MREZA I MREZNE FUNKCIJE

1 Grids and associated functions

Let the orthogonal coordinate system in $\mathbb{R}^d$ be determined by unit vectors $\boldsymbol{e}_i$, and let us define the set G_n by:

$$G_n \;=\; \{\boldsymbol{x} \;=\; h(n) \sum_{l=1}^{d} k_l\, \boldsymbol{e}_l \;:\; k_l \in \mathbb{Z}\}, \tag{1}$$

where $h(n) = 2^{-n}$ is called the grid-step. A grid-step is usually denoted by h and only if necessary by $h(n)$ or $h(n) = 2^{-n}$. Elements of G_n are called grid-knots and the constructed sets $G_n, n \in \mathbb{N}$ are called grids. Sometimes we say that G_n discretize $\mathbb{R}^d$. Accordingly, the subgrids $G_n(D) \subset G_n$ defiend by $G_n(D) = G_n \cap D$ are called discretizations of D. To each $\boldsymbol{v} \in G_n$ there corresponds a grid-cube $C_n(\boldsymbol{1}, \boldsymbol{v}) = \prod_1^d [v_j, v_j + h)$, where v_j are coordinates of $\boldsymbol{v} \in G_n$. Cubes $C_n(\boldsymbol{1}, \boldsymbol{v})$ define a decomposition of $\mathbb{R}^d$ into disjoint sets. Appart from the basic cubes, $C_n(\boldsymbol{1}, \boldsymbol{v}), \boldsymbol{v} \in G_n$, we need for constructions larger sets. Let $\boldsymbol{p} \in \mathbb{N}^d$. Then

$$C_n(\boldsymbol{p}, \boldsymbol{v}) \;=\; \prod_{i=1}^{d} [v_i, v_i + hp_i)$$

are apparently rectangles with "lower left" vertices $\boldsymbol{v}$ and edges of size hp_i. These rectangles define a partition of $\mathbb{R}^d$ as well. The considered cubes $C_n(\boldsymbol{1}, \boldsymbol{v})$ and rectangles $C_n(\boldsymbol{p}, \boldsymbol{v})$ are semi-closed in the sense that they contain only one of their 2^d vertices.

Basic cubes are defiend by their "lower left" corners. Appart from these basic cubes for our constructions we need closed rectangles,

$$S_n(\boldsymbol{p}, \boldsymbol{v}) \;=\; \prod_{i=1}^{d} [v_i - hp_i\,,\, v_i + hp_i], \tag{2}$$

which are defined by central grid-knots $\boldsymbol{v}$. Apparently, $S_n(\boldsymbol{p}, \boldsymbol{v})$ is the union of closures of those basic cubes $C_n(\boldsymbol{p}, \boldsymbol{x})$ which share the grid-knot $\boldsymbol{v}$.

The grids G_n of (1) are homogeneous with respect to translations in the direction of coordinate axes, i.e $\boldsymbol{x} \in G_n, \boldsymbol{t} = hp_i\boldsymbol{e}_i \Rightarrow \boldsymbol{x} + \boldsymbol{t} \in G_n$ for any $i \in \{1, 2, \ldots, d\}$ and $p_i \in \mathbb{Z}$. There exist subsets of G_n which are also homogeneous in the defiend sense. Let $\boldsymbol{r}_0 \in G_n$ and $\boldsymbol{r} = (r_1, r_2, \ldots, r_d) \in \mathbb{N}^d$ be fixed. The set

$$G_n(\boldsymbol{r}_0, \boldsymbol{r}) \;=\; \{\boldsymbol{r}_0 \,+\, h \sum_{l=1}^{d} k_l\, r_l\, \boldsymbol{e}_l \;:\; k_l \in \mathbb{Z}\} \tag{3}$$

is a subsets of G_n with the following feature $\boldsymbol{x} \in G_n(\boldsymbol{r}_0, \boldsymbol{r}), \boldsymbol{t} = hp_ir_i\,\boldsymbol{e}_i \Rightarrow \boldsymbol{x}+\boldsymbol{t} \in G_n(\boldsymbol{r}_0, \boldsymbol{r})$. A grid (3) is denoted by $G_n(R)$, where R stands shortly for the $2d$ parameters $\boldsymbol{r}_0, \boldsymbol{r}$.

Let $h_0 = 2^{-n_0}$ for some $n_0 \in \mathbb{N}$, $\boldsymbol{p} \in \mathbb{N}^d$ and let D be a connected set with the structure $D = \cup_{\boldsymbol{v} \in F_n} C_n(\boldsymbol{p}, \boldsymbol{v})$, where $F_n \subset G_n$. For the subgrid $G_n(D) = D \cap G_n(R)$ the set $G_n(D)$ is discrete and therefore its interior, closure and boundary are defined indirectly, $int\big(G_n(D)\big) = G_n(D) \cap int(D)$, $cls\big(G_n(D)\big) = G_n(R) \cap \overline{D_l}$ and $bnd\big(G_n(D)\big)$ is the difference of $cls\big(G_n(D)\big)$ and $int\big(G_n(l)\big)$. Apparently, $int\big(G_n(D)\big) \subseteq G_n(D) \subseteq cls\big(G_n(D)\big)$. Let a finite collection of sets $D_l, l \in \mathcal{L}$ makes a partition of $\mathbb{R}^d$, where each D_l has the structure like the described set D. Then $G(l) = D_l \cap G_n(R)$ make a partition of G_n.

Each $\boldsymbol{x} \in G_n$ can be indexed by $\boldsymbol{m} \in \mathbb{R}^d$, where $\boldsymbol{x} = h\boldsymbol{m}$. Similarily, we index grid-knots of $G_n(\boldsymbol{r}_0, \boldsymbol{r})$ by those $\boldsymbol{m} \in \mathbb{Z}^d$ for which there holds $\boldsymbol{x} = \boldsymbol{r}_0 + h\sum_l m_l r_l \boldsymbol{e}_l$. Therefore, we define the sets $I_n = \mathbb{Z}^d$ and $I_n(R) \subset I_n$, indexing the grid-knots of G_n and $G_n(R)$. In this work frequently utilized grids and the respective index sets are $G_n,\ I_n, G_n(R),\ I_n(R),\ G_n(l),\ I_n(l),\ G_n(R, D), I_n(R, D)$.

The *shift operator* $Z(\boldsymbol{x}), \boldsymbol{x} \in \mathbb{R}^d$, acting on functions $f : \mathbb{R}^d \mapsto \mathbb{R}$, is defined by $\big(Z(\boldsymbol{x})f\big)(\boldsymbol{x}) = f(\boldsymbol{x}+\boldsymbol{z})$. Similarly we define the discretized shift operator by $\big(Z_n(r,i)\mathbf{u}_n\big)_{\boldsymbol{k}} = (\mathbf{u}_n)_{\boldsymbol{l}}$, where $\boldsymbol{l} = \boldsymbol{k} + rh\boldsymbol{e}_i$.

Discretization of differential operators. A function $u \in C^{(1)}(\mathbb{R}^d)$ has continuous partial derivatives $\partial_i u, i = 1, 2, \ldots .d$. With respect to a grid step h, the partial derivatives are discretized by forward/backward finite difference operators in the usual way,

$$\begin{array}{rcl} \eth_i(t)u(\boldsymbol{x}) & = & \frac{1}{t}\big(u(\boldsymbol{x}+t\boldsymbol{e}_i) \,-\, u(\boldsymbol{x})\big), \\ \widehat{\eth}_i(t)u(\boldsymbol{x}) & = & \frac{1}{t}\big(u(\boldsymbol{x}) \,-\, u(\boldsymbol{x}-t\boldsymbol{e}_i)\big), \end{array} \quad \boldsymbol{x} \in \mathbb{R}^d,\ t \neq 0. \tag{4}$$

Let $r \in \mathbb{Z} \setminus \{0\}$. Discretizations of the functions $\partial_i u$ on G_n, denoted by $U_i(r)\mathbf{u}_n, V_i(r)\mathbf{u}_n$, are defined by:

$$\big(U_i(r)\,\mathbf{u}_n\big)_{\boldsymbol{m}} \;=\; \eth_i(rh)\,u(\boldsymbol{x}_{\boldsymbol{m}}), \quad \big(V_i(r)\,\mathbf{u}_n\big)_{\boldsymbol{m}} \;=\; \widehat{\eth}_i(rh)\,u(\boldsymbol{x}_{\boldsymbol{m}}).$$

Then

$$\begin{array}{rcl} U_i(r) & = & (rh)^{-1}(Z_n(r,i) \,-\, I), \\ V_i(r) & = & (rh)^{-1}\big(I - Z_n(-r,i)\big) \;=\; U_i(-r) \;=\; -\,U_i(r)^T. \end{array}$$

Therefore we have $U_i(-r) = U_i(r)\,Z_n(-r,i) = Z_n(-r,i)\,U_i(r)$, and similarly for $V_i(r)$.

In accordance with the previous terminology, we say that $\partial_i, \sum_{ij}\partial_i a_{ij}\partial_j$ *etc.* are differential operators on $\mathbb{R}^d$ or D. We say that their discretizations are defined on G_n or $G_n(D)$. In particular, discretizations of the differential operator $A(\boldsymbol{x})$ are denoted by A_n. Naturally, matrices A_n are the main object in this work.

1.1 Relations between $l(G_n(R))$ and W_2^1-spaces

The discretization of a function $u \in C(\mathbb{R}^d)$ on G_n is denoted by $\mathbf{u}_n$ and defined by values at grid-knots, $\big(\mathbf{u}_n\big)_{\boldsymbol{m}} = u(\boldsymbol{x}_{\boldsymbol{m}})$ where $\boldsymbol{x}_{\boldsymbol{m}} = (m_1h, m_2h, \ldots, m_dh) \in G_n$, and $\boldsymbol{m} = (m_1, m_2, \ldots, m_d)$ is a multi-index. The function $\mathbf{u}_n$ is usually called a grid function. We denote the linear spaces of discretizations by $l(G_n)$ or $l(G_n(D))$. Elements of $l(G_n)$ are also called columns. The corresponding L_p-spaces are denoted by $l_p(G_n)$ or $l_p(G_n(D))$, and their norms by $\Vert\cdot\Vert_p$. The duality pairing of $\mathbf{v} \in l_q(G_n)$ and $\mathbf{u} \in l_p(G_n)$ is denoted by $\langle\,\mathbf{v}|\mathbf{u}\,\rangle$. The scalar product in $l_2(G_n)$ is denoted by $\langle\,\cdot|\cdot\,\rangle$ and sometimes by $(\cdot|\cdot)$. The norm of $l_p(G_n(R))$ is denoted by $\Vert\cdot\Vert_{Rp}$. For $p \in [1,\infty)$ this norm is defined by:

$$\Vert\mathbf{u}\Vert_{Rp} \;=\; \Big[\,vol(R)\sum_{\boldsymbol{k}\in I_n(R)} |u_{\boldsymbol{k}}|^p\,\Big]^{1/p},$$

where $vol(R) = \prod_{i=1}^d r_i$. Finally, for $p = \infty$ we have $\Vert\mathbf{u}\Vert_{R\infty} = \sup\{|u_{\boldsymbol{k}}| : \boldsymbol{k} \in I_n(R)\}$.

Let us define the quadratic functional on $l(G_n)$ by $q(\mathbf{u}) = \sum_i^d \Vert U_i\mathbf{u}\Vert_2^2$ and $q_R(\mathbf{u}) = vol(R)\sum_i^d \Vert U_i(r_i)\mathbf{u}\Vert_{R2}^2$ on $l(G_n(R))$. It is understood $q_R = q$ for $G_n(R) = G_n$. There exist symmetric matrices Q_n on G_n such that $q_R(\mathbf{u}) = \langle\,\mathbf{u}|Q_n\mathbf{u}\,\rangle$. A discrete analog of W_2^1-spaces is the spaces $w_2^1(G_n(R))$ of those $\mathbf{u}_n \in l(G_n(R))$ for which the norm $\Vert\cdot\Vert_{R2,1}$:

$$\Vert\mathbf{u}\Vert_{R2,1}^2 \;=\; \Vert\mathbf{u}\Vert_{R2}^2 \;+\; q_R(\mathbf{u}), \tag{5}$$

is finite. By convention $\Vert\cdot\Vert_{2,1} = \Vert\cdot\Vert_{R2,1}$ for $r_i = 1$. The subspace of grid-functions $\mathbf{u} \in w_2^1(G_n(R))$ for which $\mathbf{u}_n = \mathbb{1}_{G_n(D)}\mathbf{u}_n$ is denoted by $w_2^1(G_n(R,D))$. Hence, $w_2^1(G_n(R,D))$ for $\boldsymbol{r} = \mathbf{1}$ is denoted by $w_2^1(G_n(D))$. The restriction of q_R on $G_n(R,D)$ is represented as $q_R(\mathbf{u}) = \langle\,\mathbf{u}|Q_n(D)\mathbf{u}\,\rangle$, where $Q_n(D)$ is a symmetric matrix on $G_n(R,D)$.

LEMMA **1.1** *Let D be bounded. Then the norms $\Vert\cdot\Vert_{2,1}$ and $q_R(\cdot)^{1/2}$ are equivalent in $w_2^1(G_n(R,D))$,*

$$q_R(\cdot)^{1/2} \;\geq\; \beta\,\Vert\cdot\Vert_{R2,1},$$

where β is independent of n.

PROOF: Ovo su pretpostavke $D \subset S_1(\boldsymbol{s},\mathbf{0})$ za neki $\boldsymbol{s} \in G_n(R)$ tako da $S_1(\boldsymbol{s},\mathbf{0})$ ima vrhove u skupu $G_n(R)$ za sve n. Poliedar $S_1(\boldsymbol{s},\mathbf{0})$ ima stranice duzine $4s_i$ jedinica, jer je zadan u mrezi G_1 s korakom $h = 1/2$. Za svaki n imamo q_i cvorova mreze G_n u smjeru $\boldsymbol{e}_i$ unutar poliedra. Ocito mora biti $h(n)r_iq_i < 4s_i$. Sada imamo $\boldsymbol{k} = (k, \boldsymbol{k}')$ i

$$u_{\boldsymbol{k}} \;=\; hr \sum_{l \le k} (U_1(r)\mathbf{u})_{l - r\boldsymbol{k}'}.$$

Iz CSB nejednakosti

$$\|\mathbf{u}\|_{R2}^2 \;\le\; r^2(q_1 h)^2 \;\|U_1(r)\mathbf{u}\|_{R2}^2.$$

Dakle je $\|\mathbf{u}\|_{R2} \le 4s_1 \|U_1(r)\mathbf{u}\|_{R2}$. **QED**

Let us consider a norm $\|\cdot\|_{R2,1}$ on $l(G_n(R))$ defined by (5). Any such norm is a semi-norm on $l(G_n)$. Our object of interest are quadratic functionals:

$$\begin{aligned} \|\mathbf{u}\|_{avg,2,1}^2 \;&=\; \frac{1}{vol(R)} \sum_R \|\mathbf{u}\|_{R2,1}^2 \\ &=\; \|\mathbf{u}\|_2^2 + \frac{1}{vol(R)} \sum_{i=1}^d \|U_i(r_i)\mathbf{u}\|_{R2}^2 \;\le\; \|\mathbf{u}\|_{2,1}^2, \end{aligned} \qquad \mathbf{u} \in l(G_n). \tag{6}$$

Then $\|\cdot\|_{avg,2,1}$ is a norm on $l(G_n)$. Unfortunately, it is not equivalent to $\|\cdot\|_{2,1}$ uniformly with respect to n.

An element (column) $\mathbf{u}_n \in l(G_n)$ can be associated to a continuous function on $\mathbb{R}^d$ in various ways. Here is utilized a mapping $l(G_n) \mapsto C(\mathbb{R}^d)$ which is defined in terms of hat functions. Let χ be the canonical hat function on $\mathbb{R}$, centered at the origin and having the support $[-1, 1]$. Then $z \mapsto \phi(h, x, z) = \chi(h^{-1}(z - hx))$ is the hat function on $\mathbb{R}$, centered at $x \in \mathbb{R}$ with support $[x - h, x + h]$. The functions $\boldsymbol{z} \;\mapsto\; \phi_{\boldsymbol{k}}(\boldsymbol{z}) \;=\; \prod_{i=1}^d \phi(h, x_i, z_i), x_i = hk_i$, define d-dimensional hat functions with supports $S_n(\mathbf{1}, \boldsymbol{x}) = \prod_i [x_i - h, x_i + h]$. The functions $\phi_{\boldsymbol{k}}(\cdot) \in G_n$, span a linear space, denoted by $E_n(\mathbb{R}^d)$. Let $\mathbf{u}_n \in l(G_n)$ have the entries $u_{n\boldsymbol{k}} = (\mathbf{u}_n)_{\boldsymbol{k}}$. Then the function $u(n) = \sum_{\boldsymbol{k} \in \mathbb{Z}^d} u_{n\boldsymbol{k}} \phi_{\boldsymbol{k}}$ belongs to $E_n(\mathbb{R}^d)$ and defines imbedding of grid-functions into the space of continuous functions. We denote the corresponding mapping by $\Phi_n : l(G_n) \mapsto E_n(\mathbb{R}^d)$. Obviously that there exists $\Phi_n^{-1} : E_n(\mathbb{R}^d) \mapsto l(G_n)$ and the spaces $l(G_n)$ and $E_n(\mathbb{R}^d)$ are isomorphic with respect to the pair of mappings Φ_n, Φ_n^{-1}. It is clear that $E_n(\mathbb{R}^d) \subset E_{n+1}(\mathbb{R}^d)$ and the space of functions $\cup_n E_n(\mathbb{R}^d)$ is dense in $L_p(\mathbb{R}^d), p \in [1, \infty)$, as well as in $\dot{C}(\mathbb{R}^d)$. Let us mention that $\sum_{\boldsymbol{k}} \phi_{\boldsymbol{k}} = 1$ on $\mathbb{R}^d$.

Now we consider another collection of basis functions. To each $\boldsymbol{x} = h\boldsymbol{k} \in G_n(R)$ there is associated a d-dimensional hat function

$$\psi_{\boldsymbol{k}}(\boldsymbol{x}) \;=\; \prod_{i=1}^d \chi\left(\frac{x_i - hk_i}{hr_i}\right),$$

obviously, with the support $S_n(\boldsymbol{r}, \boldsymbol{x}) = \prod_i [x_i - r_i h, x_i + r_i h]$. They span a linear space denoted by $E_n(R, \mathbb{R}^d)$. Again we have $\sum_{\boldsymbol{k}} \psi_{\boldsymbol{k}} = 1$ on $\mathbb{R}^d$. The mappings Φ_n, Φ_n^{-1} cannot be applied to elements of $l(G_n(R))$ and $E_n(R, \mathbb{R}^d)$, respectively. Therefore we define restrictions $\Phi_n(R) : l(G_n(R)) \to E_n(R, \mathbb{R}^d)$ and $\Phi_n^{-1}(R)$ by the following expression:

$$u(n) \;=\; \Phi_n(R)\,\mathbf{u}_n \;=\; \sum_{\boldsymbol{k}} \left(\mathbf{u}_n\right)_{\boldsymbol{k}} \psi_{\boldsymbol{k}}. \tag{7}$$

If we have to underline that $u(n)$ is related to a particular set of parameters R then we use an extended denotation $u(R, n)$. For two functions $v(n), u(n)$ we have $(v(n)|u(n)) = vol(R) \sum_{\boldsymbol{kl}} s_{\boldsymbol{kl}} v_{\boldsymbol{k}} u_{\boldsymbol{k}}$ where $s_{\boldsymbol{kl}} = \|\psi_{\boldsymbol{k}}\|_1^{-1} (\psi_{\boldsymbol{k}}|\psi_{\boldsymbol{l}})$. Let us notice that $\sum_{\boldsymbol{l}} s_{\boldsymbol{kl}} = 1$.

We cannot compare directly columns $\mathbf{u}_n$ with various n. An indirect comparision can be made by using $u(n) = \Phi_n(R)\mathbf{u}_n \in \cup_n E_n(R, \mathbb{R}^d)$. To compare $U_i(r_i)\mathbf{u}_n$ and $\partial_i u(n)$ we need an additional expression. Let $\mathbf{u}$ and $u(n)$ be related by (7) and $\dot{s}_{\boldsymbol{kl}}(i) = h^{-d}(\partial_i \psi_{\boldsymbol{k}} | \partial_i \psi_{\boldsymbol{l}})$. Then, for a homogeneous grid $G_n(R), \boldsymbol{r} \in \mathbb{N}^d$, there must holds

$$\sum_{\boldsymbol{kl}} v_{\boldsymbol{k}} u_{\boldsymbol{l}}\, \dot{s}_{\boldsymbol{kl}}(i) \;=\; -\frac{1}{2} \sum_{\boldsymbol{k}\,\boldsymbol{r}'\,r_i} \left(v_{\boldsymbol{k}+r_i\boldsymbol{e}_i} - v_{\boldsymbol{k}}\right) \left(u_{\boldsymbol{k}+\boldsymbol{r}'+r_i\boldsymbol{e}_i} - u_{\boldsymbol{k}+\boldsymbol{r}'}\right) s'_{\mathbf{0}\boldsymbol{r}'}\, \dot{s}_{0r_i}, \tag{8}$$

where $\boldsymbol{r}' = (r_1, r_2, \ldots, r_{i-1}, 0, r_{i+1}, \ldots, r_d)$, $\dot{s}_{0r} = (\partial_i \psi_0 | \partial_i \psi_r)$ and $s'_{\mathbf{0}\boldsymbol{r}'}$ is the $(d-1)$-dimensional quantity $s_{\boldsymbol{kl}}$. The statement follows from $\sum_{\boldsymbol{l}} s_{\boldsymbol{kl}} = 1$ and consequently $\sum_{\boldsymbol{l}} \dot{s}_{\boldsymbol{kl}} = 0$, after the sum is carried out over any particalr componet l_i of the index $\boldsymbol{l}$. Thus we have

$$(\partial_i v(n) \,|\, \partial_i u(n)) \;=\; \|\psi_{\boldsymbol{k}}\|_1 \sum_{m_i \boldsymbol{k}'\boldsymbol{l}'} s_{\boldsymbol{k}'\boldsymbol{l}'} \left(U_i(r_i)\mathbf{v}\right)_{m_i \boldsymbol{k}'} \left(U_i(r_i)\mathbf{u}\right)_{m_i \boldsymbol{l}'}, \tag{9}$$

where the indices are defined by $\boldsymbol{k}' = (k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_d)$ and analogously $\boldsymbol{l}'$.

LEMMA **1.2** *Let sequences of functions $v(n), u(n), n \in \mathbb{N}$, be defined by (7). Then*

$$\begin{aligned}
&(i) && \left| \textstyle\sum_{i=1}^d \, \left(\partial_i v(n) \,|\, \partial_i u(n) \right) \right| \;\le\; h^d \, q_R(\mathbf{v})^{1/2} \, q_R(\mathbf{u})^{1/2}. \\
&(ii) && \left| \left(\partial_i v(n) \,|\, \partial_j u(n) \right) \;-\; h^d \, vol(R) \textstyle\sum_{\boldsymbol{k} \in I_n(R)} \left(U_i(r_i)\mathbf{v}\right)_{\boldsymbol{k}} \left(U_j(r_j)\mathbf{u}\right)_{\boldsymbol{k}} \right| \\
& && \le\; h^d \, \min \left\{ \begin{array}{ll} \| U_i(r_i)\mathbf{v} \|_{R2} & \sup \left\{ \| (Z(w,j) - I)\, U_j(r_j)\mathbf{u} \|_{R2} \,:\, |w| \le r_j h \right\} \\ \| U_j(r_j)\mathbf{u} \|_{R2} & \sup \left\{ \| (Z(w,i) - I)\, U_i(r_i)\mathbf{v} \|_{R2} \,:\, |w| \le r_i h \right\} \end{array} \right.
\end{aligned}$$

PROOF: After applying the CSB-inequality to (9) and using $\sum_{\boldsymbol{l}} s_{\boldsymbol{kl}} = 1$ we get (i). Assertion (ii) is proved for $i = j = 1$. In this proof $\partial = \partial_1$. By using (9) we can calculate straight forwardly

$$\begin{aligned}
(\partial v(n) \,|\, \partial u(n)) \;&=\; \|\psi_{\boldsymbol{k}}\|_1 \sum_{\boldsymbol{k}, \boldsymbol{r}'} s_{\mathbf{0}\boldsymbol{r}'} \left(U_i(r_i)\mathbf{v}\right)_{\boldsymbol{k}} \left(U_i(r_i)\mathbf{u}\right)_{\boldsymbol{k}+\boldsymbol{r}'} \\
&=\; h^d \left(U_i(r_i)\mathbf{v} \middle| U_i(r_i)\mathbf{u}\right)_R \;+\; \delta(n),
\end{aligned}$$

where

$$\delta(n) \;=\; \|\psi_{\boldsymbol{k}}\|_1 \sum_{\boldsymbol{k}, \boldsymbol{r}'} s_{\mathbf{0}\boldsymbol{r}'} \left(U_i(r_i)\mathbf{v}\right)_{\boldsymbol{k}} \left[\left(U_i(r_i)\mathbf{u}\right)_{\boldsymbol{k}+\boldsymbol{r}'} \;-\; \left(U_i(r_i)\mathbf{u}\right)_{\boldsymbol{k}} \right].$$

By using the CBS inequality the error term $\delta(n)$ can be estimated as expressed in Assertion (ii). **QED**

LEMMA **1.3** *Let $G_n(R)$ be a homogeneous subgrid defined by (3). There exists $\sigma^2 \in (0,1)$ such that*

$$(1 \,-\, \sigma^2) \, h^d \, \| \mathbf{u}_n \|_{R2}^2 \;\le\; \| \, u(n) \, \|_2^2 \;\le\; h^d \, \| \mathbf{u}_n \|_{R2}^2$$

uniformly with respect to $n \in \mathbb{N}$.

PROOF: Let us consider first the one-dimensional case. The grid $G_n(R)$ consists of points $x_k = hrk \in \mathbb{R}, k \in \mathbb{Z}$, and $E_n(R, \mathbb{R})$ is spanned by the hat functions ψ_k centred at x_k with the supports $[-hr + x_k, x_k + hr]$. We define the matrix $S(1)$ with entries:

$$s_{kl} \;=\; \frac{1}{hr} \, (\psi_k | \psi_l) \;=\; \left\{ \begin{array}{lll} (2/3) & \text{for} & k \;=\; l, \\ (1/6) & \text{for} & k \;=\; l \pm r. \end{array} \right.$$

Obviously we have $S(1) = I - (1/3)A$, where the matrix A has the following structure $A = I + (1/2)(I_+ + I_-)$ and $I_\pm$ are the first upper and lower of diagonales. It is well known that F has a purely continuous spectrum in $[0, 2]$ so that $S(1)$ has the spectrum equal $[1/3, 1]$. Therefore

$$\|u(n)\|_2^2 \;=\; h \, \mathrm{vol}\,(R) \sum_{kl} s_{kl} \, u_k u_l \ge \frac{1}{3} \, h \, \mathrm{vol}\,(R) \sum_k u_k^2 \;=\; \frac{1}{3} \, h \, \| \mathbf{u} \|_{R2}^2.$$

Hence, we have here $1 - \sigma^2 = 1/3$.

In order to generalize this proof to d-dimensional case we proceede as follows. The symmetric matrix $S(d)$ with entries $s_{\boldsymbol{kl}}$ can be represented as the outer product of d matrices $S(1)$ with entries as in the first part of proof. Therefore its spectrum is $Sp(S(d)) = \prod_{i=1}^d Sp(S(1))$.

According to the first part of proof the matrix $S(1)$ has its spectrum in the interval $[1/3, 1]$, implying $\min Sp(d) \geq 3^{-d}$. Hence, with $\sigma^2 = 1 - 3^{-d}$ we have

$$vol(R) \sum_{\boldsymbol{kl} \in I_n(R)} s_{\boldsymbol{kl}}\, u_{\boldsymbol{k}}\, u_{\boldsymbol{l}} \;\geq\; (1-\sigma^2)\, \left\| \mathbf{u} \right\|_{R2}^2, \tag{10}$$

providing us with a proof of the general case. **QED**

Now we can get the following basic results involving the norm $\|u(n)\|_{2,1}$ and its averaged value defined by:

$$\|u(n)\|_{avg,2,1}^2 \;=\; \frac{1}{vol(R)} \sum_R \|u(R,n)\|_{2,1}^2.$$

THEOREM **1.1** *Let $u(n) = \Phi_n(R)\mathbf{u}_n$. There exists $\sigma^2 \in (0,1)$, independent of n, such that*

$$\begin{array}{rcccl} (1-\sigma^2)\, h^d \left\| \mathbf{u}_n \right\|_{R2,1}^2 & \leq & \|u(R,n)\|_{2,1}^2 & \leq & h^d \left\| \mathbf{u}_n \right\|_{R2,1}^2, \\ (1-\sigma^2)\, h^d \left\| \mathbf{u}_n \right\|_{avg,2,1}^2 & \leq & \|u(n)\|_{avg,2,1}^2 & \leq & h^d \left\| \mathbf{u}_n \right\|_{avg,2,1}^2. \end{array}$$

PROOF: It is sufficient to prove the first double inequality. The estimates from above are obvious. To get the estimates from bellow it sufficies to consider $\partial_i u$. From Expression (9) we have

$$\|\partial_i u(n)\|_2^2 \;=\; \|\psi_{\boldsymbol{k}}\|_1 \sum_m \sum_{\boldsymbol{k}'\boldsymbol{l}'} s_{\boldsymbol{k}'\boldsymbol{l}'} \left(U_i(r_i)\mathbf{u}\right)_{\boldsymbol{k}'} \left(U_i(r_i)\mathbf{u}\right)_{\boldsymbol{l}'}.$$

Then after applying (10) to the inner sum we get

$$\|\partial u(n)\|_2^2 \;\geq\; (1-\sigma^2)\, \|\psi_{\boldsymbol{k}}\|_1 \sum_{m,\boldsymbol{k}'} \left(U_i(r_i)\mathbf{u}\right)_{m\boldsymbol{k}'}^2 \;=\; (1-\sigma^2)\, h^d \left\| U_i(r_i)\mathbf{u} \right\|_{R2}^2,$$

from where follows the estimate from bellow. **QED**

An element $u \in W_2^1(\mathbb{R}^d)$ does not belong necessary to $E_n(R,\mathbb{R}^d)$. In order to approximate u with elements of $E_n(R,\mathbb{R}^d)$ we define:

$$\hat{u}(n) \;=\; \sum_{\boldsymbol{k} \in I_n(R)} \|\psi_{\boldsymbol{k}}\|_1^{-1}\, (\psi_{\boldsymbol{k}}|u)\, \psi_{\boldsymbol{k}}. \tag{11}$$

The numbers $\|\psi_{\boldsymbol{k}}\|_1^{-1}(\psi_{\boldsymbol{k}}|u)$ are called Fourier coefficients of u.

The basic result for our proof of convergence of approximate solutions is formulated by using the quantity $\Gamma_p(\boldsymbol{w}, u)$ defined by:

$$\Gamma_p(\boldsymbol{w}, u) \;=\; \|\, (Z(\boldsymbol{w}) \,-\, I)\, u\, \|_p$$

The kernels

$$\omega_n(\boldsymbol{x}, \boldsymbol{y}) \;=\; \sum_{\boldsymbol{k}} \frac{1}{\|\psi_{\boldsymbol{k}}\|_1}\, \psi_{\boldsymbol{k}}(\boldsymbol{x})\, \psi_{\boldsymbol{k}}(\boldsymbol{y}) \tag{12}$$

define an integral operator which is denoted by K_n. Actually, the kernels ω_n define a δ-sequance of functions on $\mathbb{R}^d \times \mathbb{R}^d$ and K_n converge strongly in L_p-spaces to unity:

COROLLARY **1.1** *Let $p \in [1, \infty]$. Then*

- *(i)* $\|K_n\|_p \;\leq\; 1$.
- *(ii) There is a positive number $\kappa(R)$, independent of n, such that $\|(I - K_n)u\|_p \leq \kappa(R) \sup\{\Gamma_p(\boldsymbol{w}, u) : |w_i| \leq h r_i\}$.*
- *(iii) The operator $K_n \in (\mathsf{L}_2(\mathbb{R}^d), L_2(\mathbb{R}^d))$ has the spectrum equal $Sp(K_n) = \{0\} \cup [3^{-d}, 1]$.*

PROOF: Only (iii) has to be proved. The symmetric operator K_n is reduced by $E_n(R,\mathbb{R}^d)$ and representated by an integral operator with the kernel (12). It is zero operator in the orthogonal complement $E_n(R,\mathbb{R}^d)^\perp$. With respect to the mapping $\Phi_n(R)$ the operator K_n is mapped to the symmetric matrix $\hat{K_n} = \Phi_n(R)^{-1} K_n \Phi_n(R) = S(d)$ in $l_2(I_n(R))$. **QED**

THEOREM **1.2** *Let $v, u \in W_2^1(\mathbb{R}^d)$ and $\hat{u}(n), \hat{v}(n)$ be defined by (11). Then*

$$\Big|(\hat{v}(n)|\hat{u}(n)) \;-\; (v|u)\Big| \;\leq\; c(R)\,\min\left\{\begin{array}{l} \|\,u\,\|_2\ \sup_{|\boldsymbol{w}|\leq h|\boldsymbol{r}|}\ \Gamma_2(\boldsymbol{w},v), \\ \|\,v\,\|_2\ \sup_{|\boldsymbol{w}|\leq h\boldsymbol{r}|}\ \Gamma_2(\boldsymbol{w},u), \end{array}\right.$$

$$\Big|(\partial_i\hat{v}(n)\,|\,\partial_j\hat{u}(n)) \;-\; (\,\partial_i v\,|\,\partial_j u\,)\Big| \leq\; c(R)\,\min\left\{\begin{array}{l} \|\partial_i v\|_2\left[\|\partial_j u - \eth_j u\|_2 + \displaystyle\sup_{|\boldsymbol{w}|\leq h\boldsymbol{r}|}\ \Gamma_2(\boldsymbol{w},\partial_j u)\right], \\[2ex] \|\partial_j u\|_2\left[\|\partial_i v - \eth_i v\|_2 + \displaystyle\sup_{|\boldsymbol{w}|\leq h\boldsymbol{r}|}\ \Gamma_2(\boldsymbol{w},\partial_i v)\right], \end{array}\right.$$

where $c(R)$ is n-independent.

KOMPARTMENTALNI OPERATORI

2.

For a natural r the ratio of Gamma functions, $c(k,r) = \Gamma(k+r)/\big(\Gamma(k+1)\Gamma(r)\big)$, is usually denoted by $\binom{k+r-1}{k}$. In this monography the symbol $\binom{k+r-1}{k}$ is also used for all $r \in (0,\infty)$. Let K be a bounded operator in a Banach space L and let us assume that there exists $m \in \mathbb{N}$ such that $\|K^m\| < 1$. We wish to prove that the series:

$$(I \;-\; K)^{-\alpha} \;=\; \sum_{k=0}^{\infty} \binom{k+\alpha-1}{k} K^k \tag{13}$$

converges in L.

LEMMA **1.4** *Let K be bounded in a Banach space L. If $\|K\| \leq 1$ and $\rho = \|K^m\| < 1$ for some $m \in \mathbb{N}$, then $(I-K)^{-\alpha}$ is defined in L by Expression (13) for each $\alpha > 0$ and*

$$\|(I \;-\; K)^{-\alpha}\| \;\leq\; \sum_{k=0}^{\infty} \binom{k+\alpha-1}{k} \|K^k\| \;\leq\; \rho^{1/m-1}\,(1-\rho^{1/m})^{-\alpha},$$

PROOF: Nakon prve ocjene na desnoj strani imamo za $\rho = \|K^m\|$ slijedecu ocjenu;

$$\begin{array}{l} \big[c(0,\alpha) + c(1,\alpha) + \ldots + c(m-1,\alpha)\big]\,\rho^0 + \\ \big[c(m,\alpha) + c(m+1,\alpha) + \ldots + c(2m-1,\alpha)\big]\,\rho + \\ \ldots \\ \big[c(rm,\alpha) + c(rm+1,\alpha) + \ldots + c((r-1)m-1,\alpha)\big]\,\rho^{r-1} + \\ \ldots . \end{array}$$

Sada definiramo $s = \rho^{1/m}$ i dobijemo za ovaj red ocjenu:

$$s^{-(m-1)} \sum_{k=0}^{d} c(k,\alpha,k)\,s^k,$$

t.j.:

$$\|(I \;-\; K)^{-\alpha}\| \;\leq\; s^{-(m-1)}\,(1-s)^{-\alpha} \;=\; \rho^{1/m-1}\,(1-\rho^{1/m})^{-\alpha},$$

t.j. konvergenciju po normi. QED

LEMMA **1.5 (O potenciji inverza)** *Let A be a linear operator in a Banach space L satisfying HY-conditions. Then there exists $T(\lambda,A)^\alpha$ in L such that $D(\alpha) = T(\lambda,A)^\alpha L$ is dense in L and there exists $\big(\lambda I + A\big)^\alpha$ on $D(\alpha)$ such that $\big(\lambda I + A\big)^\alpha T(\lambda,A)^\alpha = I$.*

PROOF: Let $A_n = (1/n)I - n^2 T(n, A)$ be the Yoshida approximations of A. Then there exists a sequence of bounded operators defined by:

$$
\begin{aligned}
T(\lambda, A_n)^\alpha \;&=\; \Big((\lambda + n)\, I \;-\; n^2\, T(\lambda, A)\Big)^{-\alpha} \\
= \;\frac{1}{(\lambda+n)^\alpha} \sum_{k=0}^{\infty} &c(k,\alpha)\, \left(\frac{n}{\lambda+n}\right)^k \, \big(n\, T(n, A)\big)^k,
\end{aligned}
\tag{14}
$$

where $c(k,p) = \Gamma(k+p)/\big(\Gamma(k+1)\Gamma(p)\big)$. They converg strongly in L to a bounded operator $K(\alpha)$. From $\|T(\lambda, A_n)^\alpha\| \leq 1/\lambda^\alpha$ there follows $\|K(\alpha)\| \leq 1/\lambda^\alpha$. Let $\alpha = q/p, q, p \in \mathbb{N}$. Then $\big(T(\lambda, A_n)^\alpha\big)^p = T(\lambda, A_n)^q$ implying $K(\alpha)^p = T(\lambda, A)^q$. Hence, by definition $T(\lambda, A) = K(\alpha)$ for $\alpha = q/p$. However, $\alpha \mapsto T(\lambda, A_n)^\alpha$ of (14) are analytic in $\Re\alpha > 0$ so that the established representation (14) is valid for all positive α. **QED**

3.

Sada promatramo operatore na $L_p(D, \mu)$, gdje je D otvoren skup, μ je ili Lebesqueova mjera ili suma atomskih mjera na D. Definiramo pozitivnost za neki ograniceni oprator Q. Kazemo $Q \geq 0$ ako je $Qu \geq 0$ na D za svaki $u \geq 0$. Slicno definiramo $Q > 0$ na slijedeci nacin. Operator Q je pozitivan, t.j. $Qu > 0$ na D ako za svaki $u \in L_p(D, \mu)$, $u \geq 0, \|u\|_p > 0$ slijedi $supp(Qu) = \overline{D}$. Ocito je $Q > 0$ na D akko $\langle v|u \rangle > 0$ za svaki par $v \in L_q(D, \mu)$, $u \in L_p(D, \mu)$, $v, u \geq 0$, $\|v\|_q > 0, \|u\|_p > 0$. Ova se tvrdnja dokazuje pomocu pretpostavke suprotnog za skup $F = supp(Tu) \subset \overline{D}$ i $O = D \setminus F$. Za njih se promatra $v \in L_q(O, \mu)$.

LEMMA **1.6** *Let A be a linear operator in $L(D, \mu)$ satisfying HY-conditions. Then:*

(i) If $T(\lambda, A) \geq 0$ there exists a unique $T(\lambda, A)^\alpha \geq 0$.

(ii) If L is a Hilbert space then the unique operator $T(\lambda, A)^\alpha$ of (i) is positive definite.

PROOF: From $T(\lambda, A) \geq 0$ and (14) there follows $K(\alpha) \geq 0$. **QED**

DEFINITION 1.1 (COMPARTMENTAL STRUCTURE) *A bounded operator A in $L_\infty(D, \mu)$ is said to be of positive type if $A = pI - B$, $p > 0$, $B \geq 0$ and $\|B\|_\infty \leq p$. It is called conservative if $B1 = 1$. A bounded operator A in $L_1(D, \mu)$ is said to have the compartmental structure if $A = pI - B$, $B \geq 0$ and $\|B\|_1 \leq p$. It is called conservative if for each $u \geq 0$ on D there holds $\|Bu\|_1 = p\|u\|_1$.*

In the case of $l_1(I)$ this definition and definition for matrices are in agreement.

Let us apply Lemma to compartmental operators $A = pI - B = p(I - Q)$ and $\lambda I + A = (\lambda + p)I - B$. Hence, in the present case $K = \rho Q$, where $\rho = p/(p + \lambda)$. Therefore, the operator

$$
T(\lambda, A)^\alpha \;=\; \left(\frac{1}{p+\lambda}\right)^\alpha \sum_{k=0}^{\infty} \binom{k+\alpha-1}{k} \rho^k\, Q^k \tag{15}
$$

is defined for any pair of numbers $\alpha > 0, \lambda > 0$.

Let $\lambda > 0$. The r-th power of $T(\lambda, A)$ has a representation (15). This is not the only possible representation. However this is the unique representation for which entries are non-negative. Other possibile representations can be derived from the spectral representation of A.

LEMMA **1.7** *Let A in $L_1(D, \mu)$ be compartmental. If A is conservativ then $\|B^m\|_1 = p^m$ for each $m \in \mathbb{N}$.*

PROOF: If A is conservative then the equality $\|B^m u\|_1 = p^m \|u\|_1$ must be valid for each $u \geq 0$ implying the assertion. **QED**

4.

Definira se spektar operatora A i rezolventa $R(\lambda, A) = T(\lambda, -A)$.

LEMMA **1.8** *Let A in $L_1(D, \mu)$ be a compartmental operator. Then $0 \in sp(A)$ iff A is conservative.*

PROOF: Let A be conservative. We have to construct a sequance $u_n \in L_1(D, \mu), \|u_n\|_1 = 1$, such that $Au_n \to 0$ in $L_1(D, \mu)$. It sufficies to consider the case $A = I - B$. Hence $\|B^m u_0\|_1 = 1$ for each $u_0 \geq 0, \|u_0\|_1 = 1$. Now we consider

$$u_n = \frac{1}{n}\left[I + B + B^2 + \cdots + B^{n-1}\right] u_0,$$

and calculate

$$\begin{aligned} \|u_n\|_1 &= \frac{1}{n} \sum_{k=1}^{n} \|B^{k-1} u_0\|_1 = 1, \\ Bu_n - u_n &= \tfrac{1}{n}\left[B^n u_0 - u_0\right] \to 0. \end{aligned}$$

Let us suppose now that $0 \in sp(A)$ while A is not conservative. There must exist $m \in \mathbb{N}$ such that $\rho = \|B^m\|_1 < 1$. We have

$$(I - B)^{-1} = \left[I + B + B^2 + \cdots + B^{m-1}\right] \sum_{r=0}^{\infty} B^{mr},$$

so that $\|(I - B)^{-1}\|_1 \leq m/(1 - \rho)$. Hence A^{-1} is bounded on $L_1(D, \mu)$ and $0 \in \mathbb{C}$ belongs to the resolvent set. **QED**

Primjer: Neka je F preslikavanje sa $l_2(\mathbb{Z})$ na $L_2(0, 2\pi)$ koje je ostvareno razvojem u Fourierov red po bazi $\phi_k(x) = (2\pi)^{-1/2} \exp(ikx)$. Promatramo $A = I - (1/2)(I_+ + I_-)$ u $l_2(\mathbb{Z})$ i $\hat{A}$ u $L_2(0, 2\pi)$ definiran sa $\hat{A}(x)u(x) = 2\sin^2(x/2)u(x)$. Vrijedi slijedece

$$F\,A = \hat{A}\,F.$$

Ovo se dokazuje racunom $(FA\mathbf{u})(x) = \sum_k \phi_k(x)(A\mathbf{u})_k = 2\sin^2(x/2)u(x)$. Dakle A ima neprekidni spektar u $[0, 2]$.

5.

Let I be an index set and $A = \{a_{ij}\}_{II}$ be a matrix, $A \geq 0$. We say that A is irreducible if for any finite index subset $J \subset I$ there exist $m(J) \in \mathbb{N}$ such that

$$\mathbb{1}_J \sum_{r=0}^{m(J)} A^r\, \mathbb{1}_J > 0 \quad \text{on } J.$$

The matrix $I_+ + I_-$ is irreducible, while matrices $I_{+r} + I_{-r}$ for $r \geq 2$ are not irreducible.

For a compartmental matrix $A = p(I - Q)$ we have $T(\lambda, A) \geq 0$. If $T(\lambda, A) > 0$ on I we say that the matrix A is irreducible. For instance $A = I - (1/2)(I_- + I_-)$ is irreducible while $A = I - (1/2)I_{+r} + I_{-r}$ is not irreducible.

2 Classical elliptic operator

2.1 3. Approximate solutions and $C^{(\alpha)}$-convergence

For the differential operator $A(\boldsymbol{x}) = -\sum_{ij} a_{ij}(\boldsymbol{x})\partial_i\partial_j + \sum_i b_i(\boldsymbol{x})\partial_i + c(\boldsymbol{x})$ we consider the following problem:

$$\begin{aligned} (\lambda I + A(\boldsymbol{x}))\, u(\boldsymbol{x}) &= \mu(\boldsymbol{x}),\ \boldsymbol{x} \in D, \\ u|\partial D &= 0, \end{aligned} \tag{16}$$

where the bounded and open set D has the boundary ∂D of the class $C^{(2+\alpha)}$. If the coefficents a_{ij}, b_i, c and the function μ are of the class $C^{(\alpha)}(\overline{D})$ then Problem (16) is called Dirichlet boundary value problem on D for a classical second order differential operator. For a strictly elliptic tensor-valued function $\boldsymbol{x} \mapsto a(\boldsymbol{x})$, $c \geq 0$ and $\lambda \geq 0$ this problem has a solution $u \in C^{(2+\alpha)}(\overline{D})$. This result is known as the Shauder theory.

Let the matrices A_n on G_n be discretizations of the considered differential operator $A(\boldsymbol{x})$ and let there be defined linear systems:

$$A_n \mathbf{u}_n \;=\; \boldsymbol{\mu}_n, \tag{17}$$

where $\mathbf{u}_n, \boldsymbol{\mu}_n \in l(G_n(D))$. We say that the system (17) numerically approximates the boundary value problem (16). The columns $\mathbf{u}_n$ are called grid-solutions. Obviously, $\boldsymbol{\mu}_n$ are discretizations of μ.

DEFINITION 2.1 (**Consistency**) *A matrix A_n with a finite band is called a consistent discretization with the differential operator $A(\boldsymbol{x}) = -\sum_{ij=1}^d a_{ij}(\boldsymbol{x})\partial_i\partial_j + \sum_{i=1} b_i(\boldsymbol{x})\partial_i + c(\boldsymbol{x})$ if the equalities*

$$(Au)(\boldsymbol{x}) \;=\; \sum_{\boldsymbol{l}} (A_n)_{\boldsymbol{kl}}\, u_{\boldsymbol{l}}, \quad \boldsymbol{x} = h\boldsymbol{k}, \tag{18}$$

are valid for any polynomial $\boldsymbol{x} \to u(\boldsymbol{x})$ of the second degree and the corresponding discretizations $\mathbf{u}$ defined by $u_{\boldsymbol{k}} = u(h\boldsymbol{k})$.

For grid-solutions $\mathbf{u}_n = T(\lambda, A_n)\boldsymbol{\mu}_n$ the inverse matrix $T(\lambda, A_n)$ has l_∞-norms generaly depending on n.

THEOREM **2.1** *Let A_n be discretizations of the differential operator $A(\boldsymbol{x})$ and $\mathbf{u}_n = T(\lambda, A_n)\boldsymbol{\mu}_n$ such that $\| T(\lambda, A_n) \|_\infty$ is bounded uniformly with respect to $n \in \mathbb{N}$. If A_n are consistent with $A(\boldsymbol{x})$ then the sequence of functions $\mathfrak{U} = \{u(n) : n \in \mathbb{N}\} \subset \cup_n E_n(D)$ converges in $\dot{C}^{(\alpha)}(\mathbb{R}^d)$ to the solution $u = T(\lambda, A)\mu$ to (16).*

PROOF: In this proof it is sufficient to consider the case $A(\boldsymbol{x}) = -\sum_{ij=1}^d a_{ij}(\boldsymbol{x})\partial_i\partial_j$. For any function $u \in C^{(2)}(\mathbb{R}^d)$ we have the following expression:

$$u(\boldsymbol{x} + \boldsymbol{h}) \;=\; u(\boldsymbol{x}) \,+\, \boldsymbol{h} \cdot \nabla u(\boldsymbol{x}) \,+\, \frac{1}{2}\,(\boldsymbol{h}|H(\boldsymbol{x})\boldsymbol{h}) \,+\, u_{rem}(\boldsymbol{x}), \tag{19}$$

where $H(\boldsymbol{x}) = \{\partial_i\partial_j u(\boldsymbol{x})\}_{11}^{dd}$ and the remainder $u_{rem}(\boldsymbol{x})$ of Taylor expansion has the following form:

$$u_{rem}(\boldsymbol{x}) \;=\; \int_0^1 (1-t)\left[(\boldsymbol{h}|H(\boldsymbol{x} + t\boldsymbol{h})\boldsymbol{h}) \,-\, (\boldsymbol{h}|H(\boldsymbol{x})\boldsymbol{h})\right] dt.$$

For the proof of (19) one has to evaluate $\int (1-t)\partial_t(\boldsymbol{h}|\nabla u(\boldsymbol{x} + t\boldsymbol{h}))dt$. We utilize (19) for $\boldsymbol{x}, \boldsymbol{x} + \boldsymbol{h} \in G_n(D)$. In addition there must hold $[\boldsymbol{x}, \boldsymbol{x} + \boldsymbol{h}] \subset D$.

Let $\mathbf{u}$ be discretizations of $u \in C^{(2)}(\mathbb{R}^d)$. One easily verifies

$$A(\boldsymbol{x})u(\boldsymbol{x}) \,-\, \big(A_n\, \mathbf{u}\big)_{\boldsymbol{k}} \;=\; \big(A_n\, \mathbf{u}_{rem}\big)_{\boldsymbol{k}}(\boldsymbol{x}), \tag{20}$$

where $\boldsymbol{x} = h\boldsymbol{k} \in G_n$, and $\boldsymbol{h} \in G_n$. In this proof the unique solution to (16) is denoted by u^*. If we replace u with u^* and $A(\boldsymbol{x})u(\boldsymbol{x})^*$ with $\boldsymbol{\mu}(\boldsymbol{x})$ in (20) we get

$$\boldsymbol{\mu}(\boldsymbol{x}) \,-\, \big(A_n\, \mathbf{u}_n^*\big)_{\boldsymbol{k}} \;=\; (\mathbf{s}_n)_{\boldsymbol{k}},$$

where $\mathbf{s}_n = A_n \mathbf{u}_{rem}^*$. On the other hand the discretized equations can be rewritten as

$$\boldsymbol{\mu}(\boldsymbol{x}) \,-\, \big(A_n\, \mathbf{u}_n\big)_{\boldsymbol{k}} \;=\; \mathbf{0},$$

so that

$$A_n(\mathbf{u}_n^* - \mathbf{u}_n) \;=\; -\,\mathbf{s}_n,$$

implying

$$\lVert \mathbf{u}_n^* - \mathbf{u}_n \rVert_\infty \;\le\; \lVert T(0,A_n) \rVert_\infty \, \lVert \mathbf{s}_n \rVert_\infty,$$

where $\lVert T(\lambda, A)\rVert_\infty < c(A)$ uniformly with respecy to n. Now we have to estimate $\lVert \mathbf{s}_n \rVert$. It is easy to demonstrate that there exists a number $\gamma(d)$, independent of n, so that

$$\lVert \mathbf{s}_n \rVert \;\le\; \overline{M}\,\gamma(d)\, h^\alpha \,\|\, u^* \,\|_\infty^{(2+\alpha)}.$$

In order to calculate $\|u^* - n(n)\|_\infty$ we need an auxilliary result regarding the functions u^* and $u^*(n) = \Phi_n \mathbf{u}_n^*$. The functions coincide at grid-knots of $G_n(D)$, while their difference elsewhere can be estimated as follows

$$|u^*(\boldsymbol{x}) \;-\; u^*(n, \boldsymbol{x})| \;\le\; |\boldsymbol{h}||\boldsymbol{p}|\, \|\nabla u^*\|_\infty.$$

Therefore we have

$$\|u^* - u(n)\|_\infty \;\le\; |\boldsymbol{h}||\boldsymbol{p}|\, \|\nabla u^*\|_\infty \;+\; \overline{M}\, c(A)\, \gamma(d)\, h^\alpha \,\|\, u^* \,\|_\infty^{(2+\alpha)}.$$

2.2 Construction of discretizations

1. Standard approach

A standard approach to a generation of discretizations of differential operators is based on utilization of finite difference operators approximating $\partial_i, \partial_i\partial_j$. So, by using the forward and backward difference operators $\eth_i(p_i h_i), \hat{\eth}_i(p_i h_i)$ we can define various discretizations $\eth_{ij} f(\boldsymbol{x})$ of $\partial_i\partial_j f$. Let $\eth_{ii} = \hat{\eth}_i \eth_i$ so that the operator $-a_{ii}(\partial_i)^2 - a_{jj}(\partial_j)^2$ is approximated by the standard central difference operators $-a_{ii}\eth_{ii} - a_{jj}\eth_{jj}$ with positive diagonal entries and non-positive off-diagonal entries. For $i \neq j$ finite differences are defined by:

$$\partial_i\partial_j f(\boldsymbol{x}) \to \eth_{ij} f(\boldsymbol{x}) = \frac{1}{h_1 h_2} \left\{ \begin{array}{l} f(\boldsymbol{x} \pm \boldsymbol{e}_i h_i \pm \boldsymbol{e}_j h_j) - f(\boldsymbol{x} \pm \boldsymbol{e}_i h_i) - f(\boldsymbol{x} \pm \boldsymbol{e}_j h_j) + f(\boldsymbol{x}), \\ -f(\boldsymbol{x} \pm \boldsymbol{e}_i h_i \mp \boldsymbol{e}_j h_j) + f(\boldsymbol{x} \pm \boldsymbol{e}_i h_i) + f(\boldsymbol{x} \mp \boldsymbol{e}_j h_j) - f(\boldsymbol{x}). \end{array} \right. \tag{21}$$

If $a_{ij} \geq 0$, then $a_{ij}\partial_i\partial_j$ is approximated by the half sum of the first two possibilities, otherwise by the half sum of the second two possibilities. Hence, for the d-dimensional case and $G_n \subset \mathbb{R}^d$ we have

$$\begin{array}{lcll} A_{\boldsymbol{k}\boldsymbol{k}\pm p_i \boldsymbol{e}_i} & = & -\frac{1}{h^2 p_i^2}\left[a_{ii}(\boldsymbol{x}) \;-\; \sum_{m\neq i} \frac{p_i}{p_m}\,|a_{mi}(\boldsymbol{x})|\right], & \\ A_{\boldsymbol{k}\boldsymbol{k}\pm(p_i\boldsymbol{e}_i + p_j\boldsymbol{e}_j)} & = & -\frac{1}{h^2 p_i p_j}|a_{ij}(\boldsymbol{x})|, \qquad a_{ij} \geq 0, & \boldsymbol{x} = h\boldsymbol{k}, \\ A_{\boldsymbol{k}\boldsymbol{k}\pm(p_i\boldsymbol{e}_i - p_j\boldsymbol{e}_j)} & = & -\frac{1}{h^2 p_i p_j}|a_{ij}(\boldsymbol{x})|, \qquad a_{ij} \leq 0, & \end{array} \tag{22}$$

while the diagonal entries are calculated as the negative sum of off-diagonal ones.

Discretizations of $B(\boldsymbol{x}) = \sum_i b_i(\boldsymbol{x})\partial_i + c(\boldsymbol{x})$ are simply constructed by the so called upwind method:

$$B_n \;=\; c(\boldsymbol{x})\, I \;+\; \sum_{i=1}^{d} b_i(\boldsymbol{x}) \left\{ \begin{array}{lll} U_i(p_i) & \text{for} & b_i(\boldsymbol{x}) \leq 0, \\ V_i(p_i) & \text{for} & b_i(\boldsymbol{x}) \geq 0. \end{array} \right.$$

We say that a tensor-valued function $\boldsymbol{x} \mapsto \{a_{ij}(\boldsymbol{x})\}_{11}^{dd}$ is uniformly positive definite on a set $S \subset \mathbb{R}^d$ if there exists a positive number $\underline{M}(a)$ such that $\underline{M}(a)|\boldsymbol{z}|^2 \leq (\boldsymbol{z}|a(\boldsymbol{x})\boldsymbol{z})$ for any $\boldsymbol{x} \in S$ and $\boldsymbol{z} \in \mathbb{R}^d$. To a given diffusion tensor $a(\boldsymbol{x}) = \{a_{ij}(\boldsymbol{x})\}_{11}^{dd}$ we associate an auxiliary tensor $\hat{a}(\boldsymbol{x})$ defined by:

$$\hat{a}_{ii} \;=\; a_{ii}, \quad \hat{a}_{ij} \;=\; -|a_{ij}| \quad i \neq j.$$

By using the Perron theorem one can easily prove the following assertion.

LEMMA **2.1** *There exist a parametr $\boldsymbol{p} \in \mathbb{N}^d$ such that the matrix A_n defined by (22) is of positive type iff the auxiliary diffusion tensor $\hat{a}(\boldsymbol{x})$ is uniformly positive definite on $\mathbb{R}^d$.*

In the case of $d = 2$ the tensors $a, \hat{a}$ are simultaneously positive definite or not. For higher dimensions $\hat{a}$ can be indefinite although a is positive definite. Here is an example for $d = 3$. The symmetric matrix a defined by $a_{ii} = 1, a_{12} = a_{23} = -1/\sqrt{2}$ has positive eigenvalues for $a_{13} > 0$ and a negative eigenvalue for $a_{13} < 0$.

2. General approach

Apart from this standard approach to discretizations of differential operators there can be used constructions avoiding finite difference operators.

Let us consider first a differential operator with constant coefficients, $A(\boldsymbol{x}) = -\sum_{ij=1}^{d} a_{ij}\partial_i\partial_j$, and construct its discretizations A_n on G_n so that matrix entries $(A_n)_{\boldsymbol{kl}}$ depend only on $\boldsymbol{k}-\boldsymbol{l}$. We say that the correspondinh matrices A_n are homogeneous. By demanding additionaly the symmetry of matrices A_n we have the following general stricture:

$$A_n \;=\; \sum_{\boldsymbol{r}\in J} p_{\boldsymbol{r}}\, I_{\boldsymbol{r}}, \quad p_{\boldsymbol{r}} \;=\; p_{-\boldsymbol{r}}, \quad \boldsymbol{r} \in J \subset \mathbb{Z}^d,$$

where J is a finite index set and the real numbers $p_{\boldsymbol{r}}$ have to fulfil the following three conditions:

$$\begin{array}{rcl} \sum_{\boldsymbol{r}}\; p_{\boldsymbol{r}} &=& 0, \\ \sum_{\boldsymbol{r}}\; r_i\, p_{\boldsymbol{r}} &=& 0,\; i \;=\; 1,2,\ldots,d, \\ \sum_{\boldsymbol{r}}\; r_i r_j\, p_{\boldsymbol{r}} &=& -\frac{2}{h^2}\, a_{ij},\; i,j \;=\; 1,2,\ldots,d. \end{array} \tag{23}$$

The first and second conditions imply the equality $A_n\mathbf{u} = \mathbf{0}$ for the discretizations $\mathbf{u}$ of any polynomial of the second degree $x,y \mapsto u(x,y)$ on $\mathbb{R}^d$. The third condition comes from the following demand: If $u(x,y) = \alpha x^2 + \beta y^2 + \gamma xy$ then

$$A_n\mathbf{u} \;=\; -2\left(\alpha a_{11} + \gamma a_{12} + \beta a_{22}\right)\mathbf{1},$$

i.e. it coincides with the discretizations of $A(\boldsymbol{x})u(\boldsymbol{x})$.

EXAMPLE 1 . The well known central differences of Laplacean $A(\boldsymbol{x}) = -\sigma^2\Delta$ are contained in the described class of matrices A_n. The corresponding parameters are defined as follows. $J = \{\mathbf{0}, \pm\boldsymbol{e}_i : i = 1,2,\ldots,d\} \subset \mathbb{Z}^d$, and

$$p_{\mathbf{0}} \;=\; \frac{2\sigma^2 d}{h^2},\; p_{\pm\boldsymbol{e}_i} \;=\; -\frac{\sigma^2 d}{h^2}.$$

EXAMPLE 2 . Some other discretizations with peculiar features are also contained in the considered class of discretizations. Let us consider the case $d = 1$ and discretizations with $p_0 = 0$. We can choose $\boldsymbol{p} = (p_1,p_2,p_3) \in \mathbb{R}^3_+$ so that the system

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \boldsymbol{p} \;=\; \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

has a unique solution. This solution defines $p_r, |r| \le 3$ of Sustem (23). In this example the diagonal entries of A_n have zero values. Let us point out that the central difference discretizations of Laplacean are of positive type while the discretizations of this peculiar example are not of positive type.

For the differential operator $A(\boldsymbol{x}) = -\sum_{ij=1}^{d} a_{ij}\partial_i\partial_j + \sum_{i=1} b_i\partial_i + c$ the system (23) is replaced with the following one:

$$\begin{array}{rcl} \sum_{\boldsymbol{r}}\; p_{\boldsymbol{r}} &=& c, \\ \sum_{\boldsymbol{r}}\; r_i\, p_{\boldsymbol{r}} &=& \frac{b_i}{h},\; i \;=\; 1,2,\ldots,d, \\ \sum_{\boldsymbol{r}}\; r_i r_j\, p_{\boldsymbol{r}} &=& -\frac{2}{h^2}\, a_{ij},\; i,j \;=\; 1,2,\ldots,d. \end{array} \tag{24}$$

For a general elliptic differential operator $A(\boldsymbol{x})$ with nonconstant coefficients the construction is analogous. One of possibilities is the following simple discretization of $A_0(\boldsymbol{x})$:

$$\begin{array}{rcll} A_{\boldsymbol{kk}\pm\boldsymbol{e}_i} &=& -\frac{1}{h^2}\left[a_{ii}(\boldsymbol{x}) \;-\; \sum_{m\neq i} \frac{r_i}{r_m}\,|a_{mi}(\boldsymbol{x})|\right], & \\ A_{\boldsymbol{kk}\pm(r_i\boldsymbol{e}_i + r_j\boldsymbol{e}_j)} &=& -\frac{1}{h^2 r_i r_j}|a_{ij}(\boldsymbol{x})|, & a_{ij} \ge 0, \\ A_{\boldsymbol{kk}\pm(r_i\boldsymbol{e}_i - r_j\boldsymbol{e}_j)} &=& -\frac{1}{h^2 r_i r_j}|a_{ij}(\boldsymbol{x})|, & a_{ij} \le 0, \end{array} \tag{25}$$

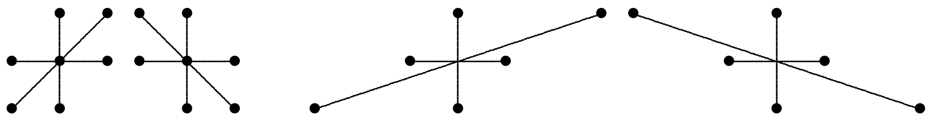
where $r_i, r_j \in \mathbb{N}$. Of course, a convex combination of the system matrices (25) is again a system matrix discretizing the classical differential operator $A_0 = -\sum_{ij} a_{ij}(\boldsymbol{x})\partial_i\partial_j$.

If we replace p_i with r_i in Lemma 2.1 we get a result about discretizations (25) of positive type. In both cases the uniform positive definitness of $\hat{a}$ on $\mathbb{R}^d$ is a sufficient condition ensuring the matrices A_n to be of positive type.

Discretizations A_n are defined in terms of its matrix entries $(A_n)_{\boldsymbol{kl}}$, where $h\boldsymbol{k}, h\boldsymbol{l} \in G_n$. For a fixed $\boldsymbol{x} = h\boldsymbol{k} \in G_n$ the set of all the grid-knots $\boldsymbol{y} = h\boldsymbol{l}$ such that $(A_n)_{\boldsymbol{kl}} \neq 0$ is denoted by $\mathcal{N}(\boldsymbol{x})$ and called the numerical neighbourhood of A_n at $\boldsymbol{x} \in G_n$:

$$\mathcal{N}(\boldsymbol{x}) \;=\; \{\boldsymbol{y} \in G_n \,:\, \boldsymbol{x} = h\boldsymbol{k},\; \boldsymbol{y} = h\boldsymbol{l},\; (A_n)_{\boldsymbol{kl}} \neq 0\}.$$

In the case of two dimensions the obtained structures of system matrix can be classified into two groups by using the corresponding numerical neighborhoods. The grid $G_n = \{h(k\boldsymbol{e}_1 + l\boldsymbol{e}_2) \,:\, k, l \in \mathbb{Z}\}$ has the corresponding index set of indices $\boldsymbol{k} = (k, l)$. Possible numerical neighborhoods $\mathcal{N}(\boldsymbol{x})$ for the respective methods (22), (25), are illustrated in Figure 2.1.



(a): $a_{12} \geq 0$, (b): $a_{12} \leq 0$ (c): $a_{12} \geq 0$ (d): $a_{12} \leq 0$

$p_1 = p_2 = 1$ $r_1 = 3, r_2 = 1$

Figure 2.1

Discretizations A_n defined by (22) and (25) have at most $1 + d + d^2$ entries in each row and the entries are linear combinations of $a_{ij}(\boldsymbol{x})$ calculated at the respective grid-knot $\boldsymbol{x} \in G_n$.

3 Operators in divergence form

3.1 Rules of construction and convergence

1. Original and discretized problems

The variational formulation for a solution $u \in \dot{W}_2^1(D)$ has the following form:

$$\lambda(v\,|\,u) \,+\, a(v,u) \;=\; \langle\, v\,|\,\mu\,\rangle, \quad \text{for any } v \in \dot{W}_2^1(D), \tag{26}$$

where $\mu \in W_2^{-1}(D)$. In the case of a problem on $\mathbb{R}^d$ the variational problem is defined by expression

$$\lambda(v\,|\,u) \,+\, a(v,u) \;=\; \langle\, v\,|\,\mu\,\rangle, \quad \text{for any } v \in \dot{W}_2^1(\mathbb{R}^d), \tag{27}$$

where $\mu \in W_2^{-1}(\mathbb{R}^d)$.

For the case of simplicity we consider the case of $D = \mathbb{R}^d$ and the problem with the form $\lambda(v|u) + a(v,u)$, $\lambda > 0$, where $W_2^1(\mathbb{R}^d) \times W_2^1(\mathbb{R}^d) \ni (v,u) \mapsto a(v,u)$ is defined by coefficients a_{ij}, b_i, c on $\mathbb{R}^d$ in the usual way.

The form a on $W_2^1(\mathbb{R}^d) \times W_2^1(\mathbb{R}^d)$ is discretized by a sequence of forms a_n on $E_n(\mathbb{R}^d) \times E_n(\mathbb{R}^d)$. Each form $a_n(\cdot,\cdot)$ determines a matrix A_n on G_n,

$$a_n(v,u) \;=\; \langle\, \mathbf{v}_n \,|\, A_n\,\mathbf{u}_n\,\rangle,$$

which is called a discretization of the original differential operator $A(\boldsymbol{x})$. In this way we arrive to the following result. The original variational problem is discretized by a sequence of discretized variational equalities:

$$\lambda\langle\, \mathbf{v}_n \,|\, \mathbf{u}_n\,\rangle \,+\, a_n(v,u) \;=\; \langle\, \mathbf{v}_n \,|\, \boldsymbol{\mu}_n\,\rangle, \quad \mathbf{v}_n \,\in\, w_{2,1}(G_n). \tag{28}$$

where $\boldsymbol{\mu}_n$ are discretizations of the functional $\mu \in W_2^{-1}(\mathbb{R}^d)$ (or $W_2^{-1}(D)$). The equalities (28) have an equivalent formulation

$$(\lambda I \,+\, A_n)\,\mathbf{u}_n \;=\; \boldsymbol{\mu}_n, \tag{29}$$

where $\mathbf{u}_n, \boldsymbol{\mu}_n$ are grid functions on G_n in case of Problem (27) and grid-functions on $G_n(D)$ in case of Problems (26).

We cannot prove the $W_2^1(\mathbb{R}^d)$-convergence of approximate solutions without a discretized version of the strict ellipticity. Descrete forms $a_n(\cdot,\cdot)$ on $l_0(G_n)\times l_0(G_n)$ are said to be strictly elliptic unuformly with respect to $n \in \mathbb{N}$ if

$$\underline{M}\,\|\mathbf{u}\|_{2,1}^2 \;\le\; \lambda\|\mathbf{u}_n\|_2^2 \,+\, a_n(\mathbf{u}_n,\mathbf{u}_n) \;\le\; \overline{M}\,\|\mathbf{u}\|_{2,1}^2, \tag{30}$$

with some positive numbers $\underline{M} \le \overline{M}$ and all $n \in \mathbb{N}$.

2. Discretizations of μ

First we have to demonstrate the existence of $\boldsymbol{\mu}_n$ such that $h^d\langle\, \mathbf{v}_n|\boldsymbol{\mu}_n\,\rangle \to \langle\, v|\mu\,\rangle$.

Beside the functions $\psi_{\boldsymbol{k}}$ we consider the functions defined by:

$$\chi_{\boldsymbol{k}i\pm}(\boldsymbol{x}) \;=\; \mathbb{1}_{[k_i,k_i\pm r_i h]}(x_i)\prod_{j\neq i}\psi_{k_j}(x_j),$$

for all the possible $i = 1,2,\dots,d$, and the linear space $F_n(R,\mathbb{R}^d)$ spanned by the defined functions $\chi_{\boldsymbol{k}i\pm}$. In particular we have

$$\partial_i\psi_{\boldsymbol{k}} \;=\; \frac{1}{r_i h}\Big[\chi_{\boldsymbol{k}i-} - \chi_{\boldsymbol{k}i+}\Big].$$

Obviously that $F_n(R,\mathbb{R}^d)$ is not a subspace of $W_2^1(\mathbb{R}^d)$. Rather we consider it as a subspace of $W_2^{-1}(\mathbb{R}^d)$ and endow it with the norm of $W_2^{-1}(\mathbb{R}^d)$-space. Hence, it is necessary to represent the elements of $F_n(R,\mathbb{R}^d)$ in the following form

$$f(n) \;=\; g_0(n) \,+\, \sum_{i=1}^{d}\partial_i g_i(n), \quad g_0 \in F_n(R,\mathbb{R}^d), \quad g_i \in E_n(R,\mathbb{R}^d), \tag{31}$$

and define the norm

$$\|f(n)\|_{2,-1}^2 \;=\; \sum_{i=0}^{d}\|g_i(n)\|_2^2. \tag{32}$$

Of course, that the representation (31) is not unique while the norm must be uniquely defined. In such cases usually one defines the norm by using the minimum value of norms for all the possible representations of the form (31). The following result is useful:

LEMMA **3.1** *Let $S \subset \mathbb{R}^d$ and $F_n(S,R,\mathbb{R}^d)$ be the linear subspace of $F_n(R,\mathbb{R}^d)$ of the functions in $F_n(R,\mathbb{R}^d)$ which are restricted to S. Then there exists a unique decomposition $F_n(S,R,\mathbb{R}^d) = F_n^{(0)}(S,R,\mathbb{R}^d)\oplus F_n^{(1)}(S,R,\mathbb{R}^d)$, where*

$$\begin{aligned} F_n^{(0)}(S,R,\mathbb{R}^d) \;&=\; \big\{f \in F_n(S,R,\mathbb{R}^d)\,:\,(1|f) = 0\big\},\\ F_n^{(1)}(S,R,\mathbb{R}^d) \;&=\; F_n(S,R,\mathbb{R}^d)\,\ominus\,F_n^{(0)}(S,R,\mathbb{R}^d). \end{aligned}$$

Each element $F_n^{(1)}(S,R,\mathbb{R}^d)$ has a unique representation in the form $\sum_i \partial_i g_i(n)$, $g_i \in E_n(R,\mathbb{R}^d)$ and (32) is the norm of $f(n) \in F_n(S,R,\mathbb{R}^d)$.

Let us define the integral operators $K_n, K_n^{(i)}$ with the respective kernels

$$\begin{aligned} \omega_n(\boldsymbol{x},\boldsymbol{y}) \;&=\; \textstyle\sum_{\boldsymbol{k}} \frac{1}{\|\psi_{\boldsymbol{k}}\|_1}\,\psi_{\boldsymbol{k}}(\boldsymbol{x})\,\psi_{\boldsymbol{k}}(\boldsymbol{y}),\\ \omega_n^{(i)}(\boldsymbol{x},\boldsymbol{y}) \;&=\; \textstyle\sum_{\boldsymbol{k}} \frac{1}{\|\chi_{\boldsymbol{k}i+}\|_1}\,\chi_{\boldsymbol{k}i+}(\boldsymbol{x})\,\chi_{\boldsymbol{k}}(\boldsymbol{y}) \;=\; \sum_{\boldsymbol{k}} \frac{1}{\|\psi_{\boldsymbol{k}}\|_1}\,\chi_{\boldsymbol{k}i-}(\boldsymbol{x})\,\chi_{\boldsymbol{k}i-}(\boldsymbol{y}). \end{aligned}$$

The following result is valid

$$\partial_i \sum_{\boldsymbol{k}} \frac{1}{\|\psi_{\boldsymbol{k}}\|_1} \psi_{\boldsymbol{k}}(\boldsymbol{x}) \chi_{\boldsymbol{k}i+}(\boldsymbol{y}) \;=\; \sum_{\boldsymbol{k}} \frac{1}{\|\psi_{\boldsymbol{k}}\|_1} \chi_{\boldsymbol{k}i-}(\boldsymbol{x}) \psi_{\boldsymbol{k}}(\boldsymbol{y}) \partial_i. \tag{33}$$

Now we define grid-functions $\boldsymbol{\mu}, \mathbf{f}$ by their components:

$$\mu_{\boldsymbol{k}} \;=\; -\frac{1}{\|\psi_{\boldsymbol{k}}\|_1} (\partial_i \psi_{\boldsymbol{k}} | f), \quad f_{\boldsymbol{k}} \;=\; (\chi_{\boldsymbol{k}i+} | f).$$

The grid function $\mathbf{f}$ is imbeded into $E_n(R, \mathbb{R}^d)$ in the usual way, $f(n) = \Phi_n(R)\mathbf{f}$. The grid function $\boldsymbol{\mu}$ is imbeded into the space $F_n(R, \mathbb{R}^d)$ by the mapping $\mu(n) = \Psi_n(R)\boldsymbol{\mu}$, where generally $\Psi_n(R)\mathbf{v} = \sum_{\boldsymbol{k}} \chi_{\boldsymbol{k}i+} v_{\boldsymbol{k}}$. By using (33) we get the basic equality

$$\mu(n) \;=\; \partial_i f(n),$$

so that $\|\mu(n)\|_{2,-1} = \|f(n)\|_2$. One calculates easily

$$\Big| \|f\|_2 - \|f(n)\|_2 \Big| \;\le\; (f|(I - K_n^{(i)})f)^{1/2} + \|f\|_2 \sup\big\{ \|(Z(w\boldsymbol{e}_e) - I, i)f\|_2 \,:\, |w| \le r_i \big\}.$$

LEMMA **3.2** *Let μ be a continuous linear functional on $W_2^1(\mathbb{R}^d)$. There exists discretizations $\boldsymbol{\mu}_n(R) \in l(G_n(R))$ such that the functions $\mu(n) = \Phi_n(R)\boldsymbol{\mu}_n(R) \in E_n(R, \mathbb{R}^d)$ converge strongly in $W_2^{-1}(\mathbb{R}^d)$ to μ and the following inequality is valid*

$$\langle\, u(n) \,|\, \mu \,\rangle \;=\; h^d \,\langle\, \mathbf{u}_n \,|\, \boldsymbol{\mu}_n \,\rangle \;=\; \|\psi_{\boldsymbol{k}}\|_1 \sum_{\boldsymbol{k}} u_{\boldsymbol{k}} \mu_{\boldsymbol{k}}. \tag{34}$$

PROOF: It suffices to consider the case $\mu = \partial f, f \in L_2(\mathbb{R}^d)$, $\|\mu\|_{2,-1} = \|f\|_2$. Let us define

$$\big(\boldsymbol{\mu}_n\big)_{\boldsymbol{k}} \;=\; -\,\|\psi_{\boldsymbol{k}}\|_1^{-1} (\partial \psi_{\boldsymbol{k}} | f).$$

The strog convergence of $\mu(n)$ towards μ in $W_2^{-1}(\mathbb{R}^d)$ is already demonstrated and the equality (34) is a result which follows straight forwardly from the definition of $\boldsymbol{\mu}_n$. **QED**

Because the sequence of functions $\mu(n) = \Psi_n(R)\boldsymbol{\mu}_n(R) \in F_n(R, \mathbb{R}^d)$ converges strongly in $W_2^{-1}(\mathbb{R}^d)$ to μ the sequence of numbers $\langle\, v(n) | \mu(n) \,\rangle$ convergences to the number $\langle\, v | \mu \,\rangle$ for any W_2^1-weakly convergent sequence of functions $v(n)$.

2. Discretized version of ellipticity

Descrete forms $a_n(\cdot, \cdot)$ on $l_0(G_n) \times l_0(G_n)$ are said to be strictly elliptic unuformly with respect to $n \in \mathbb{N}$ if

$$\underline{M} \,\| \mathbf{u} \|_{avg,2,1}^2 \;\le\; \lambda \| \mathbf{u}_n \|_2^2 + a_n(\mathbf{u}_n, \mathbf{u}_n) \;\le\; \overline{M} \,\| \mathbf{u} \|_{avg,2,1}^2. \tag{35}$$

with some positive numbers $\underline{M} \le \overline{M}$ and all $n \in \mathbb{N}$.

PROPOSITION **3.1** *Let the discretizations A_n of $A_0(\boldsymbol{x}) = -\sum \partial_i a_{ij}(\boldsymbol{x}) \partial_j$ be constructed by using either basic or extended schemes. If A_n have the compartmental structure then the descrete forms $\mathbf{v}, \mathbf{u} \mapsto \langle\, \mathbf{v} | A_n \mathbf{u} \,\rangle_R$ are strictly elliptic on $l_0(G_n(R)) \times l_0(G_n(R))$ uniformly with respect to $n \in \mathbb{N}$.*

3. W_2^1-convergence of approximate solutions

Inequality (35) and the variational equalities (28) imply the first result towards our proof of convergence of approximate solutions. If $\mathbf{u}_n$ solve (28) then (34) implies:

$$\underline{M} \,\| \mathbf{u}_n \|_{avg,2,1}^2 \;\le\; \langle\, \mathbf{u}_n \,|\, (\lambda I + A_n)\, \mathbf{u}_n \,\rangle \;\le\; \|u(n)\|_{2,1} \,\|\mu\|_{2,-1}. \tag{36}$$

By applying Theorem to the derived inequality on the left hand side we get $\|u(n)\|_{2,1} \le \underline{M}^{-1} \|\mu\|_{2,-1}$. i.e. the boundedness of sequance $\mathfrak{U} = \{u(R, n) : n \in \mathbb{N}\} \subset \cup_n E_n(R, \mathbb{R}^d)$.

COROLLARY **3.1** *Let* $\mathbf{u}_n = T(\lambda, A_n)\boldsymbol{\mu}_n$ *and* $u(R,n) = \Phi_n(R)\mathbf{u}_n$. *Then for each* R *there exists a subsequence* $\mathfrak{U}' \subset \mathfrak{U} = \{u(R,n) : n \in \mathbb{N}\} \subset \cup_n E_n(R,\mathbb{R}^d)$ *converges weakly in* $W_2^1(\mathbb{R}^d)$ *to some* $u \in W_2^1(\mathbb{R}^d)$.

In order to prove the strong convergence of constructed sequence $\mathfrak{U}$ to the unique solution to (27) we need the consistency for the analized discretized forms a_n. This property is defined in terms of sequences of functions with a particular structure:

$$\begin{aligned} \mathfrak{V} &= \{v(n) : n \in \mathbb{N}\} \subset \cup_n E_n(R,\mathbb{R}^d), \\ \mathfrak{U} &= \{u(n) : n \in \mathbb{N}\} \subset \cup_n E_n(R,\mathbb{R}^d). \end{aligned} \tag{37}$$

DEFINITION 3.1 (Consistency) *We say that forms* $a_n(\cdot,\cdot)$ *on* $E_n(R,\mathbb{R}^d) \times E_n(R,\mathbb{R}^d)$ *are consistent with the form original form* a *on* $W_2^1(\mathbb{R}^d) \times W_2^1(\mathbb{R}^d)$ *if*

$$a(v,u) = \lim_n h^d\, a_n(v(n), u(n))$$

is valid for any pair $\mathfrak{V}, \mathfrak{U}$ *of (37) such thta* $\mathfrak{V}$ *is weakly converging in* $W_2^1(\mathbb{R}^d)$ *to* v *and* $\mathfrak{U}$ *is strongly converging in* $W_2^1(\mathbb{R}^d)$ *to* u.

Let u^* be the solution to (27). Then the sequence of functions $\hat{u}^*(n)$, defined by (11), strongly converges to u^* in W_2^1. In the remaining part of this analysis we have to demonstrate the expected property $\lim_n u(R,n) = \lim_n \hat{u}^*(R,n) = u^*$ for each R. We follow the well-known finite element technique.

$$\begin{aligned} \underline{M}\, h^d \,\|\mathbf{u}_n - \hat{\mathbf{u}}_n^*\|_{avg,2,1}^2 &\leq h^d \langle \mathbf{u}_n - \hat{\mathbf{u}}_n^* \,|\, (\lambda I + A_n)(\mathbf{u}_n - \hat{\mathbf{u}}_n^*) \rangle \\ &= h^d \langle \mathbf{u}_n - \hat{\mathbf{u}}_n^* \,|\, (\lambda I + A_n)\,\mathbf{u}_n \rangle \\ &- h^d \langle \mathbf{u}_n - \hat{\mathbf{u}}_n^* \,|\, (\lambda I + A_n)\,\hat{\mathbf{u}}_n^* \rangle \\ = h^d \,\langle \mathbf{u}_n - \hat{\mathbf{u}}_n^* \,|\, \boldsymbol{\mu}_n \rangle &- h^d \langle \mathbf{u}_n - \hat{\mathbf{u}}_n^* \,|\, (\lambda I + A_n)\,\hat{\mathbf{u}}_n^* \rangle. \end{aligned} \tag{38}$$

By Lemma 3.2 the first term on the right hand side converges to $\langle u - u^* | \mu \rangle$. If the consistency property of Definition 3.1 is valid the second term converges to the same value. In this way we come to the following result:

THEOREM **3.1** *Let* $\mathfrak{U}$ *be as in Corollary 3.1. If the discretized forms* $a_n(\cdot,\cdot)$ *are consistent with the original form (TR) the sequence* $\mathfrak{U}$ *converges* $W_2^1(\mathbb{R})$*-strongly to the unique solution* u^* *to (27).*

From this result, Lemma 3.2 and Lemma 1.1 we get another important result for $\lambda = 0$.

COROLLARY **3.2** *Let* D *be a bounded domain with Lipsithz boundary and* $\mu \in W_2^{-1}(D)$. *Let* $A_n(D)$ *be the restriction to* $G_n(D)$ *of* A_n, $\boldsymbol{\mu}_n$ *on* $G_n(D)$ *satisfy (34) and* $\mathbf{u}_n = A_n(D)^{-1}\boldsymbol{\mu}_n$. *Then the sequence* $\mathfrak{U}$ *converges strongly in* $\dot{W}_2^1(D)$ *to the unique weak solution* u *of (26).*