ALGEBRAIC PROOF OF THE B–SPLINE DERIVATIVE FORMULA

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Abstract We prove a well known formula for the generalized derivatives of Chebyshev B–splines:

$$L_1 B_i^k(x) \;=\; \frac{B_i^{k-1}(x)}{C_{k-1}(i)} - \frac{B_{i+1}^{k-1}(x)}{C_{k-1}(i+1)},$$

where

$$C_{k-1}(i) \;=\; \int_{t_i}^{t_{i+k-1}} B_i^{k-1}(x)d\sigma, \tag{1}$$

in a purely algebraic fashion, and thus show that it holds for the most general spaces of splines. The integration is performed with respect to a certain measure associated in a natural way to the underlying Chebyshev system of functions. Next, we discuss the implications of the formula for some special spline spaces, with an emphasis on those that are not associated with ECC-systems.

Keywords: Chebyshev splines, divided differences

Introduction and preliminaries

The classic formula for the derivatives of polynomial B–splines

$$\frac{d}{dx}B_i^k(x) = (k-1)\,(\frac{B_i^{k-1}(x)}{t_{i+k-1}-t_i} - \frac{B_{i+1}^{k-1}(x)}{t_{i+k}-t_{i+1}}),$$

may be written in the form:

$$\frac{d}{dx}B_i^k(x) \;=\; \frac{B_i^{k-1}(x)}{C_{k-1}(i)} - \frac{B_{i+1}^{k-1}(x)}{C_{k-1}(i+1)},$$

where

$$C_{k-1}(i) \;=\; \int_{t_i}^{t_{i+k-1}} B_i^{k-1}(x)dx. \tag{2}$$

The same formula holds for Chebyshev splines if integration in (2) is performed with respect to a certain measure associated in a natural way to the underlying Chebyshev system of functions. In this way, we can define Chebyshev B-splines recursively, and inductively prove their propereties. To the best of our knowledge, the derivative formula for non–polynomial splines first appeared in [9] for one–weight Chebyshev systems. Later, special cases appear for various Chebyshev splines, like GB splines [4], tension splines [3], and Chebyshev polynomial splines [10]. The general version for Chebyshev splines, that appeared in [1], in the form of a defining recurrence relation for B-splines, is based on an indirect argument, relying on induction and uniqueness of Chebyshev B-splines. A direct proof, valid for CCC-systems and Lebesgue-Stieltjes measures, follows.

1. The derivative formula

We begin by introducing some new notation and restating some known facts, to make the proof of the derivative formula easier.

Let $\delta \subseteq [a, b]$ be measurable with respect to Lebesgue - Stieltjes measures $d\sigma_2, \ldots d\sigma_n$, and let $\boldsymbol{P}_{n-1}$ be $n-1 \times n-1$ permutation matrix, that we call *duality*:

$$(\boldsymbol{P}_{n-1})_{ij} := \delta_{i,n-j} \qquad i = 1, \ldots n-1; \quad j = 1, \ldots n-1.$$

We shall use the following notation:

measure vector :	$d\vec{\sigma} := (d\sigma_2(\delta), \ldots d\sigma_n(\delta))^T \in \mathbb{R}^{n-1}$,
reduced measure vectors:	$d\vec{\sigma}^{(j)} := (d\sigma_{j+2}, \ldots d\sigma_n)^{\mathrm{T}} \in \mathbb{R}^{n-j-1}$,
dual measure vector :	$\boldsymbol{P}_{n-1} d\vec{\sigma}$.

CCC-system $\mathcal{S}(n, d\vec{\sigma})$ of order n is a set of functions $L\{1, u_1, \ldots u_n\}$:

$$\begin{aligned} u_2(x) &= u_1(x) \int_a^x d\sigma_2(t_2), \\ &\vdots \\ u_n(x) &= u_1(x) \int_a^x d\sigma_2(t_2) \ldots \int_a^{t_{n-1}} d\sigma_n(t_n) \end{aligned}$$

(see [12] and references therein). If all of the measures $d\sigma_i$ are dominated by the Lebesgue measure, then they possess densities $\frac{1}{p_i}$, $i = 2, \ldots n$; if p_i are smooth, *i.e.* $\frac{1}{p_i} := \frac{d\sigma_i}{dt} \in C^{n-i+1}$, the functions form an *Extended Complete Chebyshev System (ECC-system)*. Reduction and duality define reduced, dual, and reduced dual Chebyshev systems as Chebyshev systems defined, respectively, by appropriate measure vectors:

j–reduced system:	$\mathcal{S}(n-j, d\vec{\sigma}^{(j)}) = \{u_{j,1}, \dots u_{j,n-j}\}$
dual system:	$\mathcal{S}(n, \boldsymbol{P}_{n-1} d\vec{\sigma}) = \{u_1^*, \dots u_j^*\}$
j–reduced dual system:	$\mathcal{S}(n-j, (\boldsymbol{P}_{n-1} d\vec{\sigma})^{(j)}) = \{u_{j,1}^*, \dots u_{j,n-j}^*\}$.

We define the *generalized derivatives* as linear operators mapping the Chebyshev space of functions spanned by $\mathcal{S}(n, d\vec{\sigma})$ to the one spanned by $\mathcal{S}(n-j, d\vec{\sigma}^{(j)})$ by $L_{j,d\vec{\sigma}} := D_j \cdots D_1$, where D_j are *measure derivatives*:

$$D_j f(x) \quad := \quad \lim_{\delta \to 0+} \frac{f(x+\delta) - f(x)}{d\sigma_{j+1}(x, x+\delta)}.$$

Generalized derivatives with respect to the dual measure vector are known as *dual generalized derivatives*. For example, if $n = 4$:

$\mathcal{S}(4, d\vec{\sigma}) = \{u_1, u_2, u_3, u_4\}$: 1 $\int_0^x d\sigma_2(t_2)$ $\int_0^x d\sigma_2(t_2) \int_0^x d\sigma_3(t_3)$ $\int_0^x d\sigma_2(t_2) \int_0^x d\sigma_3(t_3) \int_0^x d\sigma_4(t_4)$	$\mathcal{S}(4, \boldsymbol{P}_3 d\vec{\sigma}) = \{u_1^*, u_2^*, u_3^*, u_4^*\}$: 1 $\int_0^y d\sigma_4(t_4)$ $\int_0^y d\sigma_4(t_4) \int_0^{t_4} d\sigma_3(t_3)$ $\int_0^y d\sigma_4(t_4) \int_0^{t_4} d\sigma_3(t_3) \int_0^{t_3} d\sigma_2(t_2)$
$\mathcal{S}(3, d\vec{\sigma}^{(1)}) = \{u_{1,1}, u_{1,2}, u_{1,3}\}$: 1 $\int_0^x d\sigma_3(t_3)$ $\int_0^x d\sigma_3(t_3) \int_0^{t_3} d\sigma_4(t_4)$	$\mathcal{S}(3, (\boldsymbol{P}_3 d\vec{\sigma})^{(1)}) = \{u_{1,1}^*, u_{1,2}^*, u_{1,3}^*\}$: 1 $\int_0^y d\sigma_3(t_3)$ $\int_0^y d\sigma_3(t_3) \int_0^{t_3} d\sigma_2(t_2)$
$\mathcal{S}(2, d\vec{\sigma}^{(2)}) = \{u_{2,1}, u_{2,2}\}$: 1 $\int_0^x d\sigma_4(t_4)$	$\mathcal{S}(2, (\boldsymbol{P}_3 d\vec{\sigma})^{(2)}) = \{u_{2,1}^*, u_{2,2}^*\}$: 1 $\int_0^y d\sigma_2(t_2)$
$\mathcal{S}(1, d\vec{\sigma}^{(3)}) = \{u_{3,1}\}$: 1	$\mathcal{S}(1, (\boldsymbol{P}_3 d\vec{\sigma})^{(3)}) = \{u_{3,1}^*\}$: 1

Note that the dual of the reduced system is different from the reduced dual system, *i.e.:* $\boldsymbol{P}_{n-j-1}\, d\vec{\sigma}^{(j)} \neq (\boldsymbol{P}_{n-1} d\vec{\sigma})^{(j)}$.

The function $G_{n,d\vec{\sigma}}(x,y) : [a,b] \times [a,b] \to \mathbb{R}$ defined by

$$G_{n,d\vec{\sigma}}(x,y) := \begin{cases} \int_y^x d\sigma_2 \int_y^{s_2} \dots \int_y^{s_{n-1}} d\sigma_n & x \geq y, \\ 0 & \text{otherwise} \end{cases}$$

is called the *Green's function* with respect to $d\vec{\sigma}$. It follows easily that

$$L_{i,d\vec{\sigma}} G_{n,d\vec{\sigma}}(x,\cdot) = G_{n-i,d\vec{\sigma}^{(i)}}(x,\cdot), \quad \text{for} \quad i = 1,\dots,n-1. \tag{3}$$

We shall say that $\Delta = \{x_0,\dots,x_{k+1}\}$, $a \le x_i \le b$ is the *knot sequence* if

$$a = x_0 < x_1 < x_2 < \dots x_k < x_{k+1} = b$$

and $\vec{m} = (n_1,\dots,n_k)^T$ is the *multiplicity vector* if n_i are integers, and $1 \le n_i \le n$. $\{t_1 \dots t_{2n+k}\}$ is an *extended partition* if

$$\begin{aligned} t_1 = \dots = t_n &= a \\ t_{n+k+1} = \dots = t_{2n+k} &= b \end{aligned}$$

$$t_{n+1} \le \dots \le t_{n+k} = \underbrace{x_1,\dots,x_1}_{n_1},\dots,\underbrace{x_k,\dots,x_k}_{n_k}.$$

The space of Chebyshev splines of order n associated with the knot sequence Δ and vectors $\vec{m}$ and $d\vec{\sigma}$ is denoted as $\mathcal{S}(n,\vec{m},d\vec{\sigma},\Delta)$. Next, in order to define divided differences, we need to extend the Chebyshev systems by one extra function, and that means involving an additional artificial measure. To this end, let us define the *extension operator*

$$\boldsymbol{E}_i = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & \dots & 1 \\ 0 & \dots & 0 \end{pmatrix} \quad \boldsymbol{E}_i : \mathbb{R}^i \to \mathbb{R}^{i+1},$$

and the *extended measure vector*:

$$d\vec{\kappa} = (d\sigma_2,\dots d\sigma_n, d\lambda)^{\mathrm{T}} = \boldsymbol{E}_{n-1} d\vec{\sigma} + \vec{e}_n d\lambda$$

where $\vec{e}_n = [0\dots0,1]^T \in \mathbb{R}^n$, and $d\lambda$ is the artificial measure (usually taken to be the Lebesgue one). The *Chebyshev divided difference* of order n is then

$$[t_1,\dots t_{n+1}]_{\mathcal{S}(n+1,d\vec{\kappa})} f = \frac{D\begin{pmatrix} t_1,\dots,t_{n+1} \\ u_1,\dots,u_n,f \end{pmatrix}}{D\begin{pmatrix} t_1,\dots,t_{n+1} \\ u_1,\dots,u_{n+1} \end{pmatrix}}.$$

For definition of the determinants defining the divided differences, see [12]. The important thing is the *anihilation property*, which we quote for the sake of notation purposes:

$$[t_1,\dots,t_{n+1}]_{\mathcal{S}(n+1,d\vec{\kappa})} u = 0 \qquad \forall u \in \mathcal{S}(n,d\vec{\sigma}).$$

In this notation, divided differences satisfy the Mülbach's recurrence [5]:

$$[t_1, \dots, t_{n+1}]_{\mathcal{S}(n+1,d\vec{\kappa})} f = \frac{[t_2, \dots, t_{n+1}]_{\mathcal{S}(n,d\vec{\sigma})} f - [t_1, \dots, t_n]_{\mathcal{S}(n,d\vec{\sigma})} f}{[t_2, \dots, t_{n+1}]_{\mathcal{S}(n,d\vec{\sigma})} u_{n+1} - [t_1, \dots, t_n]_{\mathcal{S}(n,d\vec{\sigma})} u_{n+1}}. \tag{4}$$

Formula (4) can even be generalized to the complex case [6].

The *un-normalized Chebyshev B-splines* are then defined as

$$Q^n_{i,d\vec{\sigma}}(x) = (-1)^n [t_i, \dots t_{i+n}]_{\mathcal{S}(n+1,d\vec{\kappa})} G_{n,d\vec{\sigma}}(x, \cdot).$$

Let $K = \sum_{i=1}^{k} n_i$. B–splines $\{Q^n_{i,d\vec{\sigma}}\}_1^{n+K}$ are the basis for $\mathcal{S}(n, \vec{m}, d\vec{\sigma}, \Delta)$, and it is known [12] that they can be normalized so as to make a partition of unity, *i.e.* there are constants $\alpha_i^n(d\vec{\sigma}) > 0$ such that

$$T^n_{i,d\vec{\sigma}}(x) = \alpha_i^n(d\vec{\sigma})\, Q^n_{i,d\vec{\sigma}}(x), \tag{5}$$

and $\sum_{i=1}^{n+K} T^n_{i,d\vec{\sigma}}(x) = 1$ for $x \in [a, b]$. Moreover, $T^n_{i,d\vec{\sigma}}(x)$ do not depend on the artificial measure, that is the extension operator E_{n-1} needed to define divided differences. Indeed,

$$\alpha_i^n(d\vec{\sigma}) = \frac{D\begin{pmatrix} t_i, \dots, t_{i+n} \\ u_1^*, \dots, u_{n+1}^* \end{pmatrix} D\begin{pmatrix} t_{i+1}, \dots, t_{i+n-1} \\ u_1^*, \dots, u_{n-1}^* \end{pmatrix}}{D\begin{pmatrix} t_{i+1}, \dots, t_{i+n} \\ u_1^*, \dots, u_n^* \end{pmatrix} D\begin{pmatrix} t_i, \dots, t_{i+n-1} \\ u_1^*, \dots, u_n^* \end{pmatrix}}$$

and

$$T^n_{i,d\vec{\sigma}}(x) = \frac{D\begin{pmatrix} t_i, \dots, t_{i+n} \\ u_1^*, \dots, u_{n+1}^* \end{pmatrix} D\begin{pmatrix} t_{i+1}, \dots, t_{i+n-1} \\ u_1^*, \dots, u_{n-1}^* \end{pmatrix}}{D\begin{pmatrix} t_{i+1}, \dots, t_{i+n} \\ u_1^*, \dots, u_n^* \end{pmatrix} D\begin{pmatrix} t_i, \dots, t_{i+n-1} \\ u_1^*, \dots, u_n^* \end{pmatrix}} \cdot \frac{D\begin{pmatrix} t_i, & \dots, & & t_{i+n} \\ u_1^*, & \dots, & u_n^*, & G_{n,d\vec{\kappa}} \end{pmatrix}}{D\begin{pmatrix} t_i, \dots, t_{i+n} \\ u_1^*, \dots, u_{n+1}^* \end{pmatrix}},$$

so that the determinants involving u^*_{n+1} cancel.

Theorem 1.1. *Let $L_{1,d\vec{\sigma}}$ be the first generalized derivative with respect to the CCC-system $\mathcal{S}(n, d\vec{\sigma})$, and let the multiplicity vector $\vec{m}$ satisfy*

$n_i < n-1$ for $i = 1, \dots k$. Then for all $x \in [a,b]$ and $i = 1, \dots, n+K$:

$$L_{1,d\vec{\sigma}} T^n_{i,d\vec{\sigma}}(x) \;=\; \frac{T^{n-1}_{i,d\vec{\sigma}^{(1)}}(x)}{C_{n-1}(i)} - \frac{T^{n-1}_{i+1,d\vec{\sigma}^{(1)}}(x)}{C_{n-1}(i+1)}, \qquad (6)$$

where

$$C_{n-1}(i) \;=\; \int_{t_i}^{t_{i+n-1}} T^{n-1}_{i,d\vec{\sigma}^{(1)}} d\sigma_2.$$

PROOF. By Sylvester's determinant identity [3, p.158]:

$$T^n_{i,d\vec{\sigma}}(x) = (-1)^n \{[t_{i+1}, \cdots, t_{i+n}]_{\mathcal{S}(n,\boldsymbol{P}_{n-1} d\vec{\sigma})} G_{n,d\vec{\sigma}}(x,\cdot) \\ -[t_i, \cdots, t_{i+n-1}]_{\mathcal{S}(n,\boldsymbol{P}_{n-1} d\vec{\sigma})} G_{n,d\vec{\sigma}}(x,\cdot)\}.$$

If we apply the first generalized derivative and utilize (3), we obtain

$$L_{1,d\vec{\sigma}} T^n_{i,d\vec{\sigma}}(x) = -(\omega_{i+1} - \omega_i),$$

where

$$\omega_i := (-1)^{n-1} [t_i, \dots t_{i+n-1}]_{\mathcal{S}(n,\boldsymbol{P}_{n-1} d\vec{\sigma})} G_{n-1,d\vec{\sigma}^{(1)}}(x,\cdot).$$

The Mülbach's recurrence (4) reduces the order of divided differences:

$$\omega_i = \frac{(-1)^{n-1}}{\gamma_i} \quad \{ [t_{i+1}, \dots t_{i+n-1}]_{\mathcal{S}(n-1,\boldsymbol{P}_{n-2} d\vec{\sigma}^{(1)})} G_{n-1,d\vec{\sigma}^{(1)}} - \\ [t_i, \dots t_{i+n-2}]_{\mathcal{S}(n-1,\boldsymbol{P}_{n-2} d\vec{\sigma}^{(1)})} G_{n-1,d\vec{\sigma}^{(1)}} \},$$

where

$$\gamma_i = [t_{i+1}, \dots, t_{i+n-1}]_{\mathcal{S}(n-1,\boldsymbol{P}_{n-2} d\vec{\sigma}^{(1)})} u^*_n - [t_i, \dots, t_{i+n-2}]_{\mathcal{S}(n-1,\boldsymbol{P}_{n-2} d\vec{\sigma}^{(1)})} u^*_n.$$

Therefore, by Sylvester's identity, $\omega_i = T^{n-1}_{i,d\vec{\sigma}^{(1)}}(x)/\gamma_i$, and it remains to prove that $\gamma_i = C_{n-1}(i)$. To this end, let $d\vec{q} := \boldsymbol{E}_{n-2} \boldsymbol{P}_{n-2}\; d\vec{\sigma}^{(1)} + d\lambda\; \vec{e}_{n-1}$. Recurrence for divided differences in $\mathcal{S}(n, d\vec{q})$ applied to u^*_n yields

$$[t_i, \dots t_{i+n-1}]_{\mathcal{S}(n,d\vec{q}\,)} u^*_n = \qquad (7) \\ \frac{[t_{i+1}, \dots t_{i+n-1}]_{\mathcal{S}(n-1,\boldsymbol{P}_{n-2} d\vec{\sigma}^{(1)})} u^*_n - [t_i, \dots t_{i+n-2}]_{\mathcal{S}(n-1,\boldsymbol{P}_{n-2} d\vec{\sigma}^{(1)})} u^*_n}{[t_{i+1}, \dots t_{i+n-1}]_{\mathcal{S}(n-1,\boldsymbol{P}_{n-2} d\vec{\sigma}^{(1)})} v^*_n - [t_i, \dots t_{i+n-2}]_{\mathcal{S}(n-1,\boldsymbol{P}_{n-2} d\vec{\sigma}^{(1)})} v^*_n},$$

where v^*_n is an element of the extended dual reduced system:

$$v^*_n(y) = \int_a^y d\sigma_n(s_n) \dots \int_a^{s_4} d\sigma_3(s_3) ds_3 \int_a^{s_3} d\lambda(s_2).$$

Equation (7) implies that

$$\gamma_i = [t_i, \dots t_{i+n-1}]_{\mathcal{S}(n,d\vec{q}\,)} u_n^* \cdot$$
$$\{[t_{i+1}, \dots t_{i+n-1}]_{\mathcal{S}(n-1,\boldsymbol{P}_{n-2} d\vec{\sigma}^{(1)})} v_n^* - [t_i, \dots t_{i+n-2}]_{\mathcal{S}(n-1,\boldsymbol{P}_{n-2} d\vec{\sigma}^{(1)})} v_n^*\}.$$

By Peano representation of Chebyshev divided differences [12, p.382]

$$[t_i, \dots t_{i+n-1}]_{\mathcal{S}(n,d\vec{q})} u_n^* = \int_{t_i}^{t_{i+n-1}} Q^{n-1}_{i,d\vec{\sigma}^{(1)}} L_{n-1,d\vec{q}}\, u_n^* d\lambda,$$

and

$$L_{n-1,d\vec{q}}\, u_n^* d\lambda = d\sigma_2.$$

By yet another application of Sylvester's determinant identity, the term in $\{\}$ can be identified with the normalization constant $\alpha_i^{n-1}(d\vec{\sigma}^{(1)})$. $\square$

Theorem 1.1 may now be used to calculate, at least in theory, all derivatives of a Chebyshev spline. Generalized derivative can be factorized:

$$L_{i+1,d\vec{\sigma}} = L_{i-k+1,d\vec{\sigma}^{(k)}} L_{k,d\vec{\sigma}} \qquad \text{for} \quad k = 1, \dots i; \quad i = 1, \dots n-2, \tag{8}$$

and this fact can be used inductively to find higher derivatives as linear combination of lower order splines.

Theorem 1.2. *Let* $s(x) = \sum_{j=r-n+1}^{l-1} \delta_j T^n_{j,d\vec{\sigma}}(x)$ *be the B–representation of a Chebyshev spline* $s \in \mathcal{S}(n, \vec{m}, d\vec{\sigma}, \Delta)$ *for* $x \in [t_r, t_l]$, $0 < r < l < k+1$. *Then the B–representation of its generalized derivative* $L_{i,d\vec{\sigma}} s \in \mathcal{S}(n-i, \vec{m}, d\vec{\sigma}^{(i)}, \Delta)$ *is:*

$$L_{i,d\vec{\sigma}} s(x) = \sum_{j=r-n+i+1}^{l-1} \delta_j^i T^{n-i}_{j,d\vec{\sigma}^{(i)}}(x) \qquad \text{for} \quad i = 1, \dots n-1-\max_i n_i. \tag{9}$$

The coefficients δ_j^i *can be calculated recursively:*

$$\begin{aligned} \delta_j^0 &= \delta_j, \\ \delta_j^i &= \frac{\delta_j^{i-1} - \delta_{j-1}^{i-1}}{C_{n-i}(j)}, \end{aligned}$$

where

$$C_{n-i}(j) = \int_{t_j}^{t_{j+n-i}} T^{n-i}_{j,d\vec{\sigma}^{(i)}} d\sigma_{i+1}.$$

PROOF. We know that (9) holds for $i=1$. Let us suppose that

$$L_{i,d\vec{\sigma}}s(x) = \sum_j \delta_j^i T^{n-i}_{j,d\vec{\sigma}^{(i)}}(x). \tag{10}$$

Equation (8) for $k=i$ yields

$$L_{i+1,d\vec{\sigma}} = L_{1,d\vec{\sigma}^{(i)}} L_{i,d\vec{\sigma}},$$

whence by (10)

$$L_{i+1,d\vec{\sigma}}s(x) = L_{1,d\vec{\sigma}^{(i)}} (\sum_j \delta_j^i T^{n-i}_{j,d\vec{\sigma}^{(i)}}(x)). \tag{11}$$

Theorem 1.1 may now be applied to T^{n-i}–splines in (11) to obtain

$$L_{i+1,d\vec{\sigma}}s(x) = \sum_j \frac{\delta_j^i - \delta_{j-1}^i}{C_{n-i-1}(j)} T^{n-i-1}_{j,d\vec{\sigma}^{(i-1)}}(x) = \sum_j \delta_j^{i+1} T^{n-i-1}_{j,d\vec{\sigma}^{(i-1)}}(x). \quad \square$$

Applications

We can define, and calculate (at least theoretically) Chebyshev B-splines by a recurrence relation implied by the derivative formula:

$$T^n_{i,d\vec{\sigma}}(x) \quad = \quad \frac{1}{C_{n-1}(i)} \int_{t_i}^x T^{n-1}_{i,d\vec{\sigma}^{(1)}} d\sigma_2 - \frac{1}{C_{n-1}(i+1)} \int_{t_{i+1}}^x T^{n-1}_{i+1,d\vec{\sigma}^{(1)}} d\sigma_2.$$

From the numerical point of view, the recurrence involves dangereous subtractions resulting in the loss of significant digits, even for polynomial splines. For Chebyshev splines the numerical instability sometimes destroys the result. To illustrate it, we consider a CCC-system associated with the measure vector $d\vec{\sigma} = (t_2^{-\alpha} dt_2, dt_3, dt_4)^{\mathrm{T}}$, where $0 < \alpha < 1$. The system originates from the realistic problem concernig axially symmetric potentials, and is not an ECC-system, since the measures do not posses smooth densities [8]. The Green's function is

$$g_{4,d\vec{\sigma}}(x,y) = \begin{cases} \frac{x^{3-\alpha}-y^{3-\alpha}}{2(3-\alpha)} - \frac{y(x^{2-\alpha})-y^{2-\alpha})}{2-\alpha} + \frac{y^2(x^{1-\alpha}-y^{1-\alpha})}{2(1-\alpha)}, & x \geq y \\ 0 & \text{otherwise.} \end{cases}$$

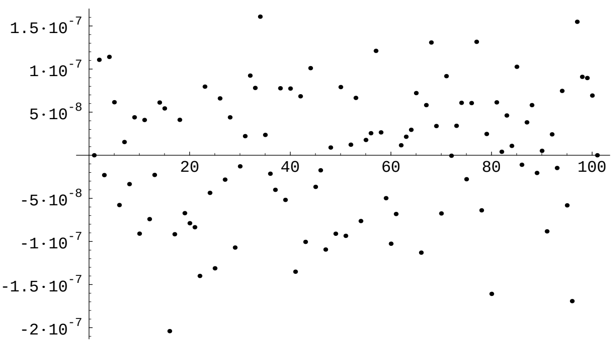
The simplest case is that of the 4^{th} order splines on triplets of knots, as the first reduced system are ordinary powers, and therefore B-splines in the first reduced system are scaled Bernstein polynomials.

The following simple `Mathematica` code generates some Chebyshev B-splines for $\alpha = \frac{1}{2}$:

```
B1[x_] := ((b - x)/(b - a))^2;
B2[x_] := 2/(b - a)^2*(x - a)*(b - x);
a = 1000; b = 1001;
C1 = Simplify[Integrate[B1[t]*Sqrt[t], {t, a, b}]]
C2 = Simplify[Integrate[B2[t]*Sqrt[t], {t, a, b}]]
first[x_] := Simplify[Integrate[B1[t]*Sqrt[t], {t, a, x}]/C1];
second[x_] := Simplify[Integrate[B2[t]*Sqrt[t], {t, a, x}]/C2];
Plot[first[x] - second[x], {x, a, b + (b - a)/3}]
```

Depending on a and b (eg. if they are either away from 0 like in the above example, or close to each other), this can lead to the loss of half of the significant digits. Indeed, recalculation in 64-bit arithmetics shows that only the first seven digits hold, and the error is shown in Fig. 1:

Figure 1. Roundoff error for the derivative formula $(\alpha = \frac{1}{2})$



Observe also that accuracy is lost if we calculate the normalization constants on small intervals by analytic formulæ. For example, the constant C1 above is:

$$\frac{2\left(-15\,a^{\frac{7}{2}}+42\,a^{\frac{5}{2}}\,b-35\,a^{\frac{3}{2}}\,b^2+8\,b^{\frac{7}{2}}\right)}{105\,(a-b)^2}.$$

It is therefore better to use a Gaussian formula with the appropriate weight. In special cases such as this, where only one measure is different from the Lebesgue one, the derivative formula and knot insertion can sometimes be used to obtain numericaly stable algorithms (at least for rational α's); Theorem 1.1 by itself is not enough.

The same qualitative behaviour also happens for some ECC-systems, like *tension powers*, where the ECC-system is determined by the measure vector $d\vec{\sigma} := (d\lambda, \cosh(px)d\lambda, \frac{1}{\cosh^2(px)}d\lambda)^{\mathrm{T}}$, and $p > 0$ is known as *tension parameter*. The tension parameter and interval length can be chosen so that straighforward application of the derivative formula leads to loss of all significant digits. There is a way out through knot insertion [11], but Theorem 1.1 still plays an important role in the construction.

Finally, the first recorded proof of the famous de Boor-Cox recurrence [7] for polynomal splines is based on the derivative formula, plus an additional algebraic fact, which does not hold in the Chebyshev setting [2].

Acknowledgments. This research was supported by Grant 0037014, by the Ministry of Science and Technology of the Republic of Croatia.

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