Numerically Stable Algorithm for Cycloidal Splines

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Abstract. We propose a knot insertion algorithm for splines that are piecewisely in $\mathrm{L}\{1, x, \sin x, \cos x\}$. Since an ECC–system on $[0, 2\pi]$ in this case does not exist, we construct a CCC–system by choosing the appropriate measures in the canonical representation. In this way, a B-basis can be constructed in much the same way as for weighted and tension splines. Thus we develop a corner cutting algorithm for lower order cycloidal curves, though a straightforward generalization to higher order curves, where ECC–systems exist, is more complex. The important feature of the algorithm is high numerical stability and simple implementation.

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1 Introduction and Preliminaries

Many objects around us come in the shapes of straight lines and circle arcs; also, in industrial applications computer guided machines are able to cut shapes like circles or helices. Therefore, it is useful to have curves piecewisely spanned by linear polynomials, sine and cosine, *i.e.* cycloidal splines (also called helix splines) to approximate such objects [2, 4, 5, 7]. The paper describes a stable algorithm for calculating with Bézier curves relying on this special Chebyshev system.

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For a given measure vector $d\boldsymbol{\sigma} := (d\sigma_2, \dots d\sigma_k)^{\mathrm{T}}$, and $x \in [a,b]$ we can define generalized powers (or Canonical Complete Chebyshev (CCC)–system) $\{1, u_2, \dots u_k\}$:

$$\begin{aligned} u_1(x) &= 1, \\ u_2(x) &= \int_a^x d\sigma_2(\tau_2), \\ &\vdots \\ u_k(x) &= \int_a^x d\sigma_2(\tau_2) \dots \int_a^{\tau_{k-1}} d\sigma_k(\tau_k). \end{aligned}$$

The i-th reduced system is defined to be a Chebyshev system corresponding to the reduced measure vector, that is

$$d\boldsymbol{\sigma}^{(i)}(\delta) := (d\sigma_{i+2}(\delta), \dots, d\sigma_k(\delta))^{\mathrm{T}} \in \mathbb{R}^{k-(i+1)}, \; i = 1, \dots k-2,$$

for $\delta \subset [a,b]$ measurable with respect to all $d\sigma_j$. Generalized derivatives $L_{j,d\boldsymbol{\sigma}} := D_j \cdots D_1$, where

$$D_j f(x) := \lim_{\delta \to 0^+} \frac{f(x+\delta) - f(x)}{\sigma_{j+1}(x+\delta) - \sigma_{j+1}(x)}, \; j = 1, \dots, k-1$$

for $f \in \mathcal{S}(k, d\boldsymbol{\sigma}) := \mathrm{span}\{1, u_2, \dots u_k\}$, are linear mappings $\mathcal{S}(k, d\boldsymbol{\sigma}) \to \mathcal{S}(k-j, d\boldsymbol{\sigma}^{(j)})$.

For a partition $\Delta = \{x_i\}_{i=0}^{\ell+1}$, $x_i < x_{i+1}$, of an interval $[a,b]$, given multiplicity vector $\boldsymbol{m} = (m_1, \dots, m_\ell)$, $(0 < m_i \le k)$, and $M := \sum_{i=1}^{\ell} m_i$, we shall denote by $\{t_1 \dots t_{2k+M}\}$, an extended partition in the usual way:

$$t_1 \le \dots \le t_k = x_0 = a,$$

$$t_{k+r} = x_i, \quad r = 1 + \sum_{j=1}^{i-1} m_j, \dots, \sum_{j=1}^{i} m_j, \quad i = 1, 2, \dots, \ell,$$

$$b = x_{\ell+1} = t_{k+M+1} \le \dots \le t_{2k+M}.$$

Elements of the extended partition are called knots, and $\mathcal{S}(k, \boldsymbol{m}, d\boldsymbol{\sigma}, \Delta)$ is the spline space spanned by functions being piecewise in $\mathcal{S}(k, d\boldsymbol{\sigma})$, with generalized derivatives up to $(k - m_i - 1)^{\text{-th}}$ order joining continuously at x_i for $i = 1, \dots, \ell$. Chebyshev B-splines $\{T_{i,d\boldsymbol{\sigma}}^k\}_{i=1}^{k+M} \in \mathcal{S}(k, \boldsymbol{m}, d\boldsymbol{\sigma}, \Delta)$ are the basis for $\mathcal{S}(k, \boldsymbol{m}, d\boldsymbol{\sigma}, \Delta)$ possessing the compact support $[t_i, t_{i+k}]$, and are unique such splines if we assume that they are normalized in such a way to make a partition of unity: $\sum_{i=1}^{k+M} T_{i,d\boldsymbol{\sigma}}^k(x) = 1$.

In the general case, we do not have the de Boor–Cox type recurrence, so we use the derivative formula [8, 9] instead:

Theorem 1 *Let $L_{1,d\boldsymbol{\sigma}}$ be the first generalized derivative with respect to CCC-system $\mathcal{S}(k, d\boldsymbol{\sigma})$, and let the multiplicity vector $\boldsymbol{m} = (m_1, \ldots, m_\ell)$ satisfy $m_i \leq k$ for $i = 1, \ldots, \ell$. Then for $x \in [a, b]$ and $i = 1, \ldots, k + \sum_{j=1}^{\ell} m_j$, the following derivative formula holds:*

$$L_{1,d\boldsymbol{\sigma}} T^k_{i,d\boldsymbol{\sigma}}(x) = \frac{T^{k-1}_{i,d\boldsymbol{\sigma}^{(1)}}(x)}{C_{k-1}(i)} - \frac{T^{k-1}_{i+1,d\boldsymbol{\sigma}^{(1)}}(x)}{C_{k-1}(i+1)}, \tag{1}$$

with $C_{k-1}(i) := \displaystyle\int_{t_i}^{t_{i+k-1}} T^{k-1}_{i,d\boldsymbol{\sigma}^{(1)}} d\sigma_2.$

We shall also implicitly make use of the following Corollary to the Lebesgue dominated convergence Theorem (the same type of argument was used for special kind of Chebyshev splines in [11]):

Lemma 1 *If the functions σ_i which define measures $d\sigma_i$ are continuous for each $i = 2, \ldots, k$, then the integrals of B-splines with respect to the Lebesgue–Stieltjes measure are continuous functions of their knots.*

2 Knot insertion

Definition 1 *Let the extended partitions $\boldsymbol{T} = \{t_j\}_{j=1}^{n+k}$, $\widetilde{\boldsymbol{T}} = \{\tilde{t}_i\}_{i=1}^{m+k}$ be associated with partitions Δ, $\widetilde{\Delta}$ of $[a, b]$ and the multiplicity vectors $\boldsymbol{m}$, $\widetilde{\boldsymbol{m}}$ respectively, such that $\mathcal{S}(k, \boldsymbol{m}, d\boldsymbol{\sigma}, \Delta) \subset \mathcal{S}(k, \widetilde{\boldsymbol{m}}, d\boldsymbol{\sigma}, \widetilde{\Delta})$. Suppose that $n \leqslant m$, and let $T^k_{j,d\boldsymbol{\sigma}}$, $\widetilde{T}^k_{i,d\boldsymbol{\sigma}}$ be B-splines in $\mathcal{S}(k, \boldsymbol{m}, d\boldsymbol{\sigma}, \Delta)$, $\mathcal{S}(k, \widetilde{\boldsymbol{m}}, d\boldsymbol{\sigma}, \widetilde{\Delta})$ respectively. A spline $f \in \mathcal{S}(k, \boldsymbol{m}, d\boldsymbol{\sigma}, \Delta)$ can be represented in this B-spline basis as*

$$f(x) = \sum_{j=1}^{n} c_j T^k_{j,d\boldsymbol{\sigma}}(x) = \sum_{i=1}^{m} d_i \widetilde{T}^k_{i,d\boldsymbol{\sigma}}(x).$$

If we denote the coefficient vectors as $\mathbf{c} := (c_1, \ldots, c_n)^{\mathrm{T}}$ and $\mathbf{d} := (d_1, \ldots, d_m)^{\mathrm{T}}$, then $m \times n$ matrix $\Gamma^k_{(d\boldsymbol{\sigma}, \boldsymbol{T}, \widetilde{\boldsymbol{T}})} = [\gamma^k_{i,j}]_{i=1,j=1}^{m,n}$ such that

$$\mathbf{d} = \Gamma^k_{(d\boldsymbol{\sigma}, \boldsymbol{T}, \widetilde{\boldsymbol{T}})} \mathbf{c},$$

is called the knot insertion matrix of order k from $\boldsymbol{T}$ to $\widetilde{\boldsymbol{T}}$.

We can calculate the non-trivial elements of the single knot insertion matrix by following theorem:

Theorem 2 *Let* $\boldsymbol{T} = (t_1 \le t_2 \le \cdots \le t_{k-1} \le a = t_k < t_{k+1} < \cdots < t_n < t_{n+1} = b \le t_{n+2} \le \cdots \le t_{n+k-1} \le t_{n+k})$ *be an extended partition of* $[a,b]$ *with all interior knots of multiplicity one. Let* $U_k = \{1, u_2, \ldots, u_k\}$ *be the CCC–system associated with the measure vector* $d\boldsymbol{\sigma} := (d\sigma_2, \ldots, d\sigma_k)^{\mathrm{T}}$*. For* $\bar{t} \in (a,b)$*, and* i *such that* $\bar{t} \in (t_i, t_{i+1})$*, let* $\bar{\boldsymbol{T}} = \boldsymbol{T} \cup \{\bar{t}\}$*. Then the nontrivial elements of the knot insertion matrix* $\Gamma^{\ell}_{(d\boldsymbol{\sigma}^{(k-l)}, \boldsymbol{T}, \bar{\boldsymbol{T}})} = [\gamma^{\ell}_{i,j}]$ *of order* ℓ *from* $\boldsymbol{T}$ *to* $\bar{\boldsymbol{T}}$ *are:*

$$\begin{aligned} \gamma^1_{j,j} &= 1 \quad \text{for } j \le i, \\ \gamma^1_{j,j-1} &= 1 \quad \text{for } j \ge i+1, \end{aligned}$$

for $\ell = 1$*, and*

$$\begin{aligned} \gamma^{\ell}_{j,j} &= 1 && \text{for } j \le i - \ell + 1, \\ \gamma^{\ell}_{j,j} &= \gamma^{\ell-1}_{j,j} \frac{\bar{C}_{\ell-1}(j)}{C_{\ell-1}(j)} && \text{for } i - \ell + 2 \le j \le i, \\ \gamma^{\ell}_{j,j-1} &= \gamma^{\ell-1}_{j+1,j} \frac{\bar{C}_{\ell-1}(j+1)}{C_{\ell-1}(j)} && \text{for } i - \ell + 2 \le j \le i, \\ \gamma^{\ell}_{j,j-1} &= 1 && \text{for } j \ge i+1, \end{aligned}$$

for $\ell = 2, \ldots, k$*, where*

$$\begin{aligned} C_{\ell-1}(j) &:= \int_{t_j}^{t_{j+\ell-1}} T^{\ell-1}_{j, d\boldsymbol{\sigma}^{(k-\ell+1)}} \, d\sigma_{k-\ell+2}, \\ \bar{C}_{\ell-1}(j) &:= \int_{\bar{t}_j}^{\bar{t}_{j+\ell-1}} \bar{T}^{\ell-1}_{j, d\boldsymbol{\sigma}^{(k-\ell+1)}} \, d\sigma_{k-\ell+2}, \end{aligned}$$

for $\ell = 2, \ldots, k$*. B-splines* $T^{\ell-1}_{j, d\boldsymbol{\sigma}^{(k-\ell+1)}}$ *and* $\bar{T}^{\ell-1}_{j, d\boldsymbol{\sigma}^{(k-\ell+1)}}$ *are associated with the extended partitions* $\boldsymbol{T}$ *and* $\bar{\boldsymbol{T}}$*, respectively.*

The proof can be found in [1]. All the other cases of $\boldsymbol{T}$, *i.e.*, when $\boldsymbol{T}$ does not have all interior knots of multiplicity one, follow from Theorem 2, and the continuity conclusion of Lemma 1.

For some special general knot insertion matrices two simple recurrences exist:

Theorem 3 *Let* $\boldsymbol{T}$ *be an extended partition of* $[a,b]$ *with all interior knots of multiplicity one. Let* $U_k = \{1, u_2, \ldots, u_k\}$ *be the CCC–system associated* *with the measure vector* $d\boldsymbol{\sigma} := (d\sigma_2, \ldots, d\sigma_k)^{\mathrm{T}}$. *Let* $\widetilde{\boldsymbol{T}} = \{\tilde{t}_j\}$ *be an extended partition with the same knots as* $\boldsymbol{T}$, *but of arbitrary multiplicity. Then*

$$\gamma_{j,i}^{\ell} = \gamma_{j-1,i}^{\ell} + \widetilde{C}_{\ell-1}(j)\left(\frac{\gamma_{j,i}^{\ell-1}}{C_{\ell-1}(i)} - \frac{\gamma_{j,i+1}^{\ell-1}}{C_{\ell-1}(i+1)}\right), \tag{2}$$

and

$$\gamma_{j,i}^{\ell} = \frac{\sum\limits_{r\leqslant j}\gamma_{r,i}^{\ell-1}\widetilde{C}_{\ell-1}(r)}{C_{\ell-1}(i)} - \frac{\sum\limits_{r\leqslant j}\gamma_{r,i+1}^{\ell-1}\widetilde{C}_{\ell-1}(r)}{C_{\ell-1}(i+1)}, \tag{3}$$

with $\Gamma^{\ell}_{(d\boldsymbol{\sigma}^{(k-l)},\boldsymbol{T},\widetilde{\boldsymbol{T}})} = [\gamma_{i,j}^{\ell}]$ *for* $\ell = 2, \ldots, k$, *where*

$$C_{\ell-1}(j) := \int_{t_j}^{t_{j+\ell-1}} T^{\ell-1}_{j,d\boldsymbol{\sigma}^{(k-\ell+1)}}\, d\sigma_{k-\ell+2},$$
$$\widetilde{C}_{\ell-1}(j) := \int_{\tilde{t}_j}^{\tilde{t}_{j+\ell-1}} \bar{T}^{\ell-1}_{j,d\boldsymbol{\sigma}^{(k-\ell+1)}}\, d\sigma_{k-\ell+2}.$$

B-splines $T^{\ell-1}_{j,d\boldsymbol{\sigma}^{(k-\ell+1)}}$ *and* $\widetilde{T}^{\ell-1}_{j,d\boldsymbol{\sigma}^{(k-\ell+1)}}$ *are associated with the extended partitions* $\boldsymbol{T}$ *and* $\widetilde{\boldsymbol{T}}$, *respectively.*

Proof: The proof of (2) is based on the properties of knot insertion matrices, the derivative formula (1), and the proof of (3) follows by recursive application of (2). ∎

Theorem 3 is a discrete version of the derivative formula, since $\gamma_{j,i}^{\ell}$ play the role of discrete Chebyshev splines. Unlike the polynomial case, we do not know of the stable recurrence. Further, the relations (2) and (3), have not to be numerically stable in general. There are however, some special cases where (3) can be rearranged to avoid potentially dangerous subtractions. One of this cases is the 'cubic' case:

Lemma 2 *Let* $T^3_{i,d\boldsymbol{\sigma}^{(1)}} \in \mathcal{S}(3, \boldsymbol{m}^{(1)}, d\boldsymbol{\sigma}^{(1)}, \Delta)$ *be a Chebyshev* 3^{rd} *order spline associated with the multiplicity vector* $\boldsymbol{m}^{(1)} = (1, \ldots, 1)^{\mathrm{T}}$, *and let us assume that* $\widetilde{T}^3_{i,d\boldsymbol{\sigma}^{(1)}} \in \mathcal{S}(3, \boldsymbol{m}^{(2)}, d\boldsymbol{\sigma}^{(1)}, \Delta)$ *are B-splines associated with multiplicity vector* $\boldsymbol{m}^{(2)} = (2, \ldots, 2)$ *on the same knot sequence. If* $\boldsymbol{T} = \{t_j\}_{j=1}^{n+4}$ *and* $\widetilde{\boldsymbol{T}} = \{\tilde{t}_j\}_{j=1}^{2n}$, *are the associated extended partitions, and* r *an index such that* $t_i = \tilde{t}_r < \tilde{t}_{r+1}$, *then for* $i = 2, \ldots, n$:

$$T^3_{i,d\boldsymbol{\sigma}^{(1)}} = \frac{\widetilde{C}_2(r)}{C_2(i)}\widetilde{T}^3_{r,d\boldsymbol{\sigma}^{(1)}} + \widetilde{T}^3_{r+1,d\boldsymbol{\sigma}^{(1)}} + \frac{\widetilde{C}_2(r+3)}{C_2(i+1)}\widetilde{T}^3_{r+2,d\boldsymbol{\sigma}^{(1)}}.$$

One of the consequences of the Theorem 2 is a possible generalization of the well known de Boor algorithm [3]:

Algorithm 1 (Generalized 4^{th} order de Boor algorithm) *Let* $\boldsymbol{T}=\{t_j\}$ *be an extended partition associated with the partition* Δ *and the multiplicity vector* $\boldsymbol{m}=(m_1,m_2,\ldots,m_\ell)$. *Let* $\bar t\in(t_i,t_{i+1})$ *and* $\bar\Delta:=\Delta\cup\bar t$, *let* $\bar{\boldsymbol{T}}=\boldsymbol{T}\cup\bar t$, $\widetilde{\boldsymbol{T}}$ *be associated with* $\bar\Delta$ *and the measure vector* $(m_1,\ldots,m_i,2,m_{i+1},\ldots,m_\ell)$, *were* $\widehat{\boldsymbol{T}}$ *is associated with* $\bar\Delta$ *and* $(m_1,\ldots,m_i,3,m_{i+1},\ldots,m_\ell)$. *The B-splines associated with these extended partitions are denoted by* $T^{4-\ell}_{j,d\boldsymbol{\sigma}^{(\ell)}}$, $\bar T^{4-\ell}_{j,d\boldsymbol{\sigma}^{(\ell)}}$, $\widetilde T^{4-\ell}_{j,d\boldsymbol{\sigma}^{(\ell)}}$, $\widehat T^{4-\ell}_{j,d\boldsymbol{\sigma}^{(\ell)}}$ *for* $\ell=0,\ldots,3$, *and their integrals by* $C_{4-\ell}(j)$, $\bar C_{4-\ell}(j)$, $\widetilde C_{4-\ell}(j)$, $\widehat C_{4-\ell}(j)$, *respectively. For* $\mathcal{S}(4,\boldsymbol{m},d\boldsymbol{\sigma},\Delta)$ *and given* $\bar t$ *the algorithm can be rearranged as*

$$f(\bar t)=\sum_{j=1}^{n}c_jT_j^4(\bar t)=\sum_{j=1}^{n+3}\hat c_j\widehat T_j^4(\bar t)=\hat c_i,$$

with

$$\begin{aligned}\hat c_i&=c_{i-3}\frac{\widehat C_3(i+1)\widetilde C_2(i+1)\bar C_1(i+1)}{C_1(i)C_2(i-1)C_3(i-2)}+c_{i-2}\left(\frac{\widehat C_3(i+1)\widetilde C_2(i+1)\bar C_3(i-2)}{\bar C_2(i)\bar C_3(i-1)C_3(i-2)}\right.\\&\left.+\frac{\widehat C_3(i+1)\widetilde C_3(i-1)\bar C_2(i+1)\bar C_3(i)}{\widetilde C_3(i)\bar C_3(i-1)C_2(i)C_3(i-1)}+\frac{\widehat C_3(i)\widetilde C_3(i+1)\bar C_2(i+1)}{\widetilde C_3(i)C_2(i)C_3(i-1)}\right)\\&+c_{i-1}\left(\frac{\widehat C_3(i+1)\widetilde C_3(i-1)\bar C_2(i-1)}{\widetilde C_3(i)C_2(i-1)C_3(i-1)}+\frac{\widehat C_3(i)\widetilde C_3(i+1)\bar C_2(i-1)\bar C_3(i-1)}{\widetilde C_3(i)\bar C_3(i)C_2(i-1)C_3(i-1)}\right.\\&\left.+\frac{\widehat C_3(i)\widetilde C_2(i)\bar C_3(i+1)}{\bar C_2(i)\bar C_3(i)C_3(i)}\right)+c_i\frac{\widehat C_3(i)\widetilde C_2(i)\bar C_1(i)}{C_1(i)C_2(i)C_3(i)}.\end{aligned}\tag{4}$$

3 Equidistant cycloidal splines

Let $\Delta=\{x_i\}_{i=0}^{\ell+1}$ be a partition of $[0,(\ell+1)\frac{\pi}{2}]$, such that $x_i=i\frac{\pi}{2}$, and let $\mathrm{Cs}:\mathbb{R}\to\mathbb{R}$ be defined by

$$\mathrm{Cs}(x):=\cos\left(x-\frac{\pi}{4}-i\frac{\pi}{2}\right)$$

for $x\in[i\frac{\pi}{2},(i+1)\frac{\pi}{2}]$, $i\in\mathbb{Z}$. Cs is continuous, periodical extension of $\cos(x-\frac{\pi}{4})|_{[0,\frac{\pi}{2}]}$. Consider a CCC-system on $\mathbb{R}$:

$$u_1(x)=1,$$

$$
\begin{aligned}
u_2(x) &= \int_0^x d\tau_2, \\
u_3(x) &= \int_0^x d\tau_2 \int_0^{\tau_2} \mathrm{Cs}(\tau_3)\, d\tau_3, \\
u_4(x) &= \int_0^x d\tau_2 \int_0^{\tau_2} \mathrm{Cs}(\tau_3)\, d\tau_3 \int_0^{\tau_3} \frac{1}{\mathrm{Cs}^2(\tau_4)}\, d\tau_4.
\end{aligned}
$$

One easily verifies that $\mathrm{span}\{u_1, u_2, u_3, u_4\} = \mathrm{span}\{1, x, \sin x, \cos x\}$ on each $[i\frac{\pi}{2}, (i+1)\frac{\pi}{2}]$.

Let $\bar{t} \in (t_i, t_{i+1})$, where $\boldsymbol{T}$, $\bar{\boldsymbol{T}}$, $\widetilde{\boldsymbol{T}}$, $\widehat{\boldsymbol{T}}$ and the rest of the notation as in Algorithm 1 with $\boldsymbol{m} = (1, \ldots, 1)$. Then by Lemma 2

$$
C_1(j) = \frac{\sin h_j}{\mathrm{Cs}(t_j)\mathrm{Cs}(t_{j+1})}, \tag{5}
$$

$$
C_2(j) = \mathrm{Cs}(t_{j+1}) \frac{\sin \frac{h_j + h_{j+1}}{2}}{\cos \frac{h_j}{2} \cos \frac{h_{j+1}}{2}}, \tag{6}
$$

$$
\begin{aligned}
C_3(j) = \cos \frac{h_{j+1}}{2} \Bigg[& \frac{\sin \frac{h_j}{2}}{\sin \frac{h_j + h_{j+1}}{2}} \left(\frac{h_j - \sin h_j}{2 \sin^2 \frac{h_j}{2}} + \frac{h_{j+1} - \sin h_{j+1}}{2 \sin^2 \frac{h_{j+1}}{2}} \right) \\
& + \frac{2}{\sin^2 \frac{h_{j+1}}{2}} \left(\sin \frac{h_{j+1}}{2} - \frac{h_{j+1}}{2} \cos \frac{h_{j+1}}{2} \right) \\
& + \frac{\sin \frac{h_{j+2}}{2}}{\sin \frac{h_{j+1} + h_{j+2}}{2}} \left(\frac{h_{j+1} - \sin h_{j+1}}{2 \sin^2 \frac{h_{j+1}}{2}} + \frac{h_{j+2} - \sin h_{j+2}}{2 \sin^2 \frac{h_{j+2}}{2}} \right) \Bigg].
\end{aligned} \tag{7}
$$

The cycloidal splines can now be calculated by de Boor algorithm (4). In order to avoid redundant operations, and also to avoid inherent numerical instabilities involved in calculating with special functions involved in (7), we define functions $CC1$, $CC2$ and $CC3$ instead of C_1, C_2, C_3:

$$
\begin{aligned}
CC1(x) &:= \sin \frac{x}{2}, \qquad CC2(x, y) := \sin \frac{x + y}{2}, \\
CC3(x, y, z) &:= \frac{1}{2} u \left(\frac{x}{2}, \frac{y}{2} \right) (f(x) + f(y)) + 2g(\frac{y}{2}) + \frac{1}{2} u \left(\frac{z}{2}, \frac{y}{2} \right) (f(y) + f(z)),
\end{aligned} \tag{8}
$$

with

$$
f(x) := \frac{x - \sin x}{\sin^2 \frac{x}{2}}, \quad g(x) := \frac{\sin x - x \cos x}{\sin^2 x}, \quad u(x, y) := \frac{\sin x}{\sin (x + y)}.
$$

The function f has to be evaluated only on interval $[0, \frac{\pi}{2}]$, with $\lim_{x \to 0} f(x) = 0$, so it is not difficult to find a Padé approximation on the whole interval.

The same is true for g, which we only need on interval $[0, \frac{\pi}{4}]$, also with $\lim_{x\to 0} g(x) = 0$. Because

$$\lim_{x\to 0} u(x, y) = 0, \qquad \lim_{y\to 0} u(x, y) = 1,$$

the function u is not continuous at $(0, 0)$, what presents no difficulties since from (8) it is obvious that $u(0, 0)$ is then multiplied by zero.

This algorithm can also be carried out for $\bar{t} = t_i$ and even with extended partition which has knots with arbitrary multiplicities, just by applying Lemma 1 in (5), (6) and (7).

The continuity of the second generalized derivative is probably unnecessary; there is no such difficulty if we use the Bézier variant of cycloidal splines.

4 Equidistant Bézier cycloidal splines

We will proceed with the Bézier case. Let $\bar{t} \in (x_i, x_{i+1}) \subset [0, (\ell+1)\frac{\pi}{2}]$, and let $\boldsymbol{T}$, $\bar{\boldsymbol{T}}$, $\widetilde{\boldsymbol{T}}$ and $\widehat{\boldsymbol{T}}$ be extended partitions of the interval $[0, (\ell+1)\frac{\pi}{2}]$ as in Algorithm 1, only with $\boldsymbol{m} = (4, \ldots, 4)$. We use the notation $\bar{T}_j^k$, $\widetilde{T}_j^k$, $\widehat{T}_j^k$, $\bar{C}_{k-1}(j)$, $\widetilde{C}_{k-1}(j)$, $\widehat{C}_{k-1}(j)$ for B-splines and their integrals accordingly. These integrals are then calculated from (5), (6) and (7) according to Lemma 1 by coalescing the knots. For computer implementation we use the generalized de Boor algorithm joint with functions defined in (8), only now more arguments are equal to zero.

Given splines are generalized Bézier splines, and analogous properties hold. If we observe an interval $[x_i, x_{i+1}] = [i\frac{\pi}{2}, (i+1)\frac{\pi}{2}]$, and a parameterized curve in $\mathbb{R}^2$:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = A\, T^4_{4i+1}(t) + B\, T^4_{4i+2}(t) + C\, T^4_{4i+3} + D\, T^4_{4(i+1)}(t)$$

on a given interval, then for some index i and de Boor points A, B, C, D, the curve passes through points A and D, *i.e.*

$$\begin{bmatrix} x(x_i) \\ y(x_i) \end{bmatrix} = A, \qquad \begin{bmatrix} x(x_{i+1}) \\ y(x_{i+1}) \end{bmatrix} = D,$$

Moreover, derivatives at the end points, by the derivative formula (1), (in this case $L_{(1,d\boldsymbol{\sigma})} = D$), are

$$\begin{bmatrix} \dot{x}(x_i) \\ \dot{y}(x_i) \end{bmatrix} = \frac{1}{C_3(4i+1)}(B - A), \qquad \begin{bmatrix} \dot{x}(x_{i+1}) \\ \dot{y}(x_{i+1}) \end{bmatrix} = \frac{1}{C_3(4i+3)}(D - C).$$

5 Nonequidistant Bézier cycloidal splines

Further generalization can be made in sense that, instead of taking an equidistant partition, we take an arbitrary partition $\{x_i\}_{i=0}^{\ell+1}$ of $[a,b]$, such that $0 < h_i = x_{i+1} - x_i \leqslant \frac{\pi}{2}$, and multiplicity vector $\boldsymbol{m} = (4,\ldots,4)$. Now, $\mathrm{Cs} : [a,b] \to \mathbb{R}$ is defined by $\mathrm{Cs}(x) := \cos(x - \frac{\pi}{4} - x_i)$ for $x \in [x_i, x_{i+1})$, $i = 0,\ldots,\ell-1$ or $x \in [x_\ell, x_{\ell+1}]$ for $i = \ell$. Generally, Cs is not continuous.

The algorithm from the previous section can be easily modified for this case; we only need to replace the knots $i\frac{\pi}{2}$ with given x_i in (5),(6) and (7).

6 Conclusion

There are some comments as to the choice of the CCC–system in Sections 3 and 5. We know that ECC–system for $\mathrm{span}\{1, x, \sin x, \cos x\}$ exists only on interval of length less than 2π (although there exists an ECC–space on interval $[a, a+2\pi]$ which contains 1, x, $\sin x$ and $\cos x$ but of a higher dimension than 4, at least 6, see [2]). We have overcome this restriction by substituting ECC–system with the CCC–system. The choice of the length of the basic subinterval $h_i = \frac{\pi}{2}$ and of the function Cs in Sections 3 and 4, is somewhat arbitrary, taken because of the simple form of CCC–system and reduced systems, and also to assure easy calculation and numerical stability. In Section 5 we put the restriction $h_i \leqslant \frac{\pi}{2}$ because in that case we can easily generalize the algorithm from Section 4.

The problem of calculating cycloidal splines already attracted some of attention, and although in [5, 6, 7] algorithms for calculating with cycloidal splines have been developed based on general theory of Chebyshev blossoming, the explicit algorithms involving only scalar products of positive quantities, and evaluation of some special functions, have not yet been discussed.

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