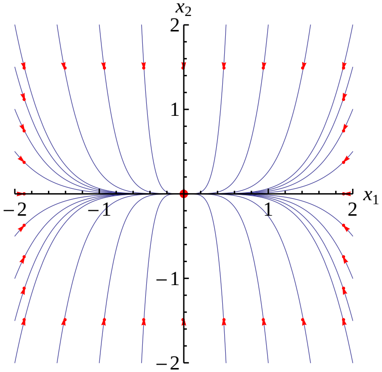
Trajectory is graph of function:

$$x_2 = x_1^5.$$

Phase portrait for $x' = \begin{bmatrix} -1 & 0 \\ 0 & -5 \end{bmatrix} x$



Phase portrait for

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i.e.

$$x_2 = \sqrt{x_1}.$$

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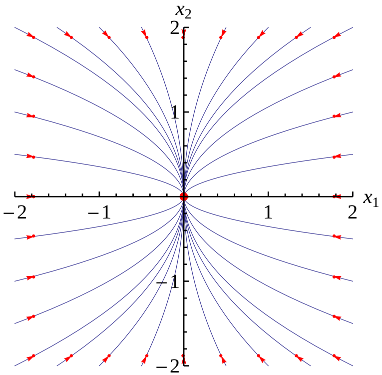
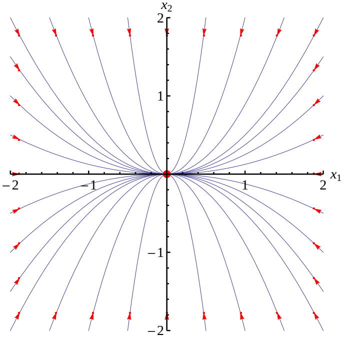
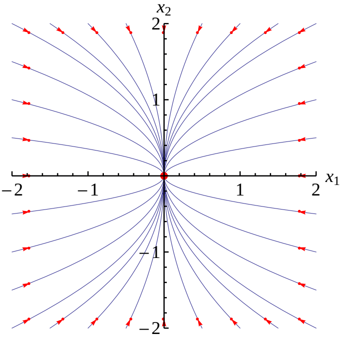


Phase portrait for

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \qquad A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$$





Parabola directed toward axis that corresponds to largest eigenvalue.

Phase portrait for

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}?$$

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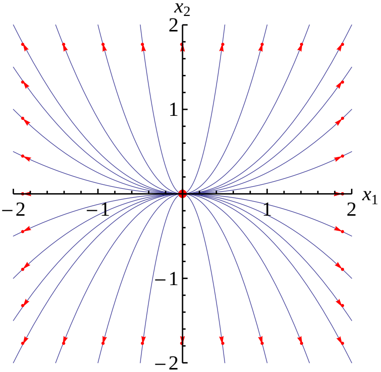
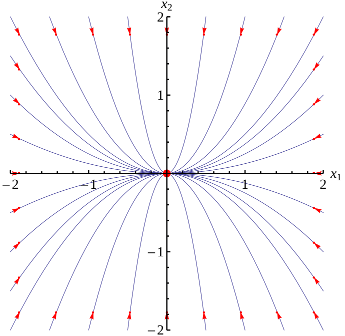
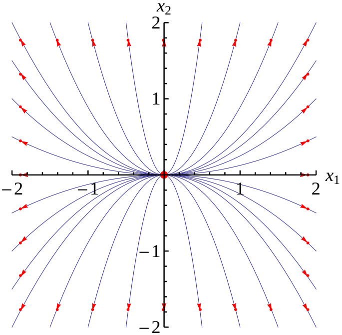
Phase portrait for $x' = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x$



Phase portrait for

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$





If eigenvalues are equal:

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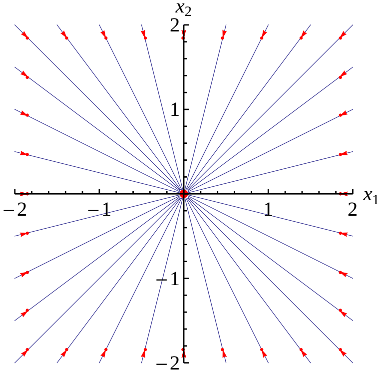
Phase portrait for $x' = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} x, \quad \lambda < 0$



Case when dimension of Jordan blok is 2×2

We consider case when

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}.$$

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From

$$x_2(t) = c_2\,\mathrm{e}^{-t}$$

it follows that

$$x_1(t) = c_1\,\mathrm{e}^{-t} + c_2 t\,\mathrm{e}^{-t} = \frac{c_1}{c_2} x_2(t) - x_2(t) \ln \frac{x_2(t)}{c_2}.$$

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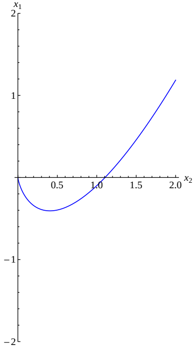
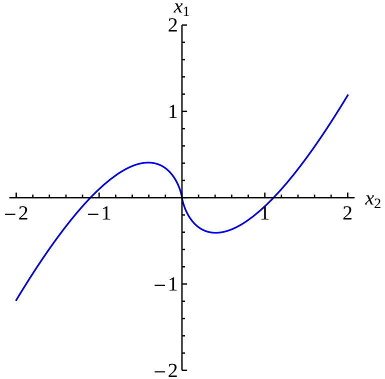
For $x_2(t) > 0$:

$$x_1 = \left(\frac{c_1}{c_2} - \ln c_2 \right) x_2 - x_2 \ln x_2 = c\,x_2 - x_2 \ln x_2.$$

Trajectory for $x_2 > 0$ and example of another trajectory for $x_2 < 0$:

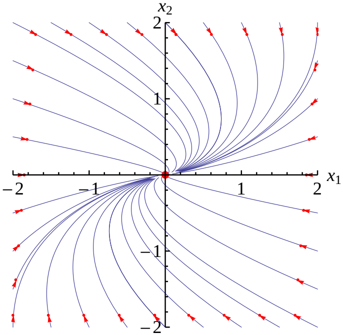


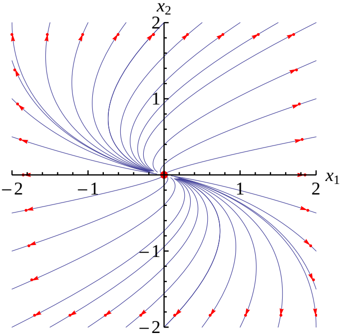


Phase portrait for

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$





Case when eigenvalues are of oposite sign

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- hyperbola

In general, for

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix},$$

$\lambda_1, \lambda_2 > 0$, solution of differential equation $x' = A\,x$ is:

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Trajectory:

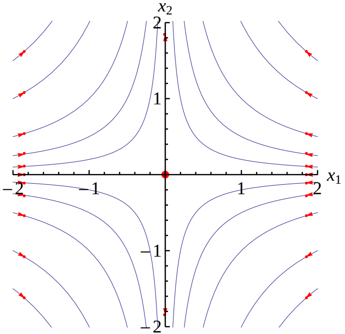
$$x_1^{\lambda_2} x_2^{\lambda_1} = c_1 c_2 = c$$

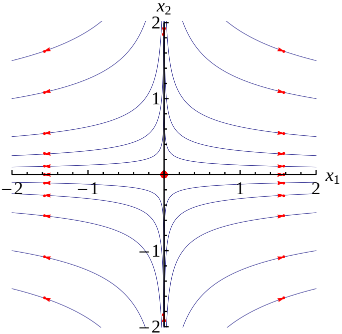
$$x_1 = \alpha\, x_2^{-\lambda_1/\lambda_2}$$

Phase portrait for

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$$





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```
{{-2(1+Sqrt[2]),2(-1+Sqrt[2])},{1,1}}
```

Eigenvalues:

$$\lambda_1 = \frac{-3-\sqrt{2}}{2}, \quad \lambda_2 = \frac{-3+\sqrt{2}}{2},$$

and eigenvectors:

$$v_1 = \begin{bmatrix} -2(1+\sqrt{2}) \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 2(-1+\sqrt{2}) \\ 1 \end{bmatrix}$$

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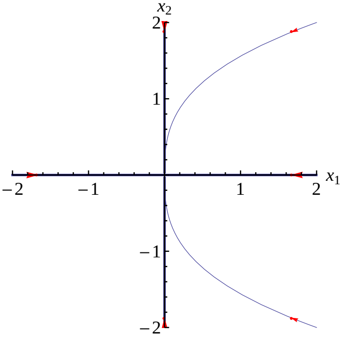
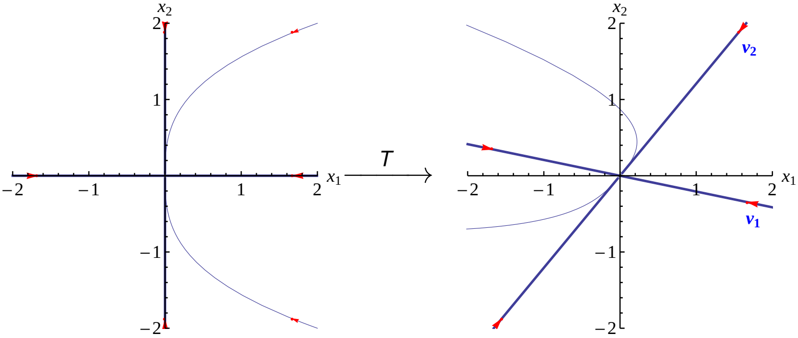
We consider differential equation $y' = D\, y$.

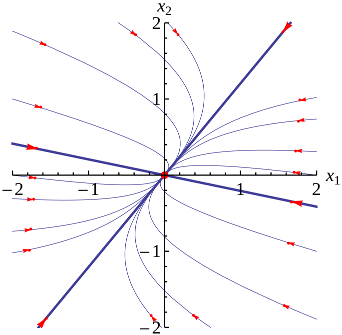
Trajectory for $y' = D\,y$:



Trajectory for $x' = A\,x, \quad x = T\,x$:



Phase portrait for $x' = \begin{bmatrix} -2 & 1 \\ \frac{1}{4} & -1 \end{bmatrix} x$:



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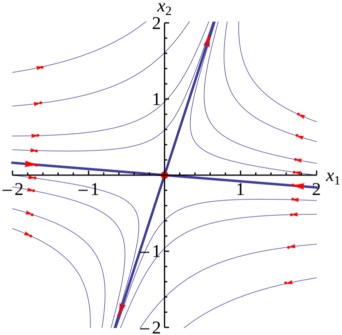

$\lambda_1 = 0,\ \lambda_2 \neq 0$

Equation:

$$x' = \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix} x$$

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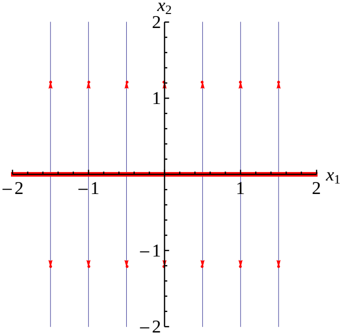

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For $x' = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} x$ solution is constant function $x(t) = c$. Therefore, each point is equilibrium.

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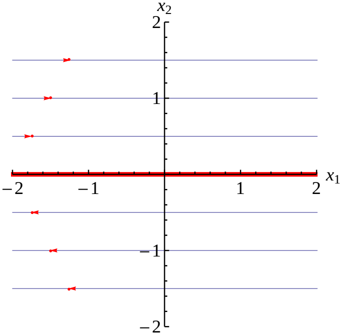
Phase portrait for $x' = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x$:



Complex eigenvalues.

Re$\lambda \neq 0$

Differential equation

$$x' = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} x$$

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$$\mathrm{e}^{\lambda_i t} = \mathrm{e}^{(a \pm i\,b)t} = \mathrm{e}^{a\,t}\,\mathrm{e}^{\pm i\,b\,t} = \mathrm{e}^{a\,t}(\cos b\,t \pm i \sin b\,t)$$

Complex eigenvalues and complex eigenvectors, but a solution is real.

...

Mathematica:

```
DSolve[{x'[t]==a x[t]+b y[t], y'[t] ==-b x[t]+a
y[t]},{x[t],y[t]},t]
```

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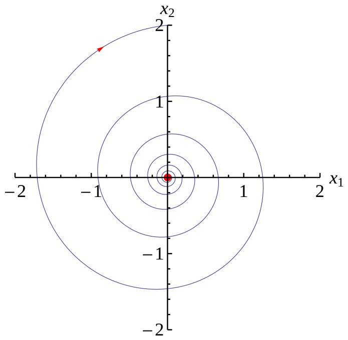
```
{{x[t]->E^(a t)C[1]Cos[b t+E^(a t)C[2]Sin[b t],
y[t]->E^(a t)C[2]Cos[b t]-E^(a t)C[1]Sin[b t]}}
```

$$\begin{aligned} x(t) &= \begin{bmatrix} c_1\, e^{a t} \cos b\, t + c_2\, e^{a t} \sin b\, t \\ c_2\, e^{a t} \cos b\, t - c_1\, e^{a t} \sin b\, t \end{bmatrix} \\ &= c_1\, e^{a t} \begin{bmatrix} \cos b\, t \\ -\sin b\, t \end{bmatrix} + c_2\, e^{a t} \begin{bmatrix} \sin b\, t \\ \cos b\, t \end{bmatrix} \end{aligned}$$

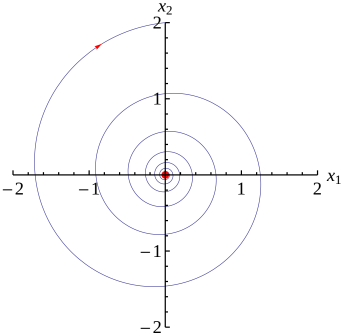
Trajectory for za $c_1 = 1$, $c_2 = 1$ and $A = \begin{bmatrix} 0.1 & 1 \\ -1 & 0.1 \end{bmatrix}$

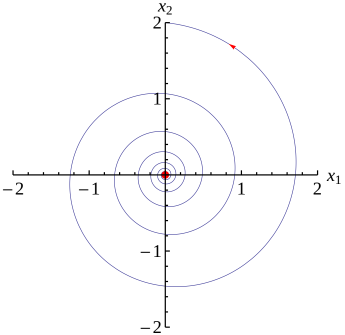


Spiral.

Phase portrait for

$$A = \begin{bmatrix} 0.1 & 1 \\ -1 & 0.1 \end{bmatrix} \qquad A = \begin{bmatrix} 0.1 & -1 \\ 1 & 0.1 \end{bmatrix}$$





$\mathbf{Re}\lambda = 0$

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$a = 0 \quad \Rightarrow$

$$x(t) = c_1 \begin{bmatrix} \cos b\,t \\ -\sin b\,t \end{bmatrix} + c_2 \begin{bmatrix} \sin b\,t \\ \cos b\,t \end{bmatrix}$$

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$$\begin{aligned} x_1(t)^2 &= c_1^2 \cos^2 b\,t + c_1 c_2 \cos b\,t \sin b\,t + c_2^2 \sin 2b\,t \\ x_2(t)^2 &= c_1^2 \sin^2 b\,t - c_1 c_2 \sin b\,t \cos b\,t + c_2^2 \cos 2b\,t \end{aligned}$$

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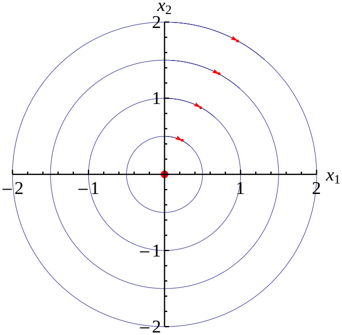
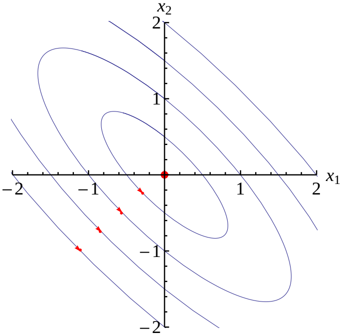
Phase portrait for $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$



Phase portrait for $B = T^{-1} A\, T = \begin{bmatrix} -\frac{4}{3} & -\frac{5}{3} \\ \frac{5}{3} & \frac{4}{3} \end{bmatrix}$

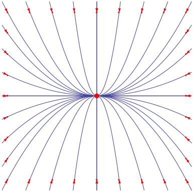
$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$



Real eigenvalues

Jordan block 2×2

$\lambda_2 < \lambda_1 < 0$



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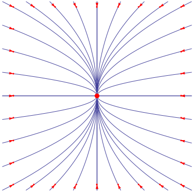

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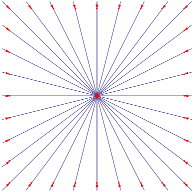

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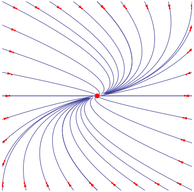

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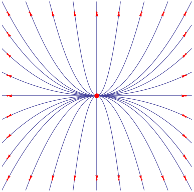

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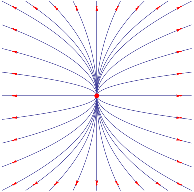

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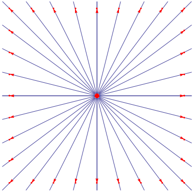

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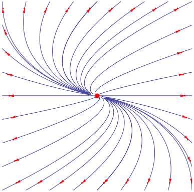

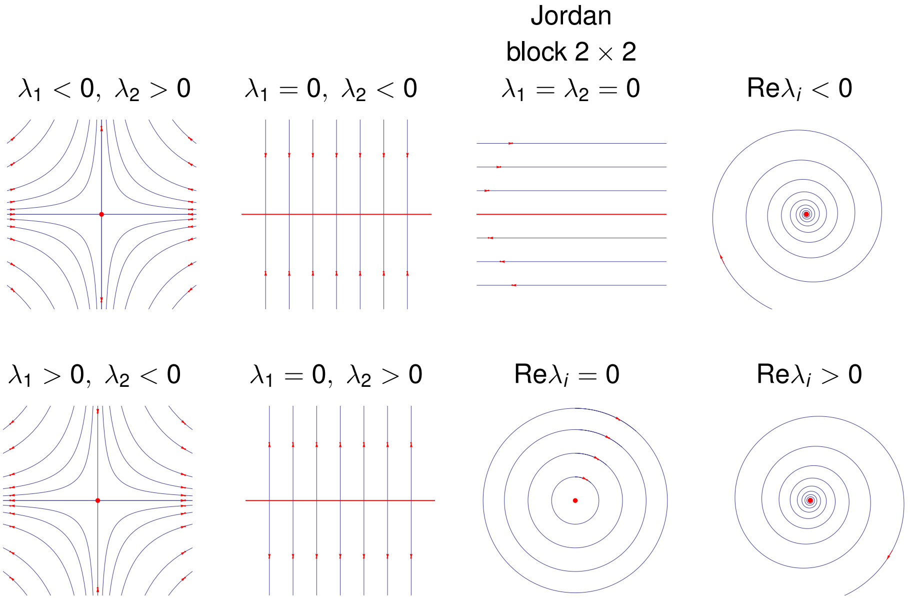
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4.2. Linearization

Consider differential equation

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$J = J_F$ is Jacobian matrix

Example

Determine Jacobian matrix for function F from chemostat model.

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$$\begin{aligned} f_1(S,P) &= -V\frac{S}{K+S}\frac{P}{Y} + \omega S_0 - \omega S \\ f_2(S,P) &= V\frac{S}{K+S}P - \omega P \end{aligned}$$

$$\frac{\partial f_1}{\partial S} = \frac{\partial}{\partial S}\left[-V\frac{S}{K+S}\frac{P}{Y} + \omega\, S_0 - \omega\, S\right]$$

$$
\begin{aligned}
\frac{\partial f_1}{\partial S} &= \frac{\partial}{\partial S}\left[-V\frac{S}{K+S}\frac{P}{Y}+\omega\,S_0-\omega\,S\right] \\
&= -\frac{V\,K}{(K+S)^2}\frac{P}{Y}-\omega
\end{aligned}
$$

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$$J_F(S, P) = \begin{bmatrix} \dfrac{\partial f_1}{\partial S} & \dfrac{\partial f_1}{\partial P} \\ \dfrac{\partial f_2}{\partial S} & \dfrac{\partial f_2}{\partial P} \end{bmatrix}$$

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Differential equation is similar to the equation for exponential model, only, $J_f(X^*)$ is (constant) matrix.

Note. Hartman-Grobman theorem justifies linearization.
Theorem shows that a solution of nonlinear differential equation

$$X' = F(X)$$

in the neighborhood of equilibrium point X^* qualitatively behaves as a solution of linear differential equation

$$X' = F'(X^*)X$$

in the neighborhood of point $X = 0$.

Hartman-Grobman theorem.

Theorem (Hartman-Grobman Theorem)

If x^* is a **hyperbolic** *equilibrium of $x' = f(x)$, $x \in \mathbb{R}^n$, then there exists a* **homeomorphism** *$z = h(x)$ defined in a neighborhood of x^* that maps trajectories of $x' = f(x)$ to those of $z' = Az$ where $A = J_f(x^*)$.*

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homeomorphism - a continuous map with a continuous inverse

Theorem

Let X^* is an equilibrium point of the system $X' = F(X)$ and all eigenvalues of $J_F(X^*)$ have nonzero real parts. Then, X^* is locally stable equilibrium if and only if all real parts of eigenvalues of the Jacobian matrix $J_F(X^*)$ are negative.

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Note. Case $\text{Re}\lambda_k = 0$ is complex and should be analyzed using some other approach.

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In the first case, all solutions converge toward 0 (unique equilibrium).

In the second case, 1 is another equilibrium and for $x_0 > 1$ solution wil not converge toward 0 (it will diverge to $+\infty$).

The hyperbolicity condition can't be removed.

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\begin{aligned}
X' &= -(X+Y)-(X-Y)\cdot(X^2+Y^2) \\
Y' &= -(X+Y)+(X-Y)\cdot(X^2+Y^2)
\end{aligned}
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$$\begin{aligned} X' &= -(X+Y) - (X-Y)\cdot(X^2+Y^2) \\ Y' &= -(X+Y) + (X-Y)\cdot(X^2+Y^2) \end{aligned}$$

Jacobian matrix at $(0,0)$:

$$J_F = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

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Eigenvalues: -2 an 0.

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Phase portrait.

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EXERCISES

We will use chemostat model in our examples:

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S' &= -V\frac{S}{K+S}\frac{P}{Y} + \omega\, S_0 - \omega\, S, \\
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Model has 5 parameters: V, K, Y, ω, S_0

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Model has 5 parameters: V, K, Y, ω, S_0

To make computation easier, we will use dedimensionalized model.

So,

Problem

Dedimensionalize chemostat model.

Hint. Introduce new variables:

$$P(t) = P^* N(\tau), \quad S(t) = S^* C(\tau), \quad t = t^* \tau$$

Constants P^*, S^*, t^* determine in the way to simplify the model (to reduce a number of parameters).

Solution.

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$$P'(t) = \frac{d}{dt}P(t) = \frac{d}{dt}P^* N(\tau)$$

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\begin{aligned}
P'(t) &= \frac{d}{dt}P(t) = \frac{d}{dt}P^* N(\tau) \\
&= P^* \frac{d}{dt} N\left(\frac{t}{t^*}\right)
\end{aligned}
$$

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P'(t) &= \frac{d}{dt}P(t) = \frac{d}{dt}P^* N(\tau) \\
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&=
\end{aligned}
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$$S'(t) =$$

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&= \frac{P^*}{t^*} N'(\tau) \\
S'(t) &= \frac{S^*}{t^*} C'(\tau)
\end{aligned}
$$

Model is of the form

$$\begin{aligned}
\frac{S^*}{t^*}C' &= -\frac{V\,S^*\,C}{K+S^*\,C}\frac{P^*\,N}{Y}+\omega\,S_0-\omega\,S^*\,C \\
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$\Rightarrow$

$$\begin{aligned}
C' &= -t^*\frac{V\,C}{K+S^*C}\frac{P^*N}{Y}+\frac{t^*\omega\,S_0}{S^*}-t^*\omega\,C\\
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C' &= -\frac{t^* V P^*}{S^* Y} \frac{C}{\frac{K}{S^*} + C} N + \frac{t^* \omega S_0}{S^*} - t^* \omega C \\
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Choose P^*, S^*, t^* to remove 3 parameters:

$$C' = -\frac{t^* V P^*}{S^* Y} \frac{C}{\frac{K}{S^*} + C} N + \frac{t^* \omega S_0}{S^*} - t^* \omega C$$

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Choose P^*, S^*, t^* to remove 3 parameters:

$$\frac{K}{S^*} = 1, \quad t^* \omega = 1, \quad \frac{t^* V P^*}{S^* Y} = 1$$

$$C' = -\frac{t^* V P^*}{S^* Y} \frac{C}{\frac{K}{S^*} + C} N + \frac{t^* \omega S_0}{S^*} - t^* \omega C$$

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Choose P^*, S^*, t^* to remove 3 parameters:

$$\frac{K}{S^*} = 1, \quad t^* \omega = 1, \quad \frac{t^* V P^*}{S^* Y} = 1$$

$$\Rightarrow \quad S^* = K, \quad t^* = \frac{1}{\omega}, \quad P^* = \frac{S^* Y}{t^* V} = \frac{Y K \omega}{V}$$

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$$\Rightarrow \quad S^* = K, \quad t^* = \frac{1}{\omega}, \quad P^* = \frac{S^* Y}{t^* V} = \frac{Y K \omega}{V}$$

Define new parameters:

$$\alpha_1 = t^* V = \frac{V}{\omega}, \quad \alpha_2 = \frac{t^* \omega S_0}{S^*} = \frac{S_0}{K}$$

Dedimensionalized chemostat model:

$$
\begin{aligned}
C' &= -\frac{C}{1+C}N + \alpha_2 - C \\
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Note. Substitution

$$
\Rightarrow \quad t^* = \frac{1}{V}, \quad S^* = t^* \omega S_0 P^* = \frac{Y K \omega}{V}
$$

also reduces number of parameters on 2.

Problem

Determine equilibrium points of chemostat model.
(Use dedimensionalized model.)

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$$\begin{aligned} C' &= -\frac{C}{1+C}N + \alpha_2 - C \\ N' &= \alpha_1 \frac{C}{1+C}N - N \end{aligned}$$

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From $F(C,N) = 0$ it follows

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$$N = 0 \quad \text{or} \quad \alpha_1 \frac{C}{1+C} = 0$$

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$$C = \alpha_2 \quad \Rightarrow \quad S = S_0.$$

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C and *N* are positive. What are conditions for the existence of positive equilibrium?

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Interpretation:

$$\alpha_1 - 1 > 0 \quad \Rightarrow \quad \frac{V}{\omega} > 1 \quad \Rightarrow \quad V > \omega$$

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Interpretation:

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Maximal growth rate should be larger then washout rate.

If washout rate is to high, loss of cells is greater then growth rate.

$$\alpha_2 - \frac{1}{\alpha_1 - 1} > 0$$

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Substrate concentration in the equilibrium:

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Substrate concentration in the equilibrium have to be smaller then inflowing substrate concentration.

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Stability of equilibrium points in chemostat model.

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$$\begin{aligned} \frac{\partial f_1}{\partial C} &= -N\frac{1}{(1+C)^2} - 1 \\ \frac{\partial f_1}{\partial N} &= -\frac{C}{1+C} \\ \frac{\partial f_2}{\partial C} &= \alpha_1 N \frac{1}{(1+C)^2} \\ \frac{\partial f_2}{\partial N} &= \alpha_1 \frac{C}{1+C} - 1 \end{aligned}$$

$$J_F = \begin{bmatrix} \frac{\partial f_1}{\partial C} & \frac{\partial f_1}{\partial N} \\ \frac{\partial f_2}{\partial C} & \frac{\partial f_2}{\partial N} \end{bmatrix} = \begin{bmatrix} -N\dfrac{1}{(1+C)^2} - 1 & -\dfrac{C}{1+C} \\ \alpha_1 N \dfrac{1}{(1+C)^2} & \alpha_1 \dfrac{C}{1+C} - 1 \end{bmatrix}$$

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Eigenvalues are on the diagonal! (Upper triangular matrix.)

$$\begin{aligned} \lambda_1 &= -1 < 0 \\ \lambda_2 &= \alpha_1 \frac{\alpha_2}{1+\alpha_2} - 1 \end{aligned}$$

$$\begin{aligned}
\lambda_2 &= \alpha_1 \frac{\alpha_2}{1+\alpha_2} - 1 \\
&= \frac{\alpha_1\alpha_2 - 1 - \alpha_2}{1+\alpha_2} \\
&= \frac{\alpha_2(\alpha_1 - 1) - 1}{1+\alpha_2} \\
&= \frac{\alpha_1 - 1}{1+\alpha_2}\left(\alpha_2 - \frac{1}{\alpha_1 - 1}\right)
\end{aligned}$$

$$\lambda_2 = \alpha_1 \frac{\alpha_2}{1+\alpha_2} - 1 = \frac{\alpha_1\alpha_2 - 1 - \alpha_2}{1+\alpha_2} = \frac{\alpha_2(\alpha_1 - 1) - 1}{1+\alpha_2} = \frac{\alpha_1 - 1}{1+\alpha_2}\left(\alpha_2 - \frac{1}{\alpha_1 - 1}\right)$$

If exists positive second equilibrium (X_2):

$$\alpha_1 - 1 \quad \text{i} \quad \alpha_2 - \frac{1}{\alpha_1 - 1} > 0$$

then

$$\lambda_2 > 0$$

and X_1 is not locally stable equilibrium.

2. equilibrium

$$X_2 = \left(\frac{1}{\alpha_1 - 1}, \alpha_1 \left(\alpha_2 - \frac{1}{\alpha_1 - 1} \right) \right)$$

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Existence of positive equilibrium$\Rightarrow$

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X_2 is locally stable equilibrium.

4.6. Phase portrait for chemostat model

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Equilibriums:

$$X_1 = (\alpha_2, 0), \quad X_2 = \left(\frac{1}{\alpha_1 - 1}, \alpha_1\left(\alpha_2 - \frac{1}{\alpha_1 - 1}\right)\right)$$

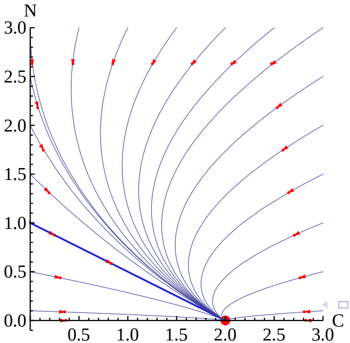
$$J_F(X_1) = J_F(\alpha_2, 0) = \begin{bmatrix} -1 & -\dfrac{\alpha_2}{1+\alpha_2} \\ 0 & \alpha_1 \dfrac{\alpha_2}{1+\alpha_2} - 1 \end{bmatrix}$$

1. One positive equilibrium

$$\alpha_1 - 1 < 0 \quad \text{or} \quad \alpha_2 - \frac{1}{\alpha_1 - 1} < 0$$

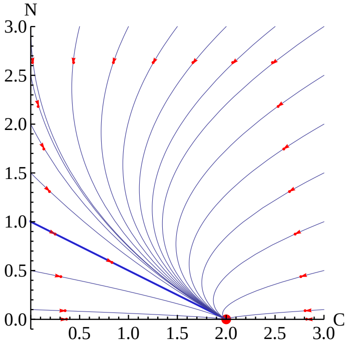
Example: $\alpha_1 = \frac{1}{2},\ \alpha_2 = 2$: $J_F(X_1) = \begin{bmatrix} -1 & -\frac{2}{3} \\ 0 & -\frac{2}{3} \end{bmatrix}$

Phase portrait of the linearized differential equatione:

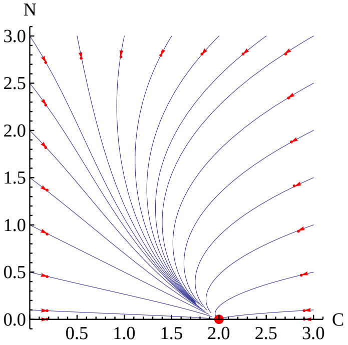


Phase portrait

Linearized equationea



Chemostat model



2. Two positive equilibriums

$$\alpha_1 - 1 > 0 \quad \text{and} \quad \alpha_2 - \frac{1}{\alpha_1 - 1} > 0$$

$$X_2 = \left(\frac{1}{\alpha_1 - 1}, \alpha_1 \left(\alpha_2 - \frac{1}{\alpha_1 - 1} \right) \right)$$

$$J_F(X_2) = \begin{bmatrix} -\left(N^* \frac{1}{(1 + C^*)^2} + 1 \right) & -\frac{C^*}{1 + C^*} \\ \alpha_1 N^* \frac{1}{(1 + C^*)^2} & 0 \end{bmatrix}$$

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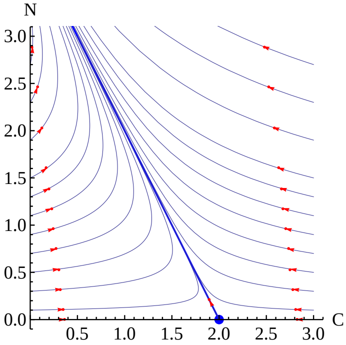
Example: $\alpha_1 = 2,\ \alpha_2 = 2$

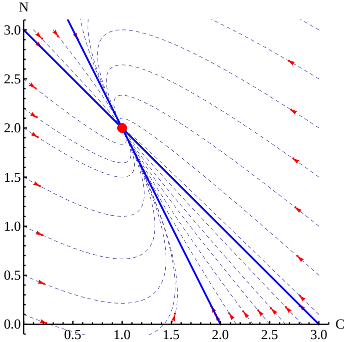
1. equilibrium: $X_1 = (2, 0)$, $J_F(X_1) = \begin{bmatrix} -1 & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix}$

Phase portrait of the linearized differential equatione:

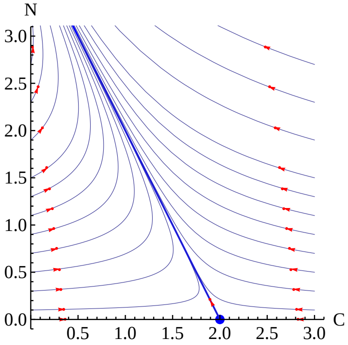


2. equilibrium: $X_2 = (1, 2)$, $F'(X_2) = \begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} \\ 1 & 0 \end{bmatrix}$

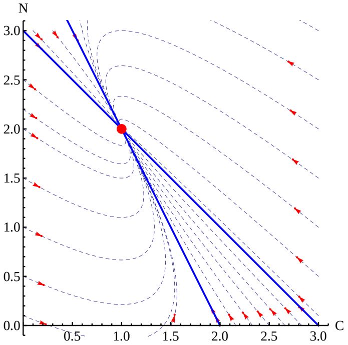
Phase portrait of the linearized differential equatione:

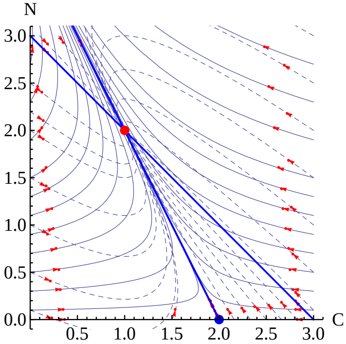


1. equilibrium

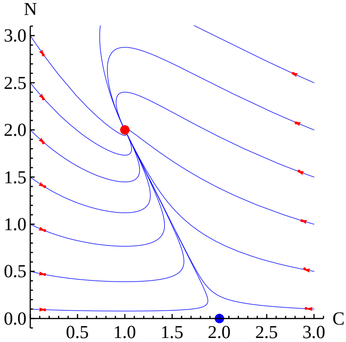


2. equilibrium




N
3.0
2.5
2.0
1.5
1.0
0.5
0.0
0.5
1.0
1.5
2.0
2.5
3.0
C

Phase portrait of the chemostat model:



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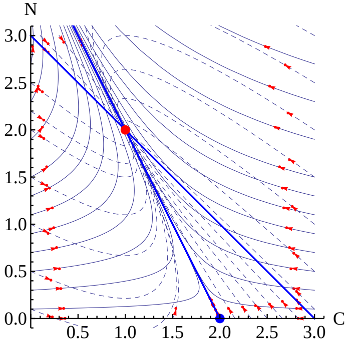

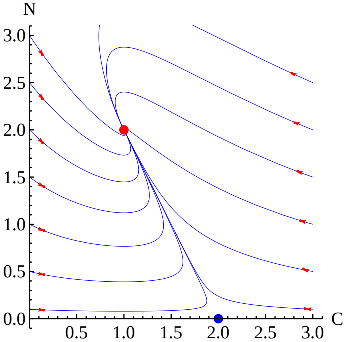

Problem

Dinamics of two populations is described by the system of differential equations:

$$\begin{aligned} x' &= x\,y - 2x - 2y + 4, \\ y' &= 4y - y^2 - x - 1. \end{aligned}$$

Sketch the phase portrait of the given differential equation.

Solution.

Equilibriums:

$$xy - 2x - 2y + 4 = 0$$

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1. $y = 2$

$$0 = 4y - y^2 - x - 1 = 3 - x$$

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$x=2$ or $y=2$.

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$$0=4y-y^2-x-1=3-x \quad\Rightarrow\quad x=3$$

Equilibrium: $E_1=(3,2)$

Solution.

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2. $x = 2$

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Solution.

Equilibriums:

$$xy - 2x - 2y + 4 = 0 \quad \Rightarrow \quad x(y-2) - 2(y-2) = (x-2)(y-2) = 0 \quad \Rightarrow$$

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2. $x = 2$

$$0 = 4y - y^2 - x - 1 = -y^2 + 4y - 3 \quad \Rightarrow \quad y_1 = 1, \quad y_2 = 3.$$

Equilibrium: $E_2 = (2, 1)$, $E_3 = (2, 3)$.

Jacobian matrix.

$$F(x, y) = \left[\begin{array}{c} x\,y - 2x - 2y + 4, \\ 4y - y^2 - x - 1. \end{array} \right]$$

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$$F(x,y) = \left[\begin{array}{c} x\,y - 2x - 2y + 4, \\ 4y - y^2 - x - 1. \end{array} \right]$$

$$J_F(x,y) = \left[\begin{array}{cc} y-2 & x-2 \\ -1 & 4-2y. \end{array} \right]$$

Jacobian matrix.

$$F(x,y) = \begin{bmatrix} x\,y - 2x - 2y + 4, \\ 4y - y^2 - x - 1. \end{bmatrix}$$

$$J_F(x,y) = \begin{bmatrix} y-2 & x-2 \\ -1 & 4-2y. \end{bmatrix}$$

1. Equilibrium

$$J_F(E_1) = J_F(3,2) = \begin{bmatrix} 0 & 1 \\ -1 & 0. \end{bmatrix}$$

Jacobian matrix.

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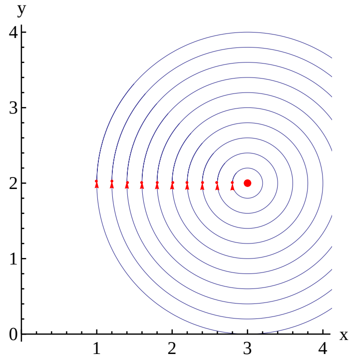
1. Equilibrium

$$J_F(E_1) = J_F(3,2) = \begin{bmatrix} 0 & 1 \\ -1 & 0. \end{bmatrix}$$

Circle!


y
4
3
2
1
0
1
2
3
4
x

2. Equilibrium

$$J_F(x,y) = \begin{bmatrix} y-2 & x-2 \\ -1 & 4-2y \end{bmatrix}$$

$$J_F(E_2) = J_F(2,3) = \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix}$$

2. Equilibrium

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$$J_F(E_2) = J_F(2,3) = \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix}$$

Saddle.

2. Equilibrium

$$J_F(x,y) = \begin{bmatrix} y-2 & x-2 \\ -1 & 4-2y \end{bmatrix}$$

$$J_F(E_2) = J_F(2,3) = \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix}$$

Saddle.

$$\lambda_2 = -2, \quad v_2 = e_2$$

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Saddle.

$$\lambda_2 = -2, \quad v_2 = e_2$$

$$J_F - \lambda_1 I = \begin{bmatrix} 0 & 0 \\ -1 & -3 \end{bmatrix}$$

2. Equilibrium

$$J_F(x, y) = \begin{bmatrix} y - 2 & x - 2 \\ -1 & 4 - 2y \end{bmatrix}$$

$$J_F(E_2) = J_F(2, 3) = \begin{bmatrix} 1 & 0 \\ -1 & -2 \end{bmatrix}$$

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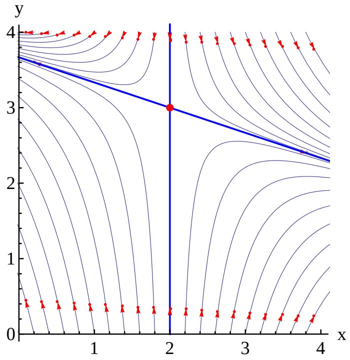
y
4
3
2
1
0
1
2
3
4
x

3. Equilibrium

$$J_F(x,y) = \begin{bmatrix} y-2 & x-2 \\ -1 & 4-2y \end{bmatrix}$$

$$J_F(E_3) = J_F(2,1) = \begin{bmatrix} -1 & 0 \\ -1 & 2 \end{bmatrix}$$

3. Equilibrium

$$J_F(x,y) = \begin{bmatrix} y-2 & x-2 \\ -1 & 4-2y \end{bmatrix}$$

$$J_F(E_3) = J_F(2,1) = \begin{bmatrix} -1 & 0 \\ -1 & 2 \end{bmatrix}$$

Saddle.

3. Equilibrium

$$J_F(x,y) = \begin{bmatrix} y-2 & x-2 \\ -1 & 4-2y \end{bmatrix}$$

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Saddle.

$$\lambda_2 = 2, \quad v_2 = e_2$$

3. Equilibrium

$$J_F(x,y) = \begin{bmatrix} y-2 & x-2 \\ -1 & 4-2y \end{bmatrix}$$

$$J_F(E_3) = J_F(2,1) = \begin{bmatrix} -1 & 0 \\ -1 & 2 \end{bmatrix}$$

Saddle.

$$\lambda_2 = 2, \quad v_2 = e_2$$

$$J_F - \lambda_1 I = \begin{bmatrix} 0 & 0 \\ -1 & 3 \end{bmatrix}$$

3. Equilibrium

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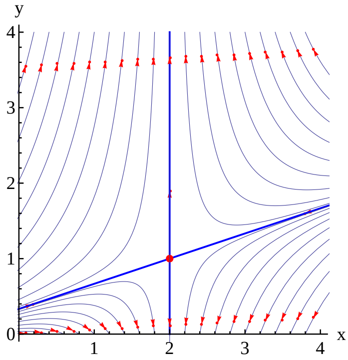
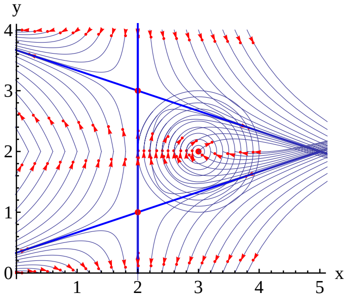
$$J_F - \lambda_1 I = \begin{bmatrix} 0 & 0 \\ -1 & 3 \end{bmatrix} \quad \Rightarrow \quad x - 1 = 3x_2 \quad \Rightarrow \quad v_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$


y
4
3
2
1
0
1
2
3
4
x

Sketch of the phase portrait



Phase portrait