On some inequalities for convex functions of higher order II

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Abstract

In this paper we prove some inequalities for convex function of a higher order. The well known Hermite interpolating polynomial leads us to a converse of Jensen inequality for a regular, signed measure and, as a consequence, a generalization of Hadamard and Petrović's inequalities. Also, we obtain a new upper bound for the error function of the Hermite interpolating polynomial $|e_H(x)|$ in terms of $\|f^{(n)}\|_\nu$, resulting in a generalization of Mahajani inequality for different norm ν.

Keywords: Jensen inequality, Hadamard inequality, Petrović inequality, Hermite's polynomial, (n)-convex function. *AMS Subject classification*:

1 Introduction

In [1] A.M.Fink has considered Fejer generalizations [9] of the left side Hadamard inequality involving convex functions of higher order. He has also proved the generalization of the right side this inequality:

Theorem A 1 *Let $f \in C^2[-1,1]$ be convex, μ a regular, non-negative measure on the collection of all Borel sets in $\mathbf{R}$, and $G(s,t)$ the homogeneous Green's function of the differential operator $\frac{d^2}{ds^2}$ on $[-1,1]$. If the function $s \to \int_{-1}^1 G(s,\cdot)d\mu$ is non-positive, then*

$$\int_{-1}^1 f\,d\mu \le P_0\frac{f(-1)+f(1)}{2} + P_1\frac{f(1)-f(-1)}{2}$$

where $P_k = \int_{-1}^1 x^k\,d\mu(x)$, $k \in \{0,1\}$.

The following Theorem corresponds to some conversions of the well known Jensen inequality for convex functions (see Lemma 1 in [4]):

Theorem A 2 *Let* $f : I = [m, M] \to \mathbf{R}$, $(-\infty < m < M < \infty)$ *be convex and* $g : [a, b] \to [m, M]$ *such that* $g \in L^1(\mu)$ *on* $[a, b]$ *for a non-negative measure* μ *satisfying* $\int_a^b d\mu = 1$. *Then*

$$\int_a^b f(g(t))d\mu(t) \le \frac{M - \int_a^b g(t)d\mu(t)}{M - m} f(m) + \frac{\int_a^b g(t)d\mu(t) - m}{M - m} f(M).$$

The following generalization of the well known Petrović inequality for convex functions [17] was given by F. Giaccardi [15] (see also [3], p. 153-154):

Theorem A 3 *Let* $\mathbf{p}$, $\mathbf{x} \in \mathbf{R^n}$ *satisfy*

$$(x_i - x_0)\left(\sum_{k=1}^n p_k x_k - x_i\right) \ \ge \ 0, \qquad i = 1, .., n, \tag{1}$$

where $x_0, \sum_{k=1}^n p_k x_k \in [a, b]$ $\sum_{k=1}^n p_k x_k \ne x_0$. *If* $f : [a, b] \to \mathbf{R}$ *is convex, then*

$$\sum_{k=1}^n p_k f(x_k) \le A f\left(\sum_{k=1}^n p_k x_k\right) + B\left(\sum_{k=1}^n p_k - 1\right) f(x_0), \tag{2}$$

where

$$A = \frac{\sum_{i=1}^n p_i(x_i - x_0)}{\sum_{i=1}^n p_i x_i - x_0}, \ \ B = \frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i x_i - x_0}.$$

G. S. Mahajani [16] proved the following results:

Theorem A 4 *Let* $f \in C^1[a, b]$ *be such that* $\int_a^b f(t)dt = 0$.

1^0 *If* $|f'(x)| \le M$ *then*

$$|\int_a^x f(t)dt| \le \frac{M(b-a)^2}{8}, \qquad \forall x \in [a, b]; \quad \textit{moreover,} \tag{3}$$

2^0 *if* (1^0) *holds and* $f(a) = f(b) = 0$, *then*

$$|\int_a^x f(t)dt| \le \frac{M(b-a)^2}{16}, \qquad \forall x \in [a, b]. \tag{4}$$

The notion of n-convexity goes back to Popoviciu [19]; we follow the definition given by Karlin [11]:

Definition 1 *A function* $f :\to \mathbf{R}$ *is said to be* (n)*-convex on* $[a, b]$, $n \ge 0$, *if for all choices of* $(n+1)$ *distinct points in* $[a, b]$, n^{th} *order divided difference of* f *satisfies*

$$[x_0, ..., x_n] f \ge 0.$$

In fact, Popoviciu proved that each continuous (n)-convex function on $[a,b]$ is the uniform limit of the sequence of (n)-convex polynomials. Many related results, as well as some important inequalities due to Favard, Berwald and Steffensen can be found in [12].

In this paper we give a proof of some inequalities for convex function of higher order $(n > 2)$. In Section 3 we prove a result of Fink generalization of Hadamard inequality for convex function of higher order, while in Section 4 we give a generalization of a converse of the Jensen inequality for regular, signed Borel measure μ, and also for a convex function of higher order. In Section 5 we obtain a generalization of Giaccardi inequality for $(2m)$-convex function, and in section 6 we give an upper bound for the error function of the Hermite interpolating polynomial in terms of $\| \, \|_\nu$, which implies a generalization of Mahajani inequality for a different ν.

2 Preliminaries

Let $-\infty < a < b < \infty$, and $a \le a_1 < a_2 \ldots < a_r \le b$, $(r \ge 2)$ be given. It is well known, that for $f \in C^n[a,b]$ a unique polynomial $P_H(t)$ of degree $(n-1)$ [10], [2] exists, such that

Hermite conditions:

$$P_H^{(i)}(a_j) = f^{(i)}(a_j); \; 0 \le i \le k_j, \; 1 \le j \le r, \; \sum_{j=1}^{r} k_j + r = n,$$

in particular:

Simple Hermite or Osculatory conditions: $(n = 2m,\ r = m,\ k_j = 1$ for all $j)$

$$P_O(a_j) = f(a_j), \; P_O'(a_j) = f'(b_j), \; 1 \le j \le m,$$

Lagrange conditions: $(r = n,\ k_j = 0$ for all $j)$

$$P_L(a_j) = f(a_j), \; 1 \le j \le n,$$

Type $(m, n-m)$ conditions: $(r = 2,\ 1 \le m \le n-1,\ k_1 = m-1,\ k_2 = n-m-1)$

$$\begin{aligned} P_{mn}^{(i)}(a) &= f^{(i)}(a_j), \; 0 \le i \le m-1 \\ P_{mn}^{(i)}(b) &= f^{(i)}(b_j), \; 0 \le i \le n-m-1, \end{aligned}$$

Two-point Taylor conditions: $(n = 2m,\ r = 2,\ k_1 = k_2 = m-1)$

$$P_{2T}^{(i)}(a) = f^{(i)}(a), \; P_{2T}^{(i)}(b) = f^{(i)}(b), \; 0 \le i \le m-1.$$

The associated error $|e_H(t)|$ can be represented in terms of the Green's function $G_H(t,s)$ for the multipoint boundary value problem $z^{(n)}(t) = 0, \; z^{(i)}(a_j) = 0 \quad 0 \le i \le k_j, \; 1 \le j \le r$, that is, the following result holds [2]:

Theorem A 5 *Let $F \in C^n[a,b]$, and let P_H be its Hermite inrepolating polynomial. Then*

$$\begin{aligned} F(t) &= P_H(t) + e_H(t) \\ &= \sum_{j=1}^{r}\sum_{i=0}^{k_j} H_{ij}(t) f^{(i)}(a_j) + \int_a^b G_H(t,s) F^{(n)}(s) ds, \end{aligned} \tag{5}$$

where H_{ij} are fundamental polynomials of the Hermite basis, defined by

$$H_{ij}(t) = \frac{1}{i!}\frac{\omega(t)}{(t-a_j)^{k_j+1-i}} \sum_{k=0}^{k_j-i} \frac{1}{k_j!}\left[\frac{(t-a_j)^{k_j+1}}{\omega(t)}\right]^{(k)}_{t=a_j} (t-a_j)^k, \tag{6}$$

where

$$\omega(t) = \prod_{j=1}^{r} (t-a_j)^{k_j+1}, \tag{7}$$

and G_H is the Green's function, defined by

$$G_H(t,s) = \begin{cases} \sum_{j=1}^{l}\sum_{i=0}^{k_j} \frac{(a_j-s)^{n-i-1}}{(n-i-1)!} H_{ij}(t), & s \le t \\ -\sum_{j=l+1}^{l}\sum_{i=0}^{k_j} \frac{(a_j-s)^{n-i-1}}{(n-i-1)!} H_{ij}(t), & s \ge t. \end{cases} \tag{8}$$

for all $a_l \le s \le a_{l+1}, l = 1, ..., r-1$.

We use the following lemma describing positivity properties of the Green's function (8), done by Levin [13] (i) and Beesack [14] (ii):

Lemma A 1 *The Green's function $G_H(t,s)$, has the following properties:*

$$(i)\ \frac{G_H(t,s)}{\omega(t)} > 0,\ a_1 \le t \le a_r,\ a_1 < s < a_r, \tag{9}$$

$$(ii)\ G_H(t,s) \le \frac{1}{(n-1)!(b-a)}|\omega(t)|, \tag{10}$$

$$(iii)\ \int_a^b |G_H(t,s)| ds = \frac{1}{n!}|\omega(t)|. \tag{11}$$

3 Generalization of Hadamard inequality

Theorem 1 *Let $f : [a,b] \to \mathbf{R}$, be (n)-convex function, let $-\infty \le a \le a_1 < a_2 ... < a_r \le b \le \infty$ be given, $(r \ge 2)$, $k_j \in N$, $j = 1, ...r$, $\sum_{j=1}^{r} k_j + r = n$, and μ a regular, signed measure on Borel sets.*

(I) If

$$\int_a^b G_H(s,t) d\mu(t) \le 0,\ \forall s \in [a,b], \tag{12}$$

then

$$\int_a^b f(t) d\mu(t) \le \sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i)}(a_j) \int_a^b H_{ij}(t) d\mu(t). \tag{13}$$

The reverse inequality in (12) implies the reverse inequality in (13).

(II) If μ is a positive measure, then

(i) if $\omega(t) > 0$, the reverse of the inequality in (13) holds;

(ii) (ii) if $\omega(t) < 0$, then inequality in (13) is always valid.

PROOF. By using (5), Theorem A5, (12), and a condition for the Green's function, we prove (I) as follows:

$$\begin{aligned}
\int_a^b f(t)d\mu(t) &= \int_a^b \sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i)}(a_j)H_{ij}(t)d\mu(t) \\
&+ \int_a^b (\int_a^b G_H(t,s)f^{(n)}(s)ds)d\mu(t) \\
&= \sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i)}(a_j)\int_a^b H_{ij}(t)d\mu(t) \\
&+ \int_a^b f^{(n)}(s)[\int_a^b G_H(s,t)d\mu(t)]ds \\
&\leq \sum_{j=1}^{r}\sum_{i=0}^{k_j} f^{(i)}(a_j)\int_a^b H_{ij}(t)d\mu(t)
\end{aligned}$$

By using (9) and lemma (i), we prove (II). ∎

For the other cases we have the following corrolaries:

Corollary 1 *(I) Let $f : [a,b] \to \mathbf{R}$, be (n)-convex, and $1 \leq m \leq n-1$ be fixed. If*

$$\int_a^b G_{mn}(s,t)d\mu(t) \leq 0,\ \forall s \in [a,b] \tag{14}$$

then

$$\int_a^b f(t)d\mu(t) \leq \sum_{i=0}^{m-1} f^{(i)}(a)\int_a^b \tau_i(t)d\mu(t) + \sum_{i=0}^{n-m-1} f^{(i)}(b)\int_a^b \nu_i(t)d\mu(t) \tag{15}$$

where

$$\tau_i(t) = \frac{1}{i!}(t-a)^i(\frac{t-b}{a-b})^{n-m}\sum_{k=0}^{m-1-i}\binom{n-m+k-1}{k}(\frac{t-a}{b-a})^k,. \tag{16}$$

$$\nu_i(t) = \frac{1}{i!}(t-b)^i(\frac{t-a}{b-a})^{n}\sum_{k=0}^{n-m-1-i}\binom{m+k-1}{k}(\frac{t-b}{a-b})^k. \tag{17}$$

The Green's function can be explicitly calculated:

$$G_{mn}(t,s) = \begin{cases} \sum_{i=0}^{m-1}[\sum_{p=0}^{m-1-i}\binom{n-m+p-1}{p}(\frac{t-a}{b-a})^p]\frac{(t-a)(a-s)^{n-i-1}}{i!(n-i-1)!}(\frac{b-t}{b-a})^{n-m}, & s \leq t \\ -\sum_{i=0}^{n-m-1}[\sum_{q=0}^{n-m-i-1}\binom{m+q-1}{q}(\frac{b-t}{b-a})^q]\frac{(t-b)(b-s)^{n-i-1}}{i!(n-i-1)!}(\frac{t-a}{b-a})^m, & s \geq t. \end{cases} \tag{18}$$

The reverse inequality in (14) implies the reverse inequality in (15).
(II) If μ is a positive measure then

(i) if m and n have the same parity, then the reverse of the inequality in (15) holds

(ii) if m and n have different parity, then inequality in (15) always holds.

PROOF. Statement (I) is a direct consequence of Theorem 1. Since the Green's function corresponding to the $(m, n-m)$ condition can be written as (18) (cf. [2]), (II) follows from the fact that $\omega(t) = (t-a)^m(t-b)^{n-m}$. ■

Corollary 2 *(I) Let $f \in C^{2m}[a,b]$, be $(2m)$-convex, and G_{2T} the Green's function of the two-point Taylor problem:*

$$G_{2T}(t,s) = \frac{(-1)^m}{(2m-1)!} \begin{cases} p^m(t,s) \sum_{j=0}^{m-1} \binom{m-1+j}{j} (t-s)^{m-1-j} q^j(t,s), & s \le t \\ q^m(t,s) \sum_{j=0}^{m-1} \binom{m-1+j}{j} (s-t)^{m-1-j} p^j(t,s), & s \ge t, \end{cases} \tag{19}$$

$$p(t,s) = \frac{(s-a)(b-t)}{b-a}, \quad q(t,s) = p(s,t), \; \forall t,s \in [a,b].$$

Suppose that that the measure μ is such that for all s

$$\int_a^b G_{2T}(s,t)d\mu(t) \le 0, \tag{20}$$

then

$$\begin{aligned} \int_a^b f(t)d\mu(t) \;&\le\; \sum_{i=0}^{m-1} \sum_{k=0}^{m-1-i} \binom{m+k-1}{k} [f^{(i)}(a) \int_a^b \frac{(t-a)^i}{i!} (\frac{t-b}{a-b})^m (\frac{t-a}{b-a})^k d\mu(t) \\ &+\; f^{(i)}(b) \int_a^b \frac{(t-b)^i}{i!} (\frac{t-a}{b-a})^m (\frac{t-b}{a-b})^k d\mu(t)] \end{aligned} \tag{21}$$

The reverse inequality in (20) implies the reverse inequality in (21).
(II) If μ is positive, then

(i) if m is even, then reverse of (21) holds;

(ii) if m is odd, (21) holds.

PROOF. By using Theorem 1 and the Green's function for the Two-point Taylor conditions [2], we first prove statement (I). Statement (II) follows from the fact that $\omega(t) = (t-a)^m(t-b)^m$. ■

Corollary 3 *(I) Let $f : [a,b] \to \mathbf{R}$, be (4)-convex and μ such that for all s*

$$\int_a^b G_O(s,t)d\mu(t) \le 0. \tag{22}$$

Then

$$\begin{aligned} \int_a^b f(t)d\mu(t) \;&\le\; \frac{f(a)}{(b-a)^2} \int_a^b [1 + \frac{2(t-a)}{b-a}](t-b)^2 d\mu(t) \\ &+\; \frac{f(b)}{(b-a)^2} \int_a^b [1 - \frac{2(t-b)}{b-a}](t-a)^2 d\mu(t) \\ &+\; \frac{f'(a)}{(b-a)^2} \int_a^b (t-a)(t-b)^2 d\mu(t) \\ &+\; \frac{f'(b)}{(b-a)^2} \int_a^b (t-b)(t-a)^2 d\mu(t), \end{aligned} \tag{23}$$

where

$$G_O(t,s)=\begin{cases}\frac{(a-s)^2}{3!}(\frac{b-t}{b-a})^2[(t-s)+2\frac{(t-a)(b-s)}{b-a}], & s\le t\\ \frac{(b-s)^2}{3!}(\frac{t-a}{b-a})^2[(s-t)+2\frac{(s-a)(b-t)}{b-a}], & s\ge t.\end{cases} \tag{24}$$

If the reverse inequality in (22) is valid, then the reverse inequality in (23) is also valid.
II. If μ is positive, then the reverse inequality in (23) holds.

PROOF. By using Theorem 1 and the Green's function for the Osculatory condition $n=4$ [2], we obtain (I). The second statement follows from the fact that for all $t\in[a,b]$ $\omega(t)=(t-a)^2(t-b)^2>0$. ∎

As the consequence of Theorem 1 for the Lagrange condition $n=2$, we obtain a generalization of the Fink inequality (1), (see also [18]).

Corollary 4 *(I) Let $f\in C^2[a,b]$ be convex and G_L the special Lagrange Greens's function defined by*

$$G_L(t,s)=\begin{cases}\frac{(a-s)(t-b)}{a-b}, & s\le t\\ \frac{-(b-s)(t-a)}{b-a,} & s\ge t.\end{cases} \tag{25}$$

If μ is such a measure that for all $s\in[a,b]$

$$\int_a^b G_L(s,t)d\mu(t)\le 0, \tag{26}$$

then

$$\begin{aligned}\int_a^b f(t)d\mu(t) &\le f(a)\int_a^b \frac{b-t}{b-a}d\mu(t)+f(b)\int_a^b\frac{t-a}{b-a}d\mu(t)\\ &= P_0[\frac{bf(b)-af(a)}{b-a}]+P_1[\frac{f(b)-f(a)}{b-a}],\end{aligned} \tag{27}$$

where $P_k=\int_a^b t^k\,d\mu(t)$, $k=0,1$. The reverse inequality in (26) implies the reverse inequality in (27).
(II) If μ is positive, then inequality in (27) holds.

PROOF. Statement (II) follows from the fact that $G_L(s,t)<0,\forall s,t\in[a,b]\times[a,b]$. ∎

4 Generalization of the conversion of the Jensen inequality

Theorem 2 *Let $f:[M_1,M_2]\to\mathbf{R}$ be (n)-convex and $m,n\in\mathbf{N}$ such that , $1\le m\le n-1$. Let $g:[a,b]\to[M_1,M_2]$ be integrable with respect to μ, If μ is such that*

$$\int_a^b G_{mn}(s,g(t))d\mu(t)\le 0,\qquad \forall s\in[M_1,M_2] \tag{28}$$

then

$$\int_a^b f(g(t))d\mu(t)\le\sum_{i=0}^{m-1}f^{(i)}(M_1)\int_a^b\tau_i(g(t))d\mu(t)+\sum_{i=0}^{n-m-1}f^{(i)}(M_2)\int_a^b\nu_i(g(t))d\mu(t) \tag{29}$$

where τ_i and ν_i are defined on $[M_1, M_2]$ by:

$$\begin{aligned}
\tau_i(x) &= \frac{1}{i!}(x-M_1)^i(\frac{x-M_2}{M_1-M_2})^{n-m}\sum_{k=0}^{m-1-i}\binom{n-m+k-1}{k}(\frac{x-M_1}{M_2-M_1})^k \\
\nu_i(x) &= \frac{1}{i!}(x-M_2)^i(\frac{x-M_1}{M_2-M_1})^{n}\sum_{k=0}^{n-m-1-i}\binom{m+k-1}{k}(\frac{x-M_2}{M_1-M_2})^k
\end{aligned}$$

If the reverse inequality in (28) is valid, then the reverse inequality in (29) holds.

PROOF. By using (5) and Theorem A 5, for $f(g(t))$ we have

$$f(g(t)) = \sum_{i=0}^{m-1} f^{(i)}(M_1)\tau_i(g(t)) + \sum_{i=0}^{n-m-1} f^{(i)}(M_2)\nu_i(g(t)) + \int_a^b G_{mn}(g(t),s)f^{(n)}(s)ds$$

By integrating and using the condition (28) we get the result (29). ■

Corollary 5 *Let $f(t) : [M_1, M_2] \to \mathbf{R}$ be $(2m)$-convex, $g : [a,b] \to [M_1, M_2]$ integrable with respect to μ,and μ such that*

$$\int_a^b G_{2T}(s, g(t))d\mu(t) \le 0, \qquad \forall s \in [M_1, M_2]. \tag{30}$$

Then

$$\begin{aligned}
\int_a^b f(g(t))d\mu(t) &\le \sum_{i=0}^{m-1}\sum_{k=0}^{m-1-i}\binom{m+k-1}{k} \\
&[\quad f^{(i)}(M_1)\int_a^b \frac{(g(t)-M_1)^i}{i!}(\frac{g(t)-M_2}{M_1-M_2})^m(\frac{g(t)-M_1}{M_2-M_1})^k d\mu(t) \\
&+ \quad f^{(i)}(M_2)\int_a^b \frac{(g(t)-M_2)^i}{i!}(\frac{g(t)-a}{M_2-M_1})^m(\frac{g(t)-M_2}{M_1-M_2})^k d\mu(t)]
\end{aligned} \tag{31}$$

where the Green's function G_2T is defined on $[M_1, M_2] \times [M_1, M_2]$. If the reverse inequality in (30) is valid, then the reverse inequality in (31) is also valid.

For the Lagrange condition $n = 2$, the following corrolary is a generalization of Theorem A2 for signed measure μ, (see also [18]).

Corollary 6 *Let $f(t)$ be convex on $[M_1, M_2]$ and $g : [a,b] \to [M_1, M_2]$ integrable with respect to μ. If μ is such that*

$$\int_a^b G_L(s, g(t))\, d\mu(t) \le 0, \qquad \forall s \in [M_1, M_2], \tag{32}$$

then

$$\int_a^b f(g(t))d\mu(t) \le f(M_1)\left(\frac{M_2 - \int_a^b g(t)d\mu(t)}{M_2 - M_1}\right) + f(M_2)\left(\frac{\int_a^b g(t)d\mu(t) - M_1}{M_2 - M_1}\right) \tag{33}$$

The reverse inequality in (32) implies the reverse inequality in (33).

By using Corrolary 6 we can obtain the results (see [18]) as in the following Corrolary:

Corollary 7 *Let the conditions of Corollary 6 be satisfied, and let J be an interval such that $J \supset f(I)$. If $\Phi(u,v)$ is a real function defined on $J \times J$, non-decreasing in u, then*

$$
\begin{aligned}
&\Phi[\int_a^b f(g(t))d\mu(t), f(\int_a^b g(t)d\mu(t))] \\
\leq\ & \max_{x\in[M_1,M_2]} \Phi\left[\frac{M_2-x}{M_2-M_1}f(M_1) + \frac{x-M_1}{M_2-M_1}f(M_2), f(x)\right] \qquad (34)\\
(=\ & \max_{\theta\in[0,1]} \Phi[\theta f(M_1) + (1-\theta)f(M_2), f(\theta M_1 + (1-\theta)M_2)]).
\end{aligned}
$$

The right-hand side of (34) is a non-decreasing function of M_2 and a non-increasing function of M_1.

5 Generalization of Giaccardi and Petrović inequality

In discrete case we also obtain some interesting inequalities for convex function of higher order, some being generalizations of the well known ones mentioned before.

Corollary 8 *Let $\mathbf{p}, \mathbf{x} \in \mathbf{R^n}$ be such that $x_0 \leq x_i \leq \sum_{k=1}^n p_k x_k \quad x_0, \sum_{k=1}^n p_k x_k \in [c,d]$ and $\sum_{k=1}^n p_k x_k \neq x_0$. If we choose $\mathbf{p} = (p_1, \ldots p_n)$ in such a way that*

$$\sum_{k=1}^n p_k G_{mN}(s, x_k) \leq 0, \qquad \forall s \in [c,d], \tag{35}$$

then for (N)-convex function $f : [c,d] \to \mathbf{R}$ and $1 \leq m \leq N-1$ we have the following inequality:

$$\sum_{i=1}^n p_i f(x_i) \quad \leq \quad \sum_{k=0}^{m-1} f^{(k)}(x_0) \sum_{i=1}^n p_i \tau_k(x_i) + \sum_{k=0}^{N-m-1} f^{(k)}(\sum_{j=1}^n p_j x_j) \sum_{i=1}^n p_i \nu_k(x_i). \tag{36}$$

The reverse inequality in (35) implies the reverse inequality in (36).

PROOF. We take a discrete measure μ in Theorem 2, and $g(t) = t$ for all $t \in [a,b]$. We can suppose that $[c,d] \supset [M_1, M_2]$, where M_1, M_2 are as in Theorem 2. For a specific choice of M_1 and M_2

$$M_1 = x_0,\ M_2 = \sum_{k=1}^n p_k x_k,$$

we have

$$x_0 \leq x_i \leq \sum_{k=1}^n p_k x_k.$$

■

In case of two-point Taylor conditions, we have the following generalization of Giaccardi inequality (2) for $(2m)$-convex function (see also [18]):

Corollary 9 *Let $\mathbf{p}, \mathbf{x} \in \mathbf{R}^n$ be such that*

$$(x_i - x_0)\left(\sum_{k=1}^{n} p_k x_k - x_i\right) \geq 0, i = 1,..,n \tag{37}$$

and $x_0, \sum_{k=1}^n p_k x_k \in [c,d]$, $\sum_{k=1}^n p_k x_k \neq x_0$. If

$$\sum_{k=1}^{n} p_k G_{2T}(s, x_k) \leq 0, \ \forall s \in [c,d], \tag{38}$$

then for a $(2m)$-convex function $f : [c,d] \to \mathbf{R}$ the following inequality holds:

$$\begin{aligned}\sum_{i=1}^{n} p_i f(x_i) \ \leq \ & \sum_{i=0}^{m-1}\sum_{k=0}^{m-1-i}\binom{m+k-1}{k} \\ [\ & f^{(i)}(x_0)\sum_{i=1}^{n} p_i \frac{(x_i-c)^i}{i!}\left(\frac{x_i-d}{c-d}\right)^m\left(\frac{x_i-c}{d-c}\right)^k \\ + \ & f^{(i)}\left(\sum_{j=1}^{n} p_j x_j\right)\sum_{i=1}^{n}\frac{(x_i-d)^i}{i!}\left(\frac{x_i-c}{d-c}\right)^m\left(\frac{x_i-d}{c-d}\right)^k].\end{aligned} \tag{39}$$

The reverse inequality in (38) also implies the reverse inequality in (39).

PROOF. As in the proof of Corrolary 8 we may suppose that $[c,d] \supset [M_1, M_2]$, and choose special M_1 and M_2 in Theorem 2, namely $M_1 = x_0, M_2 = \sum_{k=1}^n p_k x_k$, to obtain $x_0 \leq x_i \leq \sum_{k=1}^n p_k x_k$. For a different choice of M_1, M_2 ($M_2 = x_0, M_1 = \sum_{k=1}^n p_k x_k$) we have $\sum_{k=1}^n p_k x_k \leq x_i \leq x_0$, i.e the condition $(x_i - x_0)(\sum_{k=1}^n p_k x_k) \geq 0$ (37). ■

6 Generalization of Mahajani inequality

Theorem 3 *Let $f \in C^n[a,b]$, η and ν be conjugate exponents, and $1 \leq \eta \leq \infty$. Then*

$$|e_H(t)| \leq \frac{1}{[(n-1)!(b-a)]^{\frac{1}{\nu}}(n!)^{\frac{1}{\eta}}} \ |\omega(t)| \, \|f^{(n)}\|_\nu.$$

PROOF. The error function $e_H(t) = f(t) - P_H(t)$ has the following representations [2]:

$$e_H(t) = \int_a^b G_H(t,s) f^{(n)}(s) ds$$

then

$$\begin{aligned}|e_H(t)| \ &= \ |\int_a^b G_H(t,s) f^{(n)}(s) ds| \\ &\leq \ \int_a^b |G_H(t,s) f^{(n)}(s)| ds\end{aligned} \tag{40}$$

holds.

By using Hölder inequality in (40) , and (10) , (11) in Lema A we have

$$\begin{aligned}|e_H(t)| \ &\leq \ \left[\left(\frac{\omega(t)}{(n-1)!(b-a)}\right)^{\eta-1}\frac{1}{n!}\omega(t)\right]^{\frac{1}{\eta}} \|f^{(n)}\|_\nu \\ &\leq \ \frac{1}{[(n-1)!(b-a)]^{\frac{1}{\nu}}(n!)^{\frac{1}{\eta}}} \ |\omega(t)| \, \|f^{(n)}\|_\nu.\end{aligned}$$

■

As a consequence of Theorem 3 we have the error estimate as in Agarwal [2](p. 91).

Corollary 10 *Let $f \in C^n[a,b]$, η and ν be conjugate exponents , $1 \leq \eta \leq \infty$ and $p \geq 1$. Then*

$$\|e_H\|_p \leq \frac{1}{[(n-1)!(b-a)]^{\frac{1}{\nu}}(n!)^{\frac{1}{\eta}}}, \left(\int_a^b |\omega(t)|^p\right)^{\frac{1}{p}} \|f^{(n)}\|_\nu.$$

is valid.

Now we shall give a generalization of inequality by Mitrinović and Pečarić [5],[6], generalization of Theorem A4 1^o and some variant of Fink generalization of Mahajani's inequality [7],[6] :

Theorem 4 *(I) Let $f \in C^n[a,b]$, $\int_a^b f(t)dt = 0$, η and ν be conjugate exponents such that $1 \leq \eta \leq \infty$. Let $-\infty \leq a \leq a_1 < a_2 ... < a_r \leq b \leq \infty$ be the points such that $r \geq 2$, $k_j \in N$, $j = 1,...r$, $\sum_{j=1}^r k_j + r = n+1$, Then for all $x \in [a,b]$*

$$|\int_a^x f(t)dt - \sum_{j=1}^r \sum_{i=1}^{k_j} \tilde{H}_{ij}(x) f^{(i-1)}(a_j)| \leq \frac{1}{[n!(b-a)]^{\frac{1}{\nu}} \, [(n+1)!]^{\frac{1}{\eta}}} \, |\omega(x)| \, \|f^{(n)}\|_\nu \quad (41)$$

where $\tilde{H}_{ij}(x)$ are polynomials of the Hermite basis for $\int_a^x f(t)dt$.
(II) Also, if $f^{(i)}(a_j) = 0$, $1 \leq i \leq k_j - 1$, $j = 1,..r$, then for all $x \in [a,b]$

$$|\int_a^x f(t)dt| \leq \frac{1}{[n!(b-a)]^{\frac{1}{\nu}}} \frac{1}{[(n+1)!]^{\frac{1}{\eta}}} \, |\omega(x)| \, \|f^{(n)}\|_\nu \quad (42)$$

PROOF. Let $\tilde{P}_H$ be the Hermite interpolating polynomial for the primitive of the function f, and $\tilde{e}_H$ the associated error function. We can then use Theorem 3 for $\tilde{e}_H(x)$, to obtain (41). ■

Corollary 11 *Let $f \in C^n[a,b]$ such that $\int_a^b f(t)dt = 0$, and for fixed m, $1 \leq m \leq n$, holds $f^{(i)}(a) = 0$, $i = 0,...,m-2$, $f^{(i)}(b) = 0$, $i = 0,...,n-m-1$. Let η and ν be conjugate exponents such that $1 \leq \eta \leq \infty$. Then for all $x \in [a,b]$ holds*

$$\begin{aligned} |\int_a^x f(t)dt| &\leq \frac{1}{[n!(b-a)]^{\frac{1}{\nu}}} \frac{1}{[(n+1)!]^{\frac{1}{\eta}}} \, |(x-a)^m \, (x-b)^{n+1-m}| \, \|f^{(n)}\|_\nu \\ &\leq \frac{1}{[n!(b-a)]^{\frac{1}{\nu}}} \frac{(b-a)^{n+1}}{[(n+1)!]^{\frac{1}{\eta}}} \, \frac{m^m (n-m+1)^{n-m+1}}{(n+1)^{n+1}} \, \|f^{(n)}\|_\nu. \end{aligned}$$

Corollary 12 *Let $f \in C^{2m-1}[a,b]$, $\int_a^b f(t)dt = 0$, and holds $f^{(i)}(a) = f^{(i)}(b) = 0$, $i = 0,...,m-2$. Let η and ν be a conjugate exponents such that $1 \leq \eta \leq \infty$, then $\forall x \in [a,b]$*

$$\begin{aligned} |\int_a^x f(t)dt| &\leq \frac{1}{[(2m-1)!(b-a)]^{\frac{1}{\nu}}} \frac{1}{[(2m)!]^{\frac{1}{\eta}}} \, |(x-a)^m \, (x-b)^m| \, \|f^{(2m-1)}\|_\nu \\ &\leq \frac{1}{[(2m-1)!(b-a)]^{\frac{1}{\nu}}} \frac{1}{[(2m)!]^{\frac{1}{\eta}}} \, \frac{(b-a)^{2m}}{4^m} \, \|f^{(2m-1)}\|_\nu. \end{aligned}$$

Corollary 13 *Let $f \in C^1[a,b]$ and $\int_a^b f(t)dt = 0$. Let η and ν be conjugate exponents such that $1 \le \eta \le \infty$, then $\forall x \in [a,b]$*

$$|\int_a^x f(t)dt| \le \frac{(b-a)^{2-\frac{1}{\nu}}}{2^{2+\frac{1}{\eta}}} \|f'\|_\nu$$

Remark. For $\nu = \infty$ in above Corollaries we have generalization of Mahajani inequality 1^o for functions with bounded derivatives of a higher order $|f^{(n)}| \le M$ $(M \ge 0)$.
Remark. For $\nu = \infty$ and $n = 1$, $m = 1$ in the above Corollaries, we obtain Mahajani inequality 1^o (3).

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