# POLUGRUPE

Ovo je forma:

$$a(v,u) \;=\; \sum_{i,j=1}^{d} \int_D a_{ij}(\boldsymbol{x})\,\partial_i v(\boldsymbol{x})\,\partial_j u(\boldsymbol{x})\,d\boldsymbol{x}$$
$$+\sum_{i=1}^{d} \int_D b_i'(\boldsymbol{x})\,v(\boldsymbol{x})\,\partial_i u(\boldsymbol{x}) - \sum_{i=1}^{d} \int_D b_i''(\boldsymbol{x})\,\partial_i v(\boldsymbol{x})\,u(\boldsymbol{x})\,d\boldsymbol{x} + \int_D c(\boldsymbol{x})\,v(\boldsymbol{x})\,u(\boldsymbol{x})\,d\boldsymbol{x}, \tag{1}$$

a ovo je objekt kojeg trebamo iz osnova o aproksimacijama. An element $u \in W_2^1(\mathbb{R}^d)$ does not belong necessary to $E_n(R,\mathbb{R}^d)$. In order to approximate $u$ with elements of $E_n(R,\mathbb{R}^d)$ we define:

$$\hat{u}(n) \;=\; \sum_{\boldsymbol{k}\in I_n(R)} \|\psi_{\boldsymbol{k}}\|_1^{-1}\,(\psi_{\boldsymbol{k}}|u)\,\psi_{\boldsymbol{k}}. \tag{2}$$

The numbers $\|\psi_{\boldsymbol{k}}\|_1^{-1}(\psi_{\boldsymbol{k}}|u)$ are called Fourier coefficients of $u$.

### 1. Funkcionalni prostori

Prostori $X(I,L)$, gdje je $I \subset [0,\infty)$ interval a $L$ je Banachov prostor.

DEFINITION 0.1 *Let $I \subset [0,\infty)$ be an interval. A function $I \ni t \mapsto u(t) \in L$ is called finitely-valued if there exist $u_1, u_2, \ldots, u_n$ and a partition $I = \cup_{r=1}^n I_r$, where $I_r$ are the Lebesgue measurable subsets of $I$, such that $u_r(t,\cdot) = \sum_r u(\cdot)\mathbb{1}_{I_r}(t)$. A function $I \ni t \mapsto u(t) \in L$ is called separably-valued if its range $\{u(t) : t \in I\}$ is a separable set of $L$. It is called almost separably valued if there exists a subset $I_0 \subset I$ of zero measure such that $u(\cdot)$ is separably-valued on $I \setminus I_0$.*

*A function $I \ni t \mapsto u(t) \in L$ is called weakly Lebesgue measurable if $\langle\, f|u(\cdot)\,\rangle$ is a measurbale function on $I$ for each $f \in L^\dagger$. A function $I \ni t \mapsto u(t) \in L$ is called a strongly Lebesgue measurable function if there exists a sequence of finitely-valued functions $u_k(\cdot)$ convergeing to $u(\cdot)$ a.e. on $I$.*

*A function $I \ni t \mapsto u(t) \in L$ is called a strongly continuous on $I$ if this function is continuous in the norm of $L$.*

A linear space of strongly measurable functions such that $\|u\|_{(I,L)} = \int_I \|u(t)\|dt$ exists is denoted by $L_1(I,L)$. Then $L_1(I,L)$ is a Banach space with the norm $\|u\|_{(I,L)}$. Analogusly we define the linear space $W_1^1(I,L)$. We say that a function $u \in L_1(I,L)$ belongs to $W_1^1(I,L)$ if there exists a function $\dot{u} \in L_1(I,L)$ such that $u(t) - u(s) = \int_s^t \dot{u}(z)dz$ for any par $s,t \in I, s < t$. Then $W_1^1(I,L)$ is a Banach space with the norm $\|u\|_{(I,L)}^1 = \|u\|_{(I,L)} + \|\dot{u}\|_{(I,L)}$.

A linear space of strongly continuous functions on $I$ is denoted by $C(I,L)$. Analogusly we define the linear space $C^{(k)}(I,L)$. Apparently we have the following imbedding $L_1^1(I,L) \subset C(I,L)$.

LEMMA **0.1** *Let $u \in L_1(I,L)$. Then the following is valid:*

*(i)* $\| \int_I u(t)dt\| \le \|u\|_{(I,L)} = \int_I \|u(t)\|dt$.

*(ii) Let $A$ be an operator in $L$ with the domain $\mathfrak{D}(A) \subset L$ and let $\{u_n(t) \in \mathfrak{D}(A) : n \in \mathbb{N}\}$ be a sequence of finitely-valued elements of $L$. If $Au_n \to w$ in $L_1(I,L)$ then $A\int_I u(t)dt = \int_I Au(t)dt$ and $\|A\int u(t)dt\| \le \|Au\|_{I,L}$.*

*(iii) If $v \in W_1^1(I), u \in W_1^1(I,L)$ and $I = [a,b]$ then*

$$\int_I \dot{v}(t)u(t)\,dt \;+\; \int_I v(t)\dot{u}(t)\,dt \;=\; v(b)u(b) \;-\; u(a)v(a).$$

## 2. Polugrupa

1. Strongly continuous semigroup. We formaly denote it as $U(t) = \exp(-tA)$ having in mind that $A$ is positive definite and therefore expecting that $U(t)$ is not increasing in the norm.

   *EXAMPLE 1* In the first example (Semigroup of time translations) we consider $X = L_2(0,\infty)$ and

   $$t \mapsto (U(t)u)(x) = u(t+x), \quad x \in (0,\infty).$$

   This is obviously a continuous contraction semigroup in $X$. As well we can define $t \mapsto (U(t))^\dagger$ by using $(v|U(t)u) = (U(t)^\dagger v|u)$. Apparently we have $((U(t)^\dagger v|u) = \int_t^\infty v(x-t)u(x)dx$, defining a semigroup of translations in the negative direction. The second example is called the semigroup of heat spreading. Here we have $X = L_2(\mathbb{R})$ and

   $$\begin{aligned} (U(t)u)(x) &= \int_{\mathbb{R}} p(t, x-y)\, u(y)\, dy, \\ p(t,x) &= \frac{1}{\sqrt{2\pi t}\sigma} \exp\Big(-\frac{x^2}{2\sigma^2 t}\Big). \end{aligned}$$

   This is also a continuous contraction semigroup in $X$.

2. $\|U(t)\| \leq \exp(\kappa t)$.

3. The generator of a semigroup.

   *EXAMPLE 2* From the definition $A = -(d/dt)U(t)|t=0$ we have for the semigroup of time translation $A = -d/dx$ defined on $\mathfrak{D}(A) = W_2^1(0,\infty)$. Let us mention that its adjoint is $A^\dagger = d/dx$ on $\mathfrak{D}(A^\dagger) = \dot{W}_2^1(0,\infty)$. Both operators are closed on the specified domains.

4. For $f \in \dot{C}_0^{(1)}([0,\infty))$ and any $u \in L$ the following is valid: $\int f(t)U(t)u\, dt \in \mathfrak{D}(A)$ and $\mathfrak{D}(A)$ is dense in $L$.

5. $T(\lambda, A) = \int \exp(-\lambda t)U(t)dt$.

   *EXAMPLE 3* For the semigroup of time translation we have the resolvents:

   $$\begin{aligned} \big(T(\lambda,A)u\big)(x) &= \exp(\lambda x) \textstyle\int_x^\infty u(z) \exp(-\lambda z)\, dz, \\ \big(T(\lambda,A^\dagger)u\big)(x) &= \exp(-\lambda x) \textstyle\int_0^x u(z) \exp(\lambda z)\, dz. \end{aligned}$$

   From $\|T(\lambda, A)\|_p \leq 1/\lambda$ for $p = 1,\infty$ we conclude that $A$ satisfies the HY-conditions for all $p \in [1,\infty]$.

### Basic theorem

THEOREM **0.1 (Hille-Yoshida)** *Let $A$ be an (unbounded) operator in a Banach space $L$. Then $-A$ is the generator of a strongly continuous contraction semigroup $U(\cdot)$ iff $A$ satisfies the HY-conditions.*

PROOF: For the proof we need the equality:

$$U(t) - V(t) = -\int_0^t U(t-s)\,\big(A - B\big)\, V(s)\, ds, \tag{3}$$

which is valid for any two bounded operators in $L$.

LEMMA **0.2** *Let $U(\cdot), V(\cdot)$ be two strongly continuous semigroups in a Banach space $L$ with the generators $-A, -B$ on the respective domains $\mathfrak{D}(A), \mathfrak{D}(B)$. If The set $\mathfrak{D} = \mathfrak{D}(A) \cap \mathfrak{D}(B)$ is dense in $L$ and $A = B$ on $\mathfrak{D}$ then $U = V$.*

PROOF: For some $\lambda > 0$ the operators $\lambda I + A, \lambda I + B$ map the sets $\mathfrak{D}(A), \mathfrak{D}(B)$ onto $L$. Therefore

$$\mathfrak{R} \;=\; (\lambda I + A)\,\mathfrak{D} \;=\; (\lambda I + B)\,\mathfrak{D}$$

is a dense set in $L$ and $T(\lambda, A) = T(\lambda, B)$ on $\mathfrak{R}$. This implies $T(\lambda, A) = T(\lambda, B)$ on $L$ and $A = B$ on $\mathfrak{D}(A) = T(\lambda, A)L$. **QED**

COROLLARY **0.1** *Let $D \subset \mathbb{R}^d$ be an open set and $U(\cdot)$ be a strongly continuous contraction semigroup in $L_p(D)$ with the generator $-A$ on $\mathfrak{D}(A)$. Then $T(\lambda, A) \geq 0$ on $\mathfrak{D}(A)$ iff $U(t) \geq 0$ on $L_p(D)$.*

**A representation of semigroup in terms of a contour integral**

LEMMA **0.3** *Let $A$ satisfy the HY-conditons and let the operator $T(\pm i\mu, A)$ fulfil the following inequality*

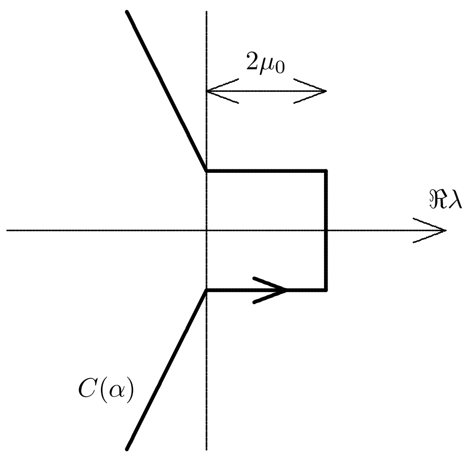
$$\|T(\pm i\mu, A)\| \;\leq\; \frac{\rho}{|\mu|}$$

*for all $|\mu| \geq \mu_0 > 0$. Then the operator valued function $\lambda \mapsto T(\lambda, A)$ is regular analytic on the set*

$$\Lambda(\mu_0, \sigma) \;=\; \Big\{\lambda \,=\, r\,\exp(i\phi),\; r \,\geq\, \mu_0,\; |\phi| \,<\, \frac{\pi}{2} + \sigma\Big\},$$

*where $\sigma = \tan^{-1}(\rho)$.*

A curve $C(\alpha) \subset \Lambda(\mu_0, \sigma)$ for $\alpha < \sigma$ is defined as in Figure.




COROLLARY **0.2** *Let the generator $-A$ of a semigroup $u(\cdot)$ has the properties as in the previous lemma. Then the operators $U(t), t > 0$, can be represented in the following form:*

$$U(t) \;=\; \frac{1}{2\pi i}\int_{C(\alpha)} \exp(\lambda t)\,T(\lambda, A)\,d\lambda.$$

The linear space $\mathfrak{D}(A, t) = U(t)L$ is dense in $L$. Actually a stronger result can be proved.

COROLLARY **0.3** *Let the generator $-A$ of a semigroup $U(\cdot)$ has the properties as in Lemma 0.3. Then for each $t > 0$ and $m \in \mathbb{N}$ the following inclusion is valid $A^m U(t)L \subset L$.*

**3. Elipticki operator**

LEMMA **0.4** *Let the linear operators $A, B$ be defined on the dense manifold $\mathfrak{D} \subset L$ such that $A^{-1}, BA^{-1}$ and $A^{-1}B$ have bounded extensions in $L$. If $\beta_1 = \|BA^{-1}\|$ and $\beta_2 = \|A^{-1}B\|$ are less than 1 then*

*(i)* $(A+B)^{-1} = A^{-1}\sum_k (-1)^k (BA^{-1})^k = \sum_k (-1)^k (BA^{-1})^k A^{-1}$,

*(ii)* $\|(A+B)^{-1}\| \leq \|A^{-1}\| \min\{(1-\beta_1)^{-1}, (1-\beta_2)^{-1}\}$,

*(iv)* $\|B(A+B)^{-1}\|, \|(A+B)^{-1}B\| \leq \min\{\beta_1(1-\beta_1)^{-1}, \beta_2(1-\beta_2)^{-1}\}$.

Here we consider a set $D \subset \mathbb{R}^d$ such that either $D = \mathbb{R}^d$ or $D$ is a Lipsitz domain.

PROPOSITION **0.1** *Let $A_0(\boldsymbol{x}) = -\sum_{ij} \partial_i a_{ij}(\boldsymbol{x})\partial_j$ and $B$ be an operator in $L_2(D)$ such that*

$$\sigma \;=\; \sup\,\{\|BT(\mu, A_0)^{1/2}\| \;:\; \mu \geq 1\} \;<\; \infty.$$

*Then*

*(i)* $A = A_0 + B \in \mathsf{L}(W_2^{-1}, W_2^1)$.

*(ii) For $\lambda$ sufficiently large, $\lambda \geq 1$ the operator $\lambda I + A$ is an one-to-one mapping between the spaces $W_2^{-1}(D)$ and $\dot{W}_2^1(D)$ having the following inverse:*

$$T(\lambda, A) \;=\; T(\lambda, A_0)^{1/2} \sum_{k=0}^{\infty} (-)^k \left(T(\lambda, A_0)^{1/2} B T(\lambda, A_0)^{1/2}\right)^k T(\lambda, A_0)^{1/2}.$$

*(iii) The following inequalities are valid:*

$$\begin{array}{lcl} \|T(\lambda, A)\|_2 & \leq & \left(1-\beta\right)^{-1} \|T(\lambda, A_0)^{1/2}\|_2^2, \\ \|T(\lambda, A)\|_{2,1} & \leq & \left(1-\beta\right)^{-1} \|T(\lambda, A_0)^{1/2}\|_2 \,\|T(\lambda, A_0)^{1/2}\|_{2,1}, \\ \|T(\lambda, A)\|_{\mathsf{L}(W_2^{-1}, W_2^1)} & \leq & \left(1-\beta\right)^{-1} \|T(\lambda, A_0)^{1/2}\|_{2,1}^2, \end{array}$$

*where $\beta = \|T(\lambda, A_0)^{1/2} B T(\lambda, A_0)^{1/2}\|_2 \leq \sigma\lambda^{-1/2}$ is less than 1 for $\lambda$ sufficiently large.*

COROLLARY **0.4** *Let $A$ on $\mathfrak{D}(A)$ be a linear operator having the properties:*

*a)* $\mathfrak{D}(A)$ *is dense in* $L$.

*b)* $\|(\lambda I + A)u\| \geq (\lambda - \kappa)\|u\|$ *for some* $\kappa > 0$ *and all* $u \in \mathfrak{D}(A)$, $\lambda \geq \kappa$.

*c)* $L = (\lambda I + A)\mathfrak{D}(A)$ *for some* $\lambda > 0$.

*Then $\kappa I + A$ fulfils the HY-conditions and $-A$ generates a continuous semigroup $U(\cdot)$ for which*

$$\|\,U(t)\,\| \;\leq\; \exp(\kappa\, t).$$

PROPOSITION **0.2** *Let $A_0(\boldsymbol{x}) = -\sum_{ij} \partial_i a_{ij}(\boldsymbol{x})\partial_j$ be an elliptic differential operator and let either $D = \mathbb{R}^d$ or $D$ be a domain in $\mathbb{R}^d$ with Lipsitz boundary. Then:*

*(i) The extensions $A_1$ to $\mathfrak{D}_1 = \dot{W}_2^1(D) \cap \{A(\boldsymbol{x})u \in L_2(D)\}$ and $A_2$ to $\mathfrak{D}_2 = \dot{W}_2^1(D) \cap \{A(\boldsymbol{x})u \in \dot{W}_2^1(D)\}$ are closed operators in $L_2(D)$ and $\dot{W}_2^1(D)$, respectively, satisfying the HY-condtions.*

*(ii) The operators $-A_1, -A_2$ generate continuous contraction semigroups in $L_2(D)$ and $\dot{W}_2^1(D)$, respectively.*

PROOF: here we consider the case $D = \mathbb{R}^d$. For the proof of existince of semigroup in $W_2^1(\mathbb{R}^d)$ we use the norm

$$|\!|\!| u |\!|\!|_A^2 \;=\; \| \, u \, \|_2^2 \,+\, a(u,u),$$

where the form $a(\cdot,\cdot)$ is defined by (1). This norm is equaivalent with the standard norm of $W_2^1(\mathbb{R}^d)$. Now we have for $u \in \mathfrak{D}(A_0)$:

$$\begin{aligned}
|\!|\!| T(\lambda,A)u |\!|\!|_A^2 \;&=\; \|T(\lambda,A)u\|_2^2 \,+\, a(T(\lambda,A)u, T(\lambda,A)u) \\
&=\; \|T(\lambda,A)\|_2^2 \,+\, \textstyle\sum_{ij} \, (\partial_i T(\lambda,A)u | a_{ij} \, \partial_j T(\lambda,A)u) \\
&=\; \|T(\lambda,A)u\|_2^2 \,+\, \| \, A^{1/2} T(\lambda,A)u \, \|_2^2 = \; \|T(\lambda,A)u\|_2^2 \,+\, \| \, T(\lambda,A)A^{1/2} \, u \, \|_2^2 \\
&\leq\; \lambda^{-2} \Big( \|u\|_2^2 \,+\, a(u,u) \Big) \;=\; \lambda^{-2} \; |\!|\!| u |\!|\!|_A^2 .
\end{aligned}$$

COROLLARY **0.5** *Let $A(\boldsymbol{x})$ be an elliptic differential operator defined by (1). Then its closures $-A_1$ on $\mathfrak{D}_1$ and $-A_2$ on $\mathfrak{D}_2$ generate continuous semigroups $U_1(\cdot)$ in $L_2(D)$ and $U_2(\cdot)$ in $\dot{W}_2^1(D)$, respectively, such that $\|U_\alpha(t)\|_2 \leq \exp(\kappa_\alpha t)$, with $\kappa_\alpha$ depending on the functions $\boldsymbol{b}', \boldsymbol{b}''$ and $c$.*

PROOF: We apply here Corollary 0.4. Hence

$$\|(A_0 + B + \lambda I)u\|^2 \;=\; \|(A_0+B)u\|^2 \,+\, \lambda^2 \, \|u\|^2 \,+\, 2\lambda \big( u \,|\, (A_0+B)u \big).$$

Now we have to estamite the last term from bellow. In the expression $\big(u \,|\, (A_0{+}B)u\big) = (u|A_0u) + (u|Bu)$ the first term is positive. Let us assume that we can estimate the second term as $|(u|Bu)| \leq (u|A_0u) + \beta\|u\|^2$. This would imply

$$\|(A_0 + B + \lambda I)u\|^2 \;\geq\; \big(\lambda^2 \,-\, 2\lambda\beta\big) \, \|u\|^2 \;\geq\; (\lambda \,-\, 2\beta)^2 \, \|u\|^2,$$

i.e. the assertion. Hence, it remains to demonstratre that the assumed estimate of $(u|Bu)$ is possible in the norm of spaces $L_2(D)$ as well as $\dot{W}_2^1(D)$. Only the case $B = \sum_i b_i \partial_i$ is considered. For the case of $L_2(D)$ we use

$$|(u|Bu)| \;\leq\; \varepsilon \, \|Bu\|_2^2 \,+\, \frac{1}{4\varepsilon} \, \|u\|_2^2 \;\leq\; 2 \, \varepsilon \, d \, \|\boldsymbol{b}'\|_\infty^2 \, \underline{M}^{-1} \, a(u,u) \,+\, \frac{1}{4\varepsilon} \, \|u\|_2^2.$$

In the case of problem in $\dot{W}_2^1(D)$ we have $(u|Bu)_{2,1} = (u|Bu)_2 + a(u,Bu)$ so that we have to estimate additionaly $a(u,Bu)$:

$$a(u,Bu) \;=\; (A_0 u \,|\, Bu) \;\leq\; \varepsilon \, a(u, A_0 u) \,+\, \frac{1}{4\varepsilon} \, \|Bu\|_2^2.$$

The quantity $\|Bu\|_2^2$ can be estimated from above by $\sigma_1 a(u,u) + \sigma_2 \|u\|_2^2$, where $\sigma_i$ depend on $\boldsymbol{b}', \boldsymbol{b}'', c$ and $\underline{M}$. **QED**

LEMMA **0.5** *Let $A_0(\boldsymbol{x})$ be given. Its closures $A_1$ in $L_2(D)$ or $A_2$ in $\dot{W}_2^1(\mathbb{R}^d)$ satisfy the following inequalities:*

$$\begin{aligned}
&\| \, T(\pm i\mu, A) \, \|_2 \;\leq\; |\mu|^{-1}, \\
&|\!|\!| \, T(\pm i\mu, A) \, |\!|\!|_A \;\leq\; \big(1 + \gamma^{-1}\big) \, |\mu|^{-1},
\end{aligned}$$

*for each $|\mu| > 0$.*

PROOF: Let $(i\mu + A)w = f, \mu > 0$, where $w = u + iv$ and $u, v \in \mathfrak{D}(A) \subset L_2(\mathbb{R}^d)$. An equaivalent formulation is:

$$\begin{aligned}
A \, u \,-\, \mu \, v \;&=\; f, \\
A \, v \,+\, \mu \, u \;&=\; 0.
\end{aligned} \tag{4}$$

From these expressions we have $\|w\|_2 \leq |\mu|^{-1}\|f\|_2$, i.e. the first assertion. Now we have to prove that (4) has solutions $u, v \in \mathfrak{D}(A)$ for each $f \in L_2(D)$. The considered system is equavalent to $(A^2+\mu^2)v = -\mu f$. Thus (4) is feasible if $A^2$ satisfies the HY-conditions. Apparently $\mathfrak{D}(A^2) = T(1,A)^2L_2(D)$ is dense and $(I+A^2)\mathfrak{D}(A^2) = L_2(D)$ becasue $A^2$ is symmetric on $\mathfrak{D}(A^2)$. It remains to prove that $A^2$ is acretive. The following inequality is valid $\|(\lambda I+A^2)u\|_2^2 \geq \lambda^2\|u\|_2^2$ because $A$ is positive semidefinite. Hence $v = T(\mu^2, A^2)(-\mu f) \in \mathfrak{D}(A^2)$ and $u = -\mu^{-1}Av \in \mathfrak{D}(A)$.

Now we use

$$\begin{aligned} A^2\,u \,-\, \mu\,A\,v \;&=\; A\,f, \\ A^2\,v \,+\, \mu\,A\,u \;&=\; 0. \end{aligned} \tag{5}$$

for a pair $w \in \mathfrak{D}(A^2), f \in \mathfrak{D}(A)$. This new pair of equalities imply $\mu(w|Aw) = -(v|Af)$ so that

$$\underline{M}\,\mu\,\|\nabla u\|_2^2 \;\leq\; -\,a(v,f).$$

By using $|a(v,f)| \leq a(v,v)^{1/2}a(f,f)^{1/2}$ we get the second assertion. Regarding System (5) we have to mention taht its feasibility can be proved by the same arguments as for Sytem (4). **QED**

COROLLARY **0.6** *Let $A(\boldsymbol{x}) = A_0(\boldsymbol{x})+B(\boldsymbol{x})$ be a general elliptic differential operator on $\mathbb{R}^d$ and $A_1, A_2$ be as in the previous lemma. Then there exist positive numbers $\rho_1, \rho_2$ and $\mu_0$ such that*

$$\|T(i\mu, A_\alpha)\|_\alpha \;\leq\; \frac{\rho_\alpha}{|\mu|}, \quad \alpha = 1, 2,$$

*are valid for $\mu \geq \mu_0$ in the norms $\|\cdot\|_1, \|\cdot\|_2$ of respective spaces $L_2(D)$ and $\dot{W}_2^1(D)$.*

Proof: We do not loose on generality by considering here the case $B(\boldsymbol{x}) = \sum_i b_i(\boldsymbol{x})\partial_i$. Now we apply Lemma 0.4 to the pair $B(\boldsymbol{x})$ and $i\mu I + A_0(\boldsymbol{x})$. In the case of $L_2(D)$ norm we start our calculations by proving the inequality

$$\|\, A_0^{1/2}T(i\mu, A_0)\,\|_2 \;\leq\; |\mu|^{-1/2}.$$

From (4) we have $\|A_0^{1/2}w\|_2^2 = (u|f)$ implying $\|A_0^{1/2}T(i\mu, A_0)f\|_2^2 \leq |\mu|^{-1}\|f\|_2^2$. Therefore

$$\|BT(i\mu, A_0)u\|_2^2 \;\leq\; \|b\|_\infty^2\, d\,\|\nabla T(i\mu, A_0)u\|_2^2 \;\leq\; \frac{\|b\|_\infty^2 d}{\underline{M}}\,\frac{1}{|\mu|}\,\|u\|_2^2,$$

so that

$$\beta \;=\; \|\, BT(i\mu, A_0)\,\|_2 \;\leq\; \frac{1}{2}$$

for $\mu| \geq \mu_0 = 2\|b\|_\infty(d/\underline{M})^{1/2}$.

In the case of $\dot{W}_2^1(D)$-space we use Lemma 0.4 by proving $\|\nabla T(i\mu, A_0)B\| \leq \sigma|\mu|^{-1/2}$ with some $\sigma > 0$.

$$\|\nabla T(i\mu, A_0)Bu\|_2^2 \;\leq\; \frac{1}{\underline{M}}\,\|A_0^{1/2}\,T(i\mu, A_0)Bu\|_2^2 \;\leq\; \frac{1}{\underline{M}}\,\|A_0^{1/2}T(i\mu, A_0)\|_2^2\,\|Bu\|_2^2,$$

implying as in the previous case

$$\|\nabla T(i\mu, A_0)Bu\|_2^2 \;\leq\; \frac{1}{2}\,\|\nabla u\|_2^2$$

for $|\mu| > \mu_0$. **QED**

COROLLARY **0.7** *There exists a continuous contraction semigroup in $W_2^{-1}(D)$ generated by the elliptic differential operator $-A(\boldsymbol{x})$. For $t > 0$ and any $\mu \in W_2^{-1}(D)$ the following inclusion must be valid: $U(t)\mu \in \mathfrak{D}(A^m)$, where $m \in \mathbb{N}$ is arbitrary.*

Only a representation of the semigroup $U(t)$ generated by $-A_0(\boldsymbol{x})$ has to be written down. Let us consider the case $W_2^{-1}(\mathbb{R}^d)$. We know that an elements $f$ of $W_2^{-1}(\mathbb{R}^d)$ has a representations $\mu = f_0 + \sum_i \partial_i f_i$. Since the closure $\lambda I + A \in \mathsf{L}(W_2^{-1}, W_2^1)$ is an one-to-one mapping between $W_2^{-1}(\mathbb{R}^d)$ and $W_2^1(\mathbb{R}^d)$ we can write $(I+A)T(1,A)\mu = \mu$. However, for $g = T(1,A)\mu$, $f_0 \in L_2(\mathbb{R}^d)$, $f_i \in W_2^1(\mathbb{R}^d)$, we have

$$U(t)\mu \;=\; (I+A)U(t)g \;=\; U(t)g \;+\; \sum_i^d \partial_i\big(a_{ij}\,\partial_j U(t)g\big).$$

Due to the fact that $W_2^1(\mathbb{R}^d)$ is dense in $L_2(\mathbb{R}^d)$ we conclude that this equality is valid for $f_i \in L_2(\mathbb{R}^d)$ as well. In addition, from the obtained representation we have $U(t)\mu \in \mathfrak{D}(A^m)$ for any $t > 0$ and $m \in \mathbb{N}$.

Let us mention that $\|(I+A)g\|_{2,1} = \|\mu\|_{2,1}$ as can be seen from

$$\langle\, \phi \,|\, (I+A)g \,\rangle \;=\; \langle\, \psi \,|\, \mu \,\rangle,$$

where $\phi \in \dot{W}_2^1(D) \Rightarrow \psi = T(\lambda, A)(I+A)\phi \in \dot{W}_2^1(D)$.

## 4. Paraboloc equations

Let $D$ be either $\mathbb{R}^d$ or a bounded domain in $D \subset \mathbb{R}^d$ with the Lipsitz boundary. The differential elliptic operator $A(\boldsymbol{x})$ has the closure $A_D$ from $\dot{W}_2^1(D)$ into $W_2^{-1}(D)$. Then the problem:

$$\begin{aligned} \partial_t u(t) \,+\, A\,u(t) \;&=\; \mu(t), \quad \text{for} \quad t > 0, \\ u(0) \;&=\; u_0, \end{aligned} \tag{6}$$

has a solution for $u_0, \mu(t)$ belonging to a certain class of spaces. If a solution exists it can be represented in a closed form as:

$$u(t) \;=\; U(t)\,u_0 \;+\; \int_0^t U(t-s)\,\mu(s)\,ds. \tag{7}$$

In order to formulate the obtained results in a convenient way we split the solution $u$ of (7) as $u = u_1 + u_2$, where $u_1$ is a linear function of the initial data $u_0$ and $u_2$ is the linear function of the right hand side $\mu$. Furthermore we denote shortly by $L(\alpha), \alpha \in \{-1, 0, 1\}$ the Hilbert spaces $\dot{W}_2^1(D), L_2(D)$ and $W_2^{-1}(D)$, respectively.

THEOREM **0.2** *There exists a unique solution (7) of (6) with the following properties:*

*(i) If* $u_0 \in L(\alpha)$ *and* $\mu \in L_1((0,\infty), L(\alpha))$ *then* $u \in L(\alpha))$ *for each* $t \geq 0$.

*(ii) If* $u_0 \in L(\alpha)$ *and* $\mu \in W_1^1((0,\infty), L(\alpha))$ *then* $u_1, u_2, u \in C((0,\infty, L(\alpha))$.

*(iii) If* $u_0 \in W_2^{-1}(D)$ *then* $A^m u_1(t) \in L_2(D)$ *for any pair* $t > 0, m \in \mathbb{N}$.

*(iv) If* $u_0 \in \dot{W}_2^1(D)$ *and* $\mu = 0$ *then* $u \in C((0,\infty, \dot{W}_2^1(D)) \cap W_2^1\big((0,\infty)\times D\big)$*, i.e.* $\dot{u} \in L_2\big((0,\infty)\times D\big)$.

*(v) If* $\mu \in W_1^1((0,\infty), W_2^{-1}(D))$ *then* $u_2 \in C((0,\infty), \dot{W}_2^1(D))$.

Proof: To prove (iv) we need the equality

$$\int_s^t \|\dot{u}(z)\|_2^2\,dz \;+\; \frac{1}{2}\,a(u(t),u(t)) \;=\; \frac{1}{2}\,a(u(s),u(s)).$$

Assuming $a(v,v) \geq \underline{M}\|\nabla v\|_2^2$ we get the assertion. **QED**

# NUMERICKI POSTUPCI

Promatramo izvorni problem i njegove diskretizacije:

$$\begin{aligned} \dot{u}^*(t) &= -A\,u^*(t) + \mu(t), & \dot{\mathbf{u}}_n(t) &= -A_n\mathbf{u}_n(t), + \boldsymbol{\mu}_n(t), \\ u^*(0) &= u_0, & \mathbf{u}_n(0) &= \mathbf{u}_{0n}, \end{aligned} \tag{8}$$

gdje je mreza ili $G_n$ ili $G_n(D)$, te je $\mathbf{u}_{0n}$ definiran ovako. Pridruzimo funkciji $u_0$ funkciju $\hat{u}_0(n)$ po izrazu (2). Tada definiramo $\mathbf{u}_{0n} = \Phi^{-1}\hat{u}_0(n)$.

LEMMA **0.6** *If $\boldsymbol{\mu}_n = \mathbf{0}$ in (8) then the grid solutions $\mathbf{u}_n(t)$ fulfil the following inequlity $\|\mathbf{u}_n(t)\|_{2,1} \le \|\mathbf{u}_n(0)\|_{2,1}$.*

PROOF: Dobijemo slijedece dvije jednakosti:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\mathbf{u}_n(t)\|_2^2 + (\mathbf{u}_n(t)\,|\,A_n\,\mathbf{u}_n(t)) &= \langle\,\mathbf{u}_n(t)\,|\,\boldsymbol{\mu}_n(t)\,\rangle, \\ \|\dot{\mathbf{u}}_n(t)\|_2^2 + \frac{1}{2}\frac{d}{dt}(\mathbf{u}_n(t)\,|\,A_n\mathbf{u}_n(t)) &= \langle\,\dot{\mathbf{u}}_n(t)\,|\,\boldsymbol{\mu}_n(t)\,\rangle. \end{aligned}$$

Integriranjem tih jednakosti u intervalu $[s,t]$ dobijemo

$$\begin{aligned} \frac{1}{2}\|\mathbf{u}_n(t)\|_2^2 + \underline{M}\int_s^t \sum_i \|U_i(r_i)\mathbf{u}_n(z)\|_2^2\,dz &\le \frac{1}{2}\|\mathbf{u}_n(s)\|_2^2 \\ &\quad + \int_s^t \langle\,\mathbf{u}_n(z)\,|\,\boldsymbol{\mu}_n(z)\,\rangle\,dz \\ \frac{1}{2}\underline{M}\sum_i \|U_i(r_i)\mathbf{u}_n(t)\|_2^2 + \int_s^t \|\dot{\mathbf{u}}_n(z)\|_2^2\,dz &\le \frac{1}{2}\underline{M}\sum_i \|U_i(r_i)\mathbf{u}_n(0)\|_2^2 \\ &\quad + \int_s^t \langle\,\dot{\mathbf{u}}_n(z)\,|\,\boldsymbol{\mu}_n(z)\,\rangle\,dz. \end{aligned}$$

Ako stavimo $\boldsymbol{\mu}_n = \mathbf{0}$, $s \downarrow 0$ i zbrojimo, dobijemo tvrdnju. **QED**

Definirajmo sada $u(n,t) = \Phi\mathbf{u}_n(t)$ i niz aproksimativnih rjesenja $\mathfrak{U} = \{u(n,\cdot) : n \in \mathbb{N}\}$. Iz predhodnog stavka i Teorema 1.1. II. slijedi: $\|u(n,t)\|_{2,1}^2 \le h^d\|\mathbf{u}_n(t)\|_{2,1} \le h^d\|\mathbf{u}_{0n}\|_{2,1}$. Sada, iz definicije $\mathbf{u}_{0n}$ imamo ovaj rezultat:

$$\limsup_n \|u(n,t)\|_{2,1} \le \|u_0\|_{2,1},$$

t.j. niz $\mathfrak{U}$ je ogranicen u $\dot{W}_2^1(D)$. To znaci da postoji slabo konvergentni podniz, $u(t) = w\text{-}\lim_n u(n,t) \in \dot{W}_2^1(D)$ za svaki $t > 0$. Treba sada dokazati jaku konvergenciju dobivenog niza $\mathfrak{U} = \{u(n,\cdot) : n \in \mathbb{N}\}$ ka rjesenju u prostoru $L_2((0,\infty), \dot{W}_2^1(D))$.

Let $u^*(t) = U(t)u_0, u_0 \in \dot{W}_2^1(D)$ and let $u^*(n,t)$ be defined by (2). Then $\mathbf{u}_n(t) = \Phi_n^{-1}u^*(n,t)$ satisfies the system

$$\begin{aligned} \dot{\mathbf{u}}_n^*(t) + A_n\,\mathbf{u}_n^*(t) &= \boldsymbol{\mu}_n(t), \\ (\boldsymbol{\mu}_n(t))_{\boldsymbol{k}} &= \big(A_n\mathbf{u}_n^*(t)\big)_{\boldsymbol{k}} - h^{-d}\,a(\phi_{\boldsymbol{k}}, u^*(t)). \end{aligned} \tag{9}$$

These equalities can be easily verified by making the scalar product of the first equality with $\mathbf{v}_n$. Hence:

$$h^d\,\langle\,\mathbf{v}_n\,|\,\boldsymbol{\mu}_n(t)\,\rangle = h^d\,a_n(v(n), u^*(n,t)) - a(v(n), u^*(t)).$$

From Theorem on consistency of the discretized forms $a_n$ we have

COROLLARY **0.8** *If $v = w{-}\lim v(n) \in \dot{W}_2^1(D)$ exists then for each $t \ge 0$ the following equality is valid:*

$$\lim_n h^d\,\langle\,\mathbf{v}_n\,|\,\boldsymbol{\mu}_n(t)\,\rangle = 0.$$

THEOREM **0.3** *Let $I = [0, T]$ and $u(n,t) = \Phi_n \mathbf{u}_n(t)$ be approximate solutions defined by ODE (8) with $\boldsymbol{\mu}_n = \mathbf{0}$. Then $u^* = s\text{–}\lim_n u(n) \in L_2(I, \dot{W}_2^1(D))$.*

Proof: The convergence of approximate solutions $u(n,t) \in E_n(D)$ to the original solution $u^*(t)$ is proved by considereing the difference $\| \mathbf{u}_n(t) - \mathbf{u}_n^*(t) \|_{2,1}$. We start to evaluate the quantity

$$
\begin{aligned}
&\delta(\mathbf{u}_n(t), \mathbf{u}_n^*(t)) \ = \\
&\tfrac{1}{2} \| \mathbf{u}_n(t) - \mathbf{u}_n^*(t) \|_2^2 \, ds \ + \ \int_0^t \langle\, \mathbf{u}_n(s) - \mathbf{u}_n^*(s) \,|\, A_n\, (\mathbf{u}_n(s) - \mathbf{u}_n^*(s)) \,\rangle \, ds.
\end{aligned} \tag{10}
$$

The squares of norms of each term can be rewritten as the sum of three terms which have to be calculated by means of the following initial value problems:

$$
\begin{aligned}
\dot{\mathbf{u}}_n(t) \ + \ A_n\, \mathbf{u}_n(t) \ &= \ \mathbf{0}, \\
\dot{\mathbf{u}}_n^*(t) \ + \ A_n\, \mathbf{u}_n^*(t) \ &= \ \boldsymbol{\mu}_n(t), \\
\mathbf{u}_n(0) \ = \ \mathbf{u}_n^*(0) \ &= \ \mathbf{u}_{0n}.
\end{aligned}
$$

The following espressions can be obtained:

$$
\begin{aligned}
&\frac{1}{2} \| \mathbf{u}_n(t) \|_2^2 \ + \ \int_0^t \langle\, \mathbf{u}_n(s) \,|\, A_n \mathbf{u}_n(s) \,\rangle \, ds \ = \ \frac{1}{2} \| \mathbf{u}_n(0) \|_2^2, \\
&\frac{1}{2} \| \mathbf{u}_n^*(t) \|_2^2 \ + \ \int_0^t \langle\, \mathbf{u}_n^*(s) \,|\, A_n \mathbf{u}_n^*(s) \,\rangle \, ds \ = \ \frac{1}{2} \| \mathbf{u}_n^*(0) \|_2^2 \ + \ \int_0^t \langle\, \mathbf{u}_n^*(s) \,|\, \boldsymbol{\mu}_n(s) \,\rangle \, ds, \\
&(\mathbf{u}_n(t) \,|\, \mathbf{u}_n^*(t)) \ + \ \int_0^t \Big[ \langle\, \mathbf{u}_n(s) \,|\, A_n \mathbf{u}_n^*(s) \,\rangle \ + \ \langle\, \mathbf{u}_n^*(s) \,|\, A_n \mathbf{u}_n(s) \,\rangle \Big] \, ds \ = \\
&(\mathbf{u}_n(0) \,|\, \mathbf{u}_n^*(0)) \ + \ \int_0^t \langle\, \mathbf{u}_n(s) \,|\, \boldsymbol{\mu}_n(s) \,\rangle \, ds.
\end{aligned}
$$

Hence

$$
h^d\, \delta(\mathbf{u}_n(t), \mathbf{u}_n^*(t)) \ = \ -\, h^d \int_0^t \langle\, \mathbf{u}_n(s) + \mathbf{u}_n^*(s) \,|\, \boldsymbol{\mu}_n(s) \,\rangle \, ds.
$$

In the expression of $\delta(\mathbf{u}_n(t), \mathbf{u}_n^*(t))$ we have an integral. Let us prove first that the function under the interal sign is bounded uniformly with respect to $n$. By using

$$
\begin{aligned}
\|u^*(n,t)\|_{2,1} \ &\le \ \|u_0\|_{2,1}, \\
\|u(n,t)\|_{2,1} \ &\le \ (1 - \sigma^2)^{-1/2}\, \|u_0\|_{2,1},
\end{aligned}
$$

we can get the following estimates

$$
h^d\, |\langle\, \mathbf{u}_n(s) + \mathbf{u}_n^*(s)) \,|\, \boldsymbol{\mu}_n(s) \,\rangle| \ \le \ \frac{4\, \overline{M}}{1 - \sigma^2}\, \|u_0\|_{2,1}^2.
$$

where we used the result of Theorem 1.1, II. part, connecting the norms of $\mathbf{u}_n$ and $u(n)$. Now, by using Lemma 0.6 and Lebesque theorem on limes under the integral sign we get:

$$
\lim_n \ h^d\, \delta(\mathbf{u}_n(t), \mathbf{u}_n^*(t)) \ = \ 0. \tag{11}
$$

Thus we have

$$
\lim_n \ \|\, u(n,t) - u^*(n,t) \,\|_2 \ = \ 0,
$$

and consequently $u^* = s\text{–}\lim u(n) \in L_1(I, L_2(D))$. To prove $u^* = s\text{–}\lim u(n) \in W_1^1(I, L_2(D))$ we have to use Lemma 1.2, II. part. Here we use the following abreviations $\mathbf{w}_n(t) = \mathbf{u}_n(t) - \mathbf{u}_n^*(t)$ and $w(n,t) = \Phi_n \mathbf{w}_n(t)$:

$$
\sum_{i=1}^{d} \|\partial_i\, w(n,s)\|_2^2 \ \le \ h^d \sum_{i=1}^{d} \| U_i\, \mathbf{w}_n(s) \|_2^2 \ \le \ \underline{M}^{-1}\, h^d\, a_n(\mathbf{w}_n(s), \mathbf{w}_n(s)).
$$

Then from (11) we heve the assertion. **QED**

Ippitni zadatak **0.1 (N. Sandric)** *Ovo je pomocni problem. Neka je $I = [0,1]$ i*

$$\begin{aligned} -u'' + \rho\, v\, u' + \lambda\, u \;&=\; f, \quad x \in (0,1), \\ u(0) \;=\; u(1) \;&=\; 0, \end{aligned} \tag{12}$$

LEMMA **0.7** *Neka su $v, f \in C(I), f \geq 0$. Ako postoji rjeswnje $u \in \dot{C}^{(2)}(I)$, tada je $u \geq 0$, i $\|u\|_\infty \leq \|f\|_\infty/\lambda$.*

LEMMA **0.8** *Ako je $\|v\|_\infty \leq \|f\|_\infty/\lambda$ i*

$$\lambda \;>\; \left(\frac{\rho^2 \|f\|_\infty^2}{2}\right)^{1/3},$$

*onda vrijedi slijedece:*

*(i)* $u \in \dot{C}^{(2)}(I)$.

*(ii)* $\|u\|_\infty \leq \|f\|_\infty/\lambda$.

Proof: Imamo $\|f\|_2 \leq \|f\|_\infty$. Oznaka $\beta = \|f\|_\infty/\lambda$. Dalje je $\|u\|_2 \leq \|u'\|_2$. Sluzimo se sa $ab \leq \varepsilon a^2 + b^2)/(4\varepsilon)$. Iz (12) imamo

$$\|u'\|_2^2\,(1 - \varepsilon\rho\beta) \;+\; \|u\|_2^2\Big(\lambda \;-\; \rho\beta\frac{1}{4\varepsilon}\Big) \;\leq\; \|u\|_2\,\|f\|_2. \tag{13}$$

Za $\varepsilon = (2\rho\beta)^{-1}$ se dobije prva () $= 1/2$. Tada je druga () jednaka $\lambda - \rho^2\|f\|_2/(2\lambda^2)$. To mora biti pozitivno. Onda iz 1. clana i desne strane, zajedno s (13) dobijemo slijedece prve dvije nejednakosti

$$\begin{aligned} \|u\|_2 \;&\leq\; 2\,\|f\|_2, \\ \|u'\|_2 \;&\leq\; 2\,\|f\|_2, \\ \|u(\cdot + h) - u(\cdot)\|_\infty \;&\leq\; 2\sqrt{h}\,\|f\|_2, \end{aligned}$$

dok je 3. nejednakost njihova posljedica. U njoj produzimo $u$ sa $I$ na $\mathbb{R}$ i promatramo normu u $L_\infty(\mathbb{R})$. Iz (12) dobijemo analognu nejednakost za $\|u'(\cdot + h) - u'(\cdot)\|_2$.

Sada promatramo niz problema:

$$\begin{aligned} -u_n'' + \rho\, u_{n-1}\, u_n' + \lambda\, u_n \;&=\; f, \quad x \in (0,1), \\ u_n(0) \;=\; u_n(1) \;&=\; 0. \end{aligned} \tag{14}$$

LEMMA **0.9** *Neka su $u_0 = 0$ i $\lambda$ kao u Lemmi 0.8. Tada niz rjesenja $\mathfrak{U} = \{u_n : n \in \mathbb{N}\}$ zadaca (14) je ravnomjerno neprekidan u $\dot{C}^{(1)}(I)$.*

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