Chapter 1

TEORIJSKI REZULTATI

1.1 ZATVORENI OPERATOR

1.

DEFINITION 1.1.1 (Zatvoreni operator) *Neka je L Banachov prostor i A linerani s podrucjem definicije $\mathcal{D}(A) \subset L$. Kazemo da je A zatvoren ako za svaki niz $\mathcal{U} = \{u_n : n \in \mathbb{N}\} \subset \mathcal{D}(A)$ i pripadni $\mathcal{V} = \{Au_n : n \in \mathbb{N}\} \subset L$ imamo slijedecu indukciju*

$$\mathcal{U},\ \mathcal{V}\ \text{ konvergiraju u } L \quad \Rightarrow \quad u = lim_n\, u_n \in \mathcal{D}(A).$$

Drugi prikaz ovog pojma je iz grafa operatora:

$$\Gamma(A) \ = \ \{(u, Au)\ :\ u \in \mathcal{D}(A),\ Au \in L\} \subset L \times L.$$

Slijedi da je A zatvoren ako mu je graf zatvoren skup u $L \times L$.

primjeri:

- $A\mathbf{u} = \{u_k/k : k \in \mathbb{N}\}$, $\mathcal{D}(A) = l_p(\mathbb{N}), \mathcal{R}(A) \neq l_p(\mathbb{N})$.
- $A\mathbf{u} = \{ku_k : k \in \mathbb{N}\}$, $\mathcal{D}(A) = \{\sum_k |ku_k|^p < \infty\}, \mathcal{R}(A) = l_p(\mathbb{N})$.
- $Au = \partial u$ u $L_p(0,1)$. Imamo za $u' = f$ ovo: $\mathcal{D}(A) = C([0,1]) \cap \{u(0) = 0,\ u' \in L_p(0,1)\}$, $\mathcal{R}(A) = L_p(0,1)$.
- $-u'' = f$ u prostoru $L_2(\mathbb{R})$. Izraz je $u(x) = (1/2)\int |x-y| f(y)\, dy$ pa $\mathcal{R}(A) \neq L_2(\mathbb{N})$.
- $-u'' = f$ u $L_2(0,1)$ ima rjesenje $u(x) = \int G(x,y)f(y)dy$ gdje su $\psi(x) = 1-x.\phi(x) = x$, te $G(x,y) = \psi(x)\phi(y)\mathbb{1}_{[0,x]} + \phi(x)\psi(y)\mathbb{1}_{[x,1]}$. sada je ocito $\mathcal{R}(A) = L_2(\mathbb{N})$.

Kazemo da je operatopr **akretivan** ako je $||(\lambda I + A)u|| \geq \lambda ||u||$ za svaki par $\lambda > 0, u \in \mathcal{D}(A)$. Kazemo da A udovoljava Teoremu Hille-Yosida, ako

$$\begin{array}{l} \mathcal{D}(A) \ \text{je gust u } L, \\ A \ \text{je akretivan}, \\ \text{za neki } \lambda > 0 \ \text{vrijedi } (\lambda I + A)\mathcal{D}(A) \ = \ L. \end{array}$$

Kratko pisemo HY-uvjetima. Istaknimo da "neki" implicira "svaki". Dokaz ide iz

$$T(\mu, A) \ = \ T(\lambda, A)\, T(1, (\mu - \lambda)T(\lambda, A)) \ = \ T(\lambda, A) \sum_k \Big((\lambda - \mu)T(\lambda, A)\Big)^k,$$

te cinjenice da je $||T(\lambda, A)|| \leq 1/\lambda$. Ujedno slijedi $\lambda \mapsto T(\lambda, A)$ je analiticka u $\Re z > 0$.

Za zatvoreni operator je podrucje vrijednosti zadano sa $\mathcal{R}(A) = (\lambda I + A)\mathcal{D}(A)$. Za zatvoreni akretivni operator i bilo koji $\lambda > 0$ to je zatvoreni skup. Ako A udovoljava HY-uvjetu to je citav prostor.

primjeri:

- Operatori su $A = \partial$ i $A = -\partial$ u prostorima $L_p(0,\infty)$, te imamo zadacu i rjesenje:

$$\begin{aligned} u' + \lambda u &= f, \quad u(x) = \exp(-\lambda x) \textstyle\int_0^x \exp(\lambda y) f(y) dy, \\ -v' + \lambda v &= g, \quad v(x) = \exp(\lambda x) \textstyle\int_x^\infty \exp(-\lambda y) g(y) dy. \end{aligned}$$

  Slijedi da inverz $T(\lambda, A)$ ima ove norme $\|T(\lambda, A)\|_1 = \|T(\lambda, A)\|_\infty = 1/\lambda$. Isto vrijedi za $T(\lambda, -A)$. Sada to vrijedi za sve L_p-prostore. Dakle je $\mathcal{D}(A) = T(\lambda, A)L_p(0,\infty), \mathcal{R}(A) = L_p(0,\infty)$.

  Pokazimo akretivnost. Za $f = f_+ - f_-$ imamo pozitivna rjesenja u_+, u_-. Dalje je $\|(A + \lambda I)u\| = \|f_+ - f_-\| = \|f_+\| + \|f_-\| \geq \lambda(\|u_+\| + \|u_-\|) \geq \lambda\|u_+ - u_-\|$.

  Dokaz da je skup $\mathcal{D}(A)$ je gust u $L_p(0,\infty)$ za $p < \infty$: Za $f \in L_p(0,\infty), g \in L_q(0,\infty)$ imamo

$$(g \,|\, T(\lambda, A)f) \;=\; (T(\lambda, -A)g \,|\, f).$$

- Za $A = -\partial^2$ na $\mathbb{R}$ imamo $T(\lambda, A)$ iz Fourierove transformacije. Sliedi da A ima HY-uvjete.

- A sada $A(x) = -\partial a(x) \partial$ gdje je $a(\cdot)$ izmjeriva ogranicena s obje strane pozitivnim konstantama.

LEMMA **1.1.1 (Yoshidina aproksimacija)** *Neka A udovoljava HY-uvjetima. Tada*

(i) Niz $\{nT(n, A) : n \in \mathbb{N}\}$ konvergira jedinici jaku u L.

(ii) Niz $\{nAT(n, A) : n \in \mathbb{N}\}$ konvergira operatoru A jaku u $\mathcal{D}(A)$.

(ii) Neka je $A_n = nAT(n, A)$. Niz $\{T(\lambda, A_n) : n \in \mathbb{N}\}$ konvergira inverzu $T(\lambda, A)$ jaku u L.

PROOF: Za dokaz (iii) koristimo

$$Q^{-1} \,-\, P^{-1} \;=\; Q^{-1}(P - Q)\, P^{-1},$$

i dio (ii). **QED**

Opertor A_n se zove **Yoshidina aproksimacija** operatora A.

VAZNA DOPUNA. Pojam zatvorenog operatora se moze prosiriti na operatore izmedju dva prostora. Neka su L', L'' Banachovi prostori i A definiran na $\mathcal{D}(A) \subset L'$. Graf je

$$\Gamma(A) \;=\; \{(u, Au) \,:\, u \in \mathcal{D}(A),\; Au \in L''\} \;\subset\; L' \times L''.$$

Kaze se da je $A \in \mathsf{L}(L', L'')$ zatvoren ako mu je graf zatvoren.

2.

For natural r the ratio $\Gamma(k + r)/\big(\Gamma(k + 1)\Gamma(r)\big)$ is usually denoted by $\binom{k+r-1}{k}$. In this monography the symbol $\binom{k+r-1}{k}$ is also used for all $r \in (0,\infty)$. Let K be an operator and let us assume that there exists $m \in \mathbb{N}$ such that $\|K^m\|_p < 1$. Then the series:

[exn1.1]

$$(I \,-\, K)^{-\alpha} \;=\; \sum_{k=0}^{\infty} \binom{k + \alpha - 1}{k} K^k \tag{1.1}$$

converges in $L_p(D, \mu)$. By using the The Rietz-Thorin interpolation theorem [?] and the explicite forms of L_1- and L_∞-norms we get:

PROOF: Nakon prve ocjene na desnoj strani imamo za $\rho = \|K^m\|_p$:

$$\begin{aligned} &\big[c(0,\alpha) + c(1,\alpha) + \ldots + c(m-1,\alpha)\big]\,\rho^0 + \\ &\big[c(m,\alpha) + c(m+1,\alpha) + \ldots + c(2m-1,\alpha)\big]\,\rho + \\ &\ldots \\ &\big[c(rm,\alpha) + c(rm+1,\alpha) + \ldots + c((r=1)m-1,\alpha)\big]\,\rho^{r-1} + \\ &\ldots. \end{aligned}$$

Sada definiramo $s = \rho^{1/m}$ i novom ocjenom dobijemo:

$$\|(I - K)^{-\alpha}\|_p \leq s^{-(m-1}(1-s)^{-\alpha},$$

t.j. konvergenciju po normi. QED

[lemn1.1] LEMMA **1.1.2** *Let K be bounded. If $\|K^m\|_p < 1$, $p = 1, \infty$, for some $m \in \mathbb{N}$, then*

$$\|(I - K)^{-\alpha}\|_p = \sum_{k=0}^{\infty} \binom{k+\alpha-1}{k} \|K^k\|_p$$

for all $p \in [1, \infty]$.

[lemLa10.3] LEMMA **1.1.3 (O potenciji inverza)** *Let A be a linear operator in a Banach space L satisfying HY-conditions. Then: there exists $T(\lambda, A)^\alpha$ in L such that $D(\alpha) = T(\lambda, A)^\alpha L$ is dense in L and there exists $(\lambda I + A)^\alpha$ on $D(\alpha)$ such that $(\lambda I + A)^\alpha T(\lambda, A)^\alpha = I$.*

PROOF: Let $A_n = (1/n)I - n^2 T(n, A)$ be the Yoshida approximations of A. Then there exists a sequence of bounded operators defined by:

$$\begin{aligned} T(\lambda, A_n)^\alpha &= \Big((\lambda + n)\, I - n^2\, T(\lambda, A)\Big)^{-\alpha} \\ &= \frac{1}{(\lambda+n)^\alpha} \sum_{k=0}^{\infty} c(k,\alpha) \left(\frac{n}{\lambda+n}\right)^k \big(n\, T(n,A)\big)^k, \end{aligned} \tag{1.2}$$

[exLa10.4]

where $c(k,p) = \Gamma(k+p)/\big(\Gamma(k+1)\Gamma(p)\big)$ is usually denoted by $\binom{k+p-1}{k}$ for natural p. They converg strongly in L to a bounded operator $K(\alpha)$. From $\|T(\lambda, A_n)^\alpha\| \leq 1/\lambda^\alpha$ there follows $\|K(\alpha)\| \leq 1/\lambda^\alpha$. Let $\alpha = q/p, q, p \in \mathbb{N}$. Then $\big(T(\lambda, A_n)^\alpha\big)^p = T(\lambda, A_n)^q$ implying $K(\alpha)^p = T(\lambda, A)^q$. Hence, by definition $T(\lambda, A) = K(\alpha)$ for $\alpha = q/p$. However, $\alpha \mapsto T(\lambda, A_n)^\alpha$ of (1.2) are analytic in $\Re\alpha > 0$ so that the established representation (1.2) is valid for all positive α. **QED**

Sada promatramo operatore na $L_p(D, \mu)$. Definiramo pozitivnost za neki ograniceni oprator Q. Kazemo $Q \geq 0$ ako je $Qu \geq 0$ na D za svaki $u \geq 0$. Slicno $Q > 0$ je operator sa svojstvom $Qu > 0$ na D za svaki netrivijalni $u \geq 0$.

LEMMA **1.1.4** *Let A be a linear operator in $L(D, \mu)$ satisfying HY-conditions. Then:*

(i) If $T(\lambda, A) \geq 0$ there exists a unique $T(\lambda, A)^\alpha \geq 0$.

(ii) If L is a Hilbert space then the unique operator $T(\lambda, A)^\alpha$ of (ii) is positive definite.

PROOF: From $T(\lambda, A) \geq 0$ there follows $K(\alpha) \geq 0$. **QED**

Let $\lambda > 0$. The r-th power of $T(\lambda, A)$ has a representation (1.3). This is not the only possible representation. However this is the unique representation for which entries are non-negative. Other possibile representations can be derived from the spectral representation of A.

[defn1.0] DEFINITION 1.1.2 (COMPARTMENTAL STRUCTURE) *A bounded operator A in $L_\infty(D, \mu)$ is said to be of positive type if $A = pI - B$, $B \geq 0$ and $\|B\|_\infty \leq p$. It is called conservative if $B1 = 1$. A bounded operator A in $L_1(D, \mu)$ is said to have the compartmental structure if $A = pI - B$, $B \geq 0$ and $\|B\|_1 \leq p$. It is called conservative if for each $u \geq 0$ on D there holds $\|Bu\|_1 = p\|u\|_1$.*

In the case of $l_1(I)$ this definition and definition for matrices are in agreement.

[lemn1.0] LEMMA **1.1.5** *Let A in $L_1(D, \mu)$ be compartmental. If A is conservativ then $\|B^m\|_1 = p^m$ for each $m \in \mathbb{N}$.*

PROOF: If A is conservative then the equality $\|B^m u\|_1 = p^m \|u\|_1$ must be valid for each $u \geq 0$ implying the assertion. **QED**

3.

Definira se spektar operatora A i rezolventa $R(\lambda, A) = T(\lambda, -A)$.

[lemn1.5] LEMMA **1.1.6** *Let A in $L_1(D,\mu)$ be a compartmental operator. Then $0 \in sp(A)$ iff A is conservative.*

PROOF: Let A be conservative. We have to construct a sequance $u_n \in L_1(D,\mu), \|u_n\|_1 = 1$, such that $Au_n \to 0$ in $L_1(D,\mu)$. It sufficies to consider the case $A = I - B$. Hence $\|B^m u_0\|_1 = 1$ for each $u_0 \geq 0, \|u_0\|_1 = 1$. Now we consider

$$u_n = \frac{1}{n}\left[I + B + B^2 + \cdots + B^{n-1}\right] u_0,$$

and calculate

$$\begin{aligned}\|u_n\|_1 &= \frac{1}{n}\sum_{k=1}^{n} \|B^{k-1}u_0\|_1 = 1,\\ Bu_n - u_n &= \tfrac{1}{n}\left[B^n u_0 - u_0\right] \to 0.\end{aligned}$$

Let us suppose now that $0 \in sp(A)$ while A is not conservative. There must exist $m \in \mathbb{N}$ such that $\rho = \|B^m\|_1 < 1$. We have

$$(I - B)^{-1} = \left[I + B + B^2 + \cdots + B^{m-1}\right] \sum_{r=0}^{\infty} B^{mr},$$

so that $\|(I-B)^{-1}\|_1 \leq m/(1-\rho)$. Hence A^{-1} is bounded on $L_1(D,\mu)$ and $0 \in \mathbb{C}$ belongs to resolvent set. **QED**

Let us apply Lemma to compartmental operators $A = pI - B = p(I - Q)$ and $\lambda I + A = (\lambda + p)I - B$. Hence, in the present case $K = \rho Q$, where $\rho = p/(p+\lambda)$. Therefore, the operator

[exn1.2]
$$T(\lambda, A)^\alpha = \left(\frac{1}{p+\lambda}\right)^\alpha \sum_{k=0}^{\infty} \binom{k+\alpha-1}{k} \rho^k Q^k \tag{1.3}$$

is defined for any pair of numbers $\alpha > 0, \lambda > 0$.

1.2 ELIPTICKI OPERATOR NA SVOM PROSTORU

1.

Za diferencijalni operator $A = -\sum_{ij} \partial_i a_{ij} \partial_j + \sum_i b'_i \partial_i + \sum_i (\partial_i b''_i \cdot) + c$ definiramo bilinearnu formu

[ex2.3]
$$\begin{aligned} a(v,u) &= \sum_{i,j=1}^{d} \int_D a_{ij}(\boldsymbol{x})\, \partial_i v(\boldsymbol{x})\, \partial_j u(\boldsymbol{x})\, d\boldsymbol{x} \\ &+ \sum_{i=1}^{d} \int_D b'_i(\boldsymbol{x})\, v(\boldsymbol{x})\, \partial_i u(\boldsymbol{x}) - \sum_{i=1}^{d} \int_D b''_i(\boldsymbol{x})\, \partial_i v(\boldsymbol{x})\, u(\boldsymbol{x})\, d\boldsymbol{x} + \int_D c(\boldsymbol{x})\, v(\boldsymbol{x})\, u(\boldsymbol{x})\, d\boldsymbol{x}. \end{aligned} \tag{1.4}$$

te promatramo zadace:

Jednadzba za funkciju $\boldsymbol{x} \mapsto u(\boldsymbol{x})$:

$$(\lambda + A(\boldsymbol{x}))\, u(\boldsymbol{x}) = f(\boldsymbol{x}),$$

gdje je $f \in W_2^{-1}(\mathbb{R}^d)$ se zovu zadaca za elipticki diferencijalni operator na $W_2^1(\mathbb{R}^d)$. Funkcija $u(\cdot)$ se zove jako rjesenje zadace. Jednadzbe za funkciju $\boldsymbol{x} \mapsto u(\boldsymbol{x})$:

$$\lambda\,(v|u) + a(v,u) = \langle\, v \,|\, f \,\rangle, \quad v \in W_2^1(\mathbb{R}^d)$$

se zove variaciona zadaca za elipticki diferencijalni operator. Funkcija $u(\cdot)$ se zove slabo rjesenje zadace.

[Thm2.1] THEOREM **1.2.1 (O rjesenju zadace na $\mathbb{R}^d$)** *Postoji $\lambda_0 \geq 0$ takav da obje zadace imaju jedinstvena rjesenja, koja se podudaraju, kada god je $\lambda > \lambda_0$.*

2.

Operator $\lambda I + H = \lambda I - \sigma^2 \Delta$ na $\mathbb{R}^d$ je zadan ovako

$$\mathcal{D}(H) = \{u = F[\hat{u}] \,:\, \boldsymbol{p} \mapsto (1+\sigma^2 \boldsymbol{p}^2)\,\hat{u}(\boldsymbol{p}) \in L_2(\mathbb{R}^d)\}$$

te vrijedi

$$(\lambda - \sigma^2 \Delta)\,\mathcal{D}(H) \;=\; L_2(\mathbb{R}^d),$$

kada gofd je $\lambda \notin (-\infty, 0] \subset \mathbb{C}$. To znaci da za $f \in L_2(\mathbb{R}^d)$ imamo rjesenje $u = T(\lambda, H) f = F[\hat{w}]$ gdje je

$$\hat{w}(\boldsymbol{p}) \;=\; \frac{\hat{f}(\boldsymbol{p})}{\lambda + \sigma^2 \boldsymbol{p}^2}.$$

U $\boldsymbol{x}$-reprezentaciji je to integralni operator s jezgrom:

$$t(\alpha, \lambda, H, \boldsymbol{z}) \;=\; \left(\frac{\lambda}{\pi\sigma^2}\right)^{d/2} \frac{2^{1-\alpha}}{\lambda^\alpha \Gamma(\alpha)} \, \frac{K_{(d-2\alpha)/2}(z)}{z^{(d-2\alpha)/2}}, \quad z \;=\; \frac{\sqrt{2\lambda}|\boldsymbol{z}|}{\sigma}, \tag{1.5}$$

za $\alpha = 1$. Here $K_\nu(z)$ are Bessel functions. Properties of the Bessel function K_ν are well known, $K_{-\nu}(z) = K_\nu(z)$ for $\Re z > 0$, $K_\nu(z) > 0$ for $z > 0$ and:

$$\begin{array}{llll} K_\nu(z) \;\le\; c(\nu)\,\dfrac{\exp(-z)}{\sqrt{z}} & \text{for} \quad z > 1 & \text{any} \quad \nu, \\ K_0(z) \;\le\; c_0(\nu)\,\big(1 \,+\, |\ln z|\big) & \text{for} \quad 0 < z < 1, & \\ K_\nu(z) \;\le\; c(\nu)\, z^{-\nu} & \text{for} \quad 0 < z < 1 & \text{any} \quad \nu > 0, \end{array}$$

where $c_0(\nu)$ are z-independent.

Jos treba reci da je linearni prostor $C_0^{(2)}(\mathbb{R}^d)$ gust podrostor u $L_2(\mathbb{R}^d)$ i ujedno je $C_0^{(2)}(\mathbb{R}^d) \subset \mathcal{D}(H) \subset L_2(\mathbb{R}^d)$.

3.

Jezgro $t(\lambda, H, \cdot)$ je dobro i za slucaj da promatramo preslikavanje $T(\lambda, H)$ sa $L_p(\mathbb{R}^d)$ u $L_p(\mathbb{R}^d), 1 \le p \le \infty$. Stoga definiramo

$$\mathcal{D}(H) \;=\; T(\lambda, H)\, L_p(\mathbb{R}^d), \quad \lambda > 0,$$

te operator H za prostore $L_p(\mathbb{R}^d)$. Za $p = \infty$ linerani prostor $\mathcal{D}(H)$ nije gust, dok je za sve ostale gust u osnovnom prostoru. Uvjek imamo rjesenje jednadzbe $(\lambda + H) u = f, f \in L_p(\mathbb{R}^d)$

4.

Jezgro $t(1, \lambda, H, \cdot)$ je dobro poslije 1. deriviranja, te za derivaciju imamo Youngovu nejednakost:

$$\begin{array}{rcl} g(\boldsymbol{x}) &=& \int \partial_i t(\lambda, H, \boldsymbol{x} - \boldsymbol{y})\, f(\boldsymbol{y})\, d\boldsymbol{y}, \\ ||f||_p &\le& ||\partial_i T(\lambda, H)||_1\, ||g||_p. \end{array}$$

Dakle se $L_p(\mathbb{R}^d)$ preslikava u $W_p^1(\mathbb{R}^d)$.

5.

Jezgro $t(\alpha, \lambda, H)$ definira $T(\lambda, H)^\alpha$. Ovo treba znati:

$$\begin{array}{rcll} \partial_i T(\lambda, H)^\alpha \;=\; T(\lambda, H)^\alpha \partial_i &:& L_2(\mathbb{R}^d) \mapsto L_2(\mathbb{R}^d) & \alpha > 0, \\ T(\lambda, H)^\alpha &:& W_2^1(\mathbb{R}^d) \mapsto L_2(\mathbb{R}^d), & \alpha \ge 1/2 \\ T(\lambda, H)^\alpha &:& L_2(\mathbb{R}^d) \mapsto W_2^1(\mathbb{R}^d) & \alpha \ge 1/2. \end{array}$$

as follows from

$$\partial_i T(\lambda, H)^\alpha u \;=\; F[\hat{w}], \quad \hat{w}(\boldsymbol{p}) \;=\; \frac{p_i}{(\lambda + \sigma^2\, \boldsymbol{p}^2)^\alpha}\, \hat{u}(\boldsymbol{p}).$$

Zatim je

$$\|T(\lambda,H)^\alpha\|_2 \;=\; \lambda^{-\alpha}, \qquad \sum_{i=1}^{d} \|\partial_i T(\lambda,H)^\alpha\|_2^2 \;\le\; \sigma^{-2}\,\lambda^{1-2\alpha}.$$

Iz F-transformacije slijedi postojanje operatora $B = (\lambda I + H)^\alpha$ na $\mathcal{D}(B) = T(\lambda,H)^\alpha L_2(\mathbb{R}^d)$. Operator $(\lambda I + H)^\alpha$ ima svojstvo $(\lambda I + H)^\alpha T(\lambda,H)^\alpha = I$.

6.

Koristimo formalno:

$$\begin{aligned} T(\lambda,A) \;&=\; \big(\lambda I + H + (A-H)\big)^{-1} \;=\; T(\lambda,H)^{1/2}\Big(I - T(\lambda,H)^{1/2}(H-A)T(\lambda,H)^{1/2}\Big)^{-1} T(\lambda,H)^{1/2} \\ &=\; T(\lambda,H)^{1/2}\left(\textstyle\sum_{k=0}^{\infty} W^k\right) T(\lambda,H)^{1/2}, \end{aligned}$$

gdje su

$$W \;=\; T(\lambda,H)^{1/2}\,(H-A)\,T(\lambda,H)^{1/2}.$$

i $H = -\sigma^2\Delta$, te

$$H - A \;=\; -\sum_{ij} \partial_i\left(\sigma^2\delta_{ij} - a_{ij}(\boldsymbol{x})\right)\partial_j.$$

Treba izracunati normu operatora W. Imamo

$$\begin{aligned} (u|Wu) \;&=\; \textstyle\sum_{ij}\int \partial_i u(\boldsymbol{x})\left(\sigma^2\delta_{ij} - a_{ij}(\boldsymbol{x})\right)\partial_j u(\boldsymbol{x})\,d\boldsymbol{x}, \\ (\sigma^2 - \overline{M})\textstyle\sum_i \|\partial_i u\|_2^2 \;&\le (u|Wu) \;\le\; (\sigma^2 - \underline{M})\textstyle\sum_i \|\partial_i u\|_2^2, \end{aligned}$$

gdje je $v = T(\lambda,A)^{1/2}$. Za $\sigma^2 = \overline{M}$ imamo pozitivno definitni W, tako da je

$$0 \;\le\; (u|Wu) \;\le\; (1-\gamma)\,\|u\|_2^2,$$

gdje je $\gamma = \underline{M}/\overline{M} < 1$. Dakle je $\|W\|_2 \le (1-\gamma)$. Ocito postoji $(I-W)^{-1}$, $\|(I-W)^{-1}\|_2 \le \gamma^{-1}$.

Provjeriti da je $T(\lambda,H)^{1/2}(I-W)^{-1}T(\lambda,H)^{1/2}$ jednako $T(\lambda,A)$ iz izraza

$$(v|(\lambda I + A)\,T(\lambda,H)^{1/2}(I-W)^{-1}T(\lambda,H)^{1/2}u) = (v|u)$$

za svaki par (v,u) iz gustog skupa. Uzimamo skup $W_2^1(\mathbb{R}^d)$ te racunamo

$$\begin{aligned} &(v|(\lambda I + H)\,T(\lambda,H)^{1/2}(I-W)^{-1}T(\lambda,H)^{1/2}u), \\ &(v|(A-H)\,T(\lambda,H)^{1/2}(I-W)^{-1}T(\lambda,H)^{1/2}u). \end{aligned}$$

i zbrojimo. Rezultat je $(v|u)$.

THEOREM **1.2.2 (Reprezentacija inverza eliptickog operatora)** *Za elipticki operatoe $A(\boldsymbol{x}) = -\sum_{ij}\partial_i a_{ij}(\boldsymbol{x})\partial_j + \lambda$ se inverz reprezentira sa:*

$$T(\lambda,A) \;=\; T(\lambda,H)^{1/2}\,(I-W)^{-1}\,T(\lambda,H)^{1/2}$$

na $L_2(\mathbb{R}^d)$. Povrh toga je

$$\begin{aligned} &T(\lambda,A) \;:\; W_2^{-1}(\mathbb{R}^d) \mapsto W_2^1(\mathbb{R}^d), \\ &\|T(\lambda,A)\|_{\mathsf{L}(W_2^{-1},W_2^1)} \le \frac{1}{\sqrt{\overline{M}\underline{M}}}\,\frac{1}{\lambda}\left(\lambda + \overline{M}\right)^{1/2}, \end{aligned}$$

gdje je $\mathsf{L}(W_2^{-1},W_2^1)$ linerani prostor ogranicenih operatora sa $W_2^{-1}(\mathbb{R}^d)$ u $W_2^1(\mathbb{R}^d)$. Povrh toga operator $T(\lambda,A)$ je bijekcija izmedju prostora $W_2^{-1}(\mathbb{R}^d)$ i $W_2^1(\mathbb{R}^d)$.

[isp2.1] IPPITNI ZADATAK **1.1** *Za dokaz glavnog teorema treba pokazati da zatvarac A promatranog diferencijalnog operatora $A(\boldsymbol{x})$ ima HY-uvjete za $\lambda > \lambda_0$.*

1.3 ELIPTICKI OPERATOR NA OBLASTI

Literatura (osnovnas):

D. Gilberg, N. S. Trudinger, *Elliptic Partial Differential Equations*, Springer, Berlin, 1983.

O. A. Ladyzhenskaya, N. N. Ural'tseva , *Linear and Quasilinear Elliptic Partial Differential Equations*, Academic Press, N.Y., 1968

Promatramo slucaj ogranicene oblasti D sa Lipsicxevom granicom ∂D. Jednadzba za funkciju $D \ni \boldsymbol{x} \mapsto u(\boldsymbol{x}) \in \dot{W}_2^1(D)$:

[exn3.1] (1.6)
$$\begin{aligned} (\lambda + A(\boldsymbol{x}))\, u(\boldsymbol{x}) \; &= \; f(\boldsymbol{x}), \quad \boldsymbol{x} \in D, \\ u|\partial D \; &= \; 0, \end{aligned} ,$$

gdje je $f \in W_2^{-1}(D)$ se zove zadaca za elipticki diferencijalni operator na D uz homogene Dirichletove rubne uvjete. Funkcija $u(\cdot) \in \dot{W}_2^1(D)$ se zove jako rjesenje zadace. Jednadzbe za funkciju $\boldsymbol{x} \mapsto u(\boldsymbol{x}) \in \dot{W}_2^1(D)$:

[exn3.1a] (1.7)
$$\lambda\,(v|u) \;+\; a(v,u) \;=\; \langle\, v \,|\, f\,\rangle, \quad v \in \dot{W}_2^1(D)$$

se zove variaciona zadaca za elipticki diferencijalni operator uz homogene Dirichletove rubne uvjete. Funkcija $u(\cdot)$ se zove slabo rjesenje zadace.

[Thm3.1] THEOREM **1.3.1 (O rjesenju zadace na oblasti)** *Neka je zadan elipticki diferencijalni operator $A(\boldsymbol{x})$ na oblasti D. Tada postoji $\lambda_0 \geq 0$ takav da postoje jedinstvena jaka i slaba rjesenje elipticke zadace uz homogene Dirichletove rubne uvjetu kada god je $\lambda \geq \lambda_0$, te su ta rjesenja medjusobno jednaka.*

1.

Najprije promatramo $H(\boldsymbol{x}) \;=\; -\sigma^2\Delta$ i pripadnu zadacu (1.6). Pocinjemo s pomocnim rezultatom.

Let the functions $u, \hat{u} \in L_2(\mathbb{R}^d)$ be mutually related by the Fourier transformation, $u = F[\hat{u}]$, $\hat{u} = F^{-1}[u]$. If we write down an apparent equality

$$\hat{u}(\boldsymbol{p}) \;=\; \frac{1}{1+\boldsymbol{p}^2}\,\hat{u}(\boldsymbol{p}) \;+\; \sum_{i=1}^{d} p_i\,\frac{1}{1+\boldsymbol{p}^2}\,p_i\,\hat{u}(\boldsymbol{p}),$$

and perform the transformation F to both sides we get

[exp3.1] (1.8)
$$u \;=\; T(1,H)u \;-\; \sum_{i=1}^{d} \partial_i\, T(1,H)\, \partial_i u,$$

being a representation of u in terms of its first partial derivatives. Now we can extend this representation to hold for a wider class of functions.

LEMMA **1.3.1** *Let $u \in W_p^1(\mathbb{R}^d), p \in [1,\infty]$. Then (1.8) is valid.*

A proof follows from Young inequality for d-dimesional spaces.

The representation (1.8) makes possible to demonstrate some of results which are known as Sobolev lemma:

$$\begin{aligned} &u \in W_p^1(\mathbb{R}^d),\; p > d \;\Rightarrow\; u \;=\; T(\lambda,H)f \;\in\; \dot{C}^{(\tau)}(\mathbb{R}^d), \quad \tau = 1 - d/p, \\ &u \in W_p^1(\mathbb{R}^d),\; p < d \;\Rightarrow\; u \;=\; T(\lambda,H)f \;\in\; L_r(\mathbb{R}^d),\; r = \frac{dp}{d-p}. \end{aligned}$$

A proof of the first result follows from the previous Corollary. Elsewhere, proofs based on properties of kernels are in 115, 117 of vol 5. Smirnov. A simple proof of the second relation is given in GT. A proof based on non-homogeneous L_p-spaces and Hardy Littlewood theorem can be taken from Besov.

Another important part of Sobolev lemma is the compact imbedding:

[lem3.5] LEMMA **1.3.2** *Let $\mathcal{B} \subset \dot{W}_p^1(D), p \in [1,\infty]$ be bounded. Then $\mathcal{B}$ is conditionaly compact in $L_p(D)$.*

PROOF: Let us rewrite (1.8) in the following way:

$$u(\boldsymbol{x}) \;=\; \int_D t(\lambda, H, \boldsymbol{x}-\boldsymbol{y})\, u(\boldsymbol{y})\, d\boldsymbol{y} \;+\; \sum_{i=1}^d \int_D \partial_i t(\lambda, H, \boldsymbol{x}-\boldsymbol{y})\, \partial_i u(\boldsymbol{y}) d\boldsymbol{y}.$$

We have

$$\begin{aligned} u(\boldsymbol{x}+\boldsymbol{h}) - u(\boldsymbol{x}) \;&=\; \int_D \big[t(\lambda, H, \boldsymbol{x}+\boldsymbol{h}-\boldsymbol{y}) - t(\lambda, H, \boldsymbol{x}-\boldsymbol{y})\big]\, u(\boldsymbol{y})\, d\boldsymbol{y} \\ &+\; \sum_{i=1}^d \int_D \big[\partial_i t(\lambda, H, \boldsymbol{x}+\boldsymbol{h}-\boldsymbol{y}) - \partial_i t(\lambda, H, \boldsymbol{x}-\boldsymbol{y})\big]\, \partial_i u(\boldsymbol{y}) d\boldsymbol{y}, \end{aligned}$$

The functions $t(\lambda, H, \cdot)$ and $\partial_i t(\lambda, H, \cdot)$ are elements of $L_1(\mathbb{R}^d)$. By using the global continuity of elements in L_1-space we conclude that the brackets in the considered expression have small $L_1(\mathbb{R}^d)$-norms for $\boldsymbol{h}$ small. Let us say these norms are not larger than ε. Hence, the square brackets are kernels depending on $\boldsymbol{x}-\boldsymbol{y}$ with the L_1-norm less than ε. The domain D is contained in a ball $B(r)$ of radius r centred at the origin. In the following $\|\cdot\|_p$ denotes the norm of $L_p(B(2r))$. By using Young inequality we get

$$\|u(\cdot+\boldsymbol{h}) - u(\cdot)\|_p \;\le\; \varepsilon\, \|u\|_{p,1},$$

i.e. the equicontinuity and thus proving Lemma. **QED**

2.

Sada rjesavamo (1.6) za Laplaceov operator. Pitamo se kako se definira rjesenje. Krecemo od problema minimuma za bilinearnu formu $h(v,u) = \sigma^2 \sum_{i=1}^d (\partial_i v|\partial_i u)$ i pripadni kvadratni funkcional $q(u) = h(u,u) - 2\langle\, u|f\,\rangle$.

[exp3.2] (1.9) $$\inf\{q(u) \;:\; u \in \dot{W}_2^1(D),\; \|u\|_{2,1} = 1\}.$$

[lem3.2] LEMMA **1.3.3** *Postoji jedinstveni $u_0 \in \dot{W}_2^1(D)$ koji ostvaruje rjesenje zadace (1.9). To se rjesenje podudara sa slabim rjesenjem diferencijalnog operatora $H(\boldsymbol{x})$ zadace (1.7). Za $f \in L_2(D),\; f \ge 0$ rjesenje je $u_0 \ge 0$.*

PROOF: OGRANICENOST FUNKCIONALA ODOZDO. Za ovaj dio dokaza trebamo $\|\nabla u\|_2^2 \ge \kappa(D)\|u\|_2^2$ za neki $\kappa(D)$. Neka je $\sigma^2 = 1$. Imamo ovo:

$$\langle\, u|f\,\rangle \;=\; (u|f_0) \;+\; \sum_{i=1}^d (\partial_i u|f_i).$$

Koristimo $ab \le \varepsilon a^2 + (4\varepsilon)^{-1} b^2$ pa dobijemo:

$$|\langle\, u|f\,\rangle| \;\le\; \varepsilon\Big(\|u\|_2^2 + \|\nabla u\|_2^2\Big) \;+\; \frac{1}{4\varepsilon}\Big(\|f_0\|_2^2 + \|\nabla f\|_2^2\Big).$$

Dakle je

$$q(u) \;\ge\; \Big[\kappa(D)\,(1-2\varepsilon) \;-\; 2\varepsilon\Big] \;-\; \frac{1}{2\varepsilon}\,\|f\|_{2,-1}^2.$$

KOMPAKTNI PODNIZ U $L_2(D)$. Neka je $\mathfrak{U} \;=\; \{u_n \;:\; n \in \mathbb{N}\} \subset \dot{W}_2^1(D)$ minimizirajuci niz, t.j. $\mu = \lim_n q(u_n)$. On je ogranicen u $\dot{W}_2^1(D)$ jer je:

$$\|u\|_{2,1}^2 \;\le\; \big(1+\kappa(D)^{-1}\big)\,\|\nabla u\|_2^2 \;\le\; \big(1+\kappa(D)^{-1}\big)\Big[q(u) + \varepsilon\|u\|_{2,1}^2 + (4\varepsilon)^{-1}\|f\|_{2,-1}^2\Big].$$

Neka je $\mathfrak{U}'$ slabo konvergentni podniz $u_0 = w - \lim u_n \in \dot{W}_2^1(D)$, te neka je $\mathfrak{U}''$ jako konvergentni podniz u $L_2(D)$ prema Lemmi 1.3.2, t.j. $u_0 = s - \lim u_n \in L_2(D)$. Sigurno je $\|u_0\|_{2,1} \leq 1$. U nasem dokazu svi nizovi imaju istu oznaku $\mathfrak{U}$.

Jako konvergentni podniz u $\dot{W}_2^1(D)$. Sada treba dokazari $\|u_0\|_{2,1} = 1$. Iz toga slijedi $u_0 = s - \lim u_n$ u $\dot{W}_2^1(D)$. Kako je $\|u_n\|_{2,1} = 1$ slijedi

$$1 \;=\; \mu \,+\, \|u_0\|_2 \,+\, 2\langle\, u_0|f\,\rangle.$$

Dalje je

$$\mu \;\leq\; q(u_0) \;=\; \|\nabla u_0\|_2^2 \;-\; 2\langle\, u_0|f\,\rangle.$$

Dodajmo lijevo i desno $\|u_0\|_2^2 + 2\langle\, u_0|f\,\rangle$, tako da je na lijevo 1, t.j.:

$$1 \;\leq\; \|u_0\|_{2,1}^2,$$

odakle slijedi jaka konvergencija podniza $\mathfrak{U}$.

Nenegativnost Ako je $u_0 = u_+ - u_-$ onda je ovo istina: Za $|u_0| = u_+ + u_-$ je

$$q(|u_0|) \;=\; \|\nabla|u_0|\|_2^2 \;-\; 2(u_+|f) \;-\; 2(u_-|f) \;\leq\; q(u_0),$$

pa zbog jedinstvenosti rjesenja mora biti $u_0 = |u_0|$ u $\dot{W}_2^1(D)$. **QED**

Zadacom minimizacije je ostvareno preslikavanje $f \mapsto u$ iz $W_2^{-1}(D)$ u $\dot{W}_2^1(D)$. Ono je linearno i oznacenos s $T(0, H_D)$. Analogno se definira $T(\lambda, H_D)$ za zadace s operatorom $\lambda I + H(\boldsymbol{x})$.

COROLLARY **1.3.1** *Neka je u rjesenje zadace*

$$\inf\{\lambda\|u\|_2^2 \,+\, \|\nabla u\|_2^2 \,-\, 2\langle\, u|f\,\rangle \;:\; \|u\|_{2,1} = 1\}$$

za neki $\lambda > 0$. Tada vrijedi:

(i) $f \in C^{(\infty)}(D) \to u \in C^{(\infty)}(D)$.

(ii) $f \in L_\infty(D) \to u \in C^{(1+\alpha)}(D)$ za $\alpha < 1$.

Proof: Neka je χ funkcija izgladjivanja s vrijednostima u $[0,1]$, $supp(\chi) \subset D$, $\chi = 1$ na $S \subset D$. Dokazimo da je funkcija $\boldsymbol{x} \mapsto \chi(\boldsymbol{x})u(\boldsymbol{x})$ klase $C^{(\infty)}$ na S. Formalno je $(\lambda I + H(\cdot))\chi u = \chi f - (\Delta\chi)u - 2\nabla\chi\nabla u$. Oznacimo to sa f'. Ocito je $f' \in L_2(\mathbb{R}^d)$, a isto tako je $f' = f$ na S. Dakle je $w = T(\lambda, H)f' \in \mathcal{D}(H)$ i $\lambda w(\boldsymbol{x}) + H(\boldsymbol{x})w(\boldsymbol{x}) \;=\; f'(\boldsymbol{x})$ na $\mathbb{R}^d$. Odavde imamo za test-funkciju v:

$$\lambda(v|w) \,+\, (v|Hw) \;=\; \lambda(v|w) \,+\, (\nabla v\,|\,\nabla w) \;=\; (v\,|\,f').$$

Desna se strana moze napisati kao

$$(v\,|\,f') \;=\; \lambda(v|\chi u) \,+\, (\nabla v\,|\,\nabla(\chi u)).$$

Na taj nacin slijedi $\lambda(v|w - \chi u) + h(v, w - \chi u) = 0, \to \chi u = w$, pa svojstva funkcije χu slijede iz svojstava $w = T(\lambda, H)f' = \int t(\lambda, H, \cdot - \boldsymbol{y})f'(\boldsymbol{y})\,d\boldsymbol{y}$. Za $\boldsymbol{x} \in S$ integral doprinosi od vrijednosti f i vrijednosti funkcija $u, \partial_i u$ izvan S, t.j. sa skupa $D \setminus S$. Time imamo glatkost na S klase $C^{(\infty)}$. Dio (ii) se dokazuje analogno. **QED**

[Th3.1] THEOREM **1.3.2** *Za svaki $\lambda \geq 0$ je $T(\lambda, H_D)$ kompaktan u $L_2(D)$ i $T(\lambda, H_D) > 0$. Za svaki $\alpha \in (0,1)$ postoji kompaktni $T(\lambda, H_D)^\alpha > 0$.*

Proof: Kompaktnost slijedi iz Lemme 1.3.3. Pozitivnost se moze dokazati klasicnim metodama ovako. Neka postoji $f \in L_2(D)$, $0 \leq f \leq 1$, takav da je $u(\boldsymbol{x}) = T(\lambda, H_D)f(\boldsymbol{x}) = 0$ za neki $\boldsymbol{x} \in D$ i neki $\lambda > 0$. Ovu tocku i dio nosaca funkcije f obuhvatiom plohom ∂S klase $C^{(2)}$ u skupu D. Ploha definira slup $S \subset D$. Iz prethodnog rezultata znamo da je rjesenje klsae $C^{(1+\alpha)}$ na S, t.j. to je klasicno rjesenje na S. Ako je rjesenje pozitivno na dijelu ∂S

ono mora biti pozitivno u S. Pozitivnost na nekoj tocki granice ∂S mora biti ispunjena za netrivijalni u.

Kompaktni operator u Banachovom prostoru ima cisto diskretan spektar i t.d. Vidi U Kurepinoj knjizi Funkcionalna analiza. Kako je $T(\lambda, H_D)$ simetrican, imamo ortonormirane svojstvene funkcije ϕ_k i pripadne svojstvene vrijednosti λ_k, a $T(\lambda, H_D)$ je integralni operator s jezgrom:

$$t(\lambda, H_D, \boldsymbol{x}, \boldsymbol{y}) \;=\; \sum_k \frac{1}{\lambda + \lambda_k}\, \phi_k(\boldsymbol{x})\, \phi_k(\boldsymbol{y}).$$

Iz $\|\nabla u\|_2^2 \geq \kappa(D)\|u\|_2^2$ slijedi ga je najmanja svojstvena vrijednost pozitivna, pa ovaj izraz vrijedi i za $\lambda = 0$. Tada je $T(\lambda, H_D)^\alpha$ integralni operator zadan s jezgrom:

$$t(\alpha, \lambda, H_D, \boldsymbol{x}, \boldsymbol{y}) \;=\; \sum_k (\lambda + \lambda_k)^{-\alpha}\, \phi_k(\boldsymbol{x})\, \phi_k(\boldsymbol{y}).$$

Biramo pozitivne korjene $(\lambda + \lambda_k)^\alpha$ i na taj nacin dobijemo jedinstveni pozitivno definitni $T(\lambda, H_D)^\alpha$. Pozitivnost operatora $T(\lambda, H_D)^\alpha$ u $L_2(D)$ se dokazuje pomocu Yoshidinih aproksimacija iz izraza (1.2).

Promatramo preostali slucaj $\lambda = 0$.

$$\frac{1}{\lambda_k} - \frac{1}{\lambda + \lambda_k} \;=\; \lambda\, \frac{1}{\lambda + \lambda_k}\, \frac{1}{\lambda_k}$$

pomnzomo s $\phi_k(\boldsymbol{x})\phi_k(\boldsymbol{y})$ i zbrojimo dobijemo jednakost za jezgre pa onda i za operatore:

$$T(0, H_D) - T(\lambda, H_D) \;=\; \lambda\, T(\lambda, H_D)\, T(0, H_D),$$

odakle slijedi $T(0, H_D) \geq T(\lambda, H_D) > 0$. **QED**

3.

PROPOSITION **1.3.1** *Let A be selfadjoint in $L_2(D)$, positive definite, $T(\lambda, A)$ be compact and $T(\lambda, A) \geq 0$. Let ϕ be an eigenfunction of A corresponding to the minimal eigenvalue $\mu > 0$.*

(i) If ϕ has both signs $\phi = \phi_+ - \phi_-$, $\phi_\pm > 0$ on open sets $D_\pm$, $\operatorname{supp}(\phi_\pm) = \overline{D}_\pm$, $\|\phi_\pm\|_2 > 0$, then $T(\lambda, A)$ is reduced by both $L_2(D_\pm)$.

(ii) Let $S = \operatorname{supp}(\phi)$. If S^c has a positive measure then $T(\lambda, A)$ is reduced in both $L_2(S)$ and $L_2(S^c)$.

(iii) If $T(\lambda, A) > 0$ then $|\phi|$ is positive on D and μ is a single eigenvalue of A.

PROOF: Let $\phi = \phi_+ - \phi_-$ as assumed in (i). From the principle

$$\sup_{\|u\|=1} (u|T(\lambda, A)u) \;=\; (\lambda + \mu)^{-1}\, (\phi|T(\lambda, A)\phi)$$

and $T(\lambda, A) \geq 0$, one gets that $|\phi| = \phi_+ + \phi_-$ is also an eigenfuction corresponding to μ, implying $\phi_\pm$ are also eigenfunctions. Hence $T(\lambda, A)(\lambda+\mu)^{-1}\phi_+ = \phi_+$ and $(u, T(\lambda, A)\phi_+) = 0$ for all $u \in L_2(D_+^c)$. Because $T(\lambda, A)$ is symmetric $L_2(D_+)$ reduces A as well. (ii) is proved analogously. To prove (iii) let us assume that μ is not single. Then at least one eigenfunction has both signs on D and (i) is valid. However, $T(\lambda, A)\phi_+ > 0$ on D contradicting the results of (i). **QED**

COROLLARY **1.3.2** *Neka je u svojstvena funkcija operatora H_D uz najmanju svojstvenu vrijednost. Tada je $|u| > 0$ na D, a svojstvena vrijednost je jednostruka.*

4.

Pokusajmo konstruirati inverzni operator $T(\lambda, A)$ za $A_0(\boldsymbol{x}) = -\sum_{ij} \partial_i a_{ij}(\boldsymbol{x})\partial_j$ kao u prethodnom slucaju pomocu inverza za odgovarajuci diferencijalni operator $H = -\sigma^2\Delta$ na D. Nas zanimaju $\partial_i T(\lambda, H_D)^{1/2}$ i $T(\lambda, H_D)^{1/2}\partial_i$. Imamo

$$\sum_i \|\partial_i T(\lambda, H_D)^{1/2}u\|_2^2 \;=\; \sigma^{-2} \sum_{kl} \frac{u_k}{(\lambda+\lambda_k)^\alpha}\,\frac{u_l}{(\lambda+\lambda_l)^\alpha}\,(H(\boldsymbol{x})\,\phi_k|\phi_l) \;\le\; \sigma^{-2}\,\|u\|_2^2.$$

Dakle $\partial_i T(\lambda, H_D)^{1/2}$ je ogranicen, te je i $T(\lambda, H_D)^{1/2}\partial_i$ takodjer ogranicen

$$\sum_{i=1}^d \|\partial_i T(\lambda, H_D)^{1/2}\|_2^2 \;=\; \sum_{i=1}^d \|T(\lambda, H_D)^{1/2}\partial_i\|_2^2 \;\le\; \sigma^{-2}.$$

Stoga mora biti

$$W_D \;=\; T(\lambda, H_D)^{1/2} \sum_{ij} \partial_i \left(\sigma^2\delta_{ij} - a_{ij}(\cdot)\right) \partial_j T(\lambda, H_D)^{1/2}$$

ogranicen s normom $\|W_D\|_2 \;=\; 1-\gamma \;<\; 1$ za $\sigma^2 \;=\; \overline{M}$ i sve $\lambda \ge 0$. Dakle postoji $T(\lambda, H_D)^{1/2}(I - W_D)^{-1}T(\lambda, H_D)^{1/2}$ s normom kao za sav prostor.

THEOREM **1.3.3** *Za dferencijalni operator $A(\boldsymbol{x}) = -\sum_{ij} \partial_i a_{ij}(\boldsymbol{x})\partial_j$ na D s Dirichletovim rubnim uvjetima na ∂D postoji inverz:*

$$T(\lambda, A_D) \;=\; T(\lambda, H_D)^{1/2}\,(I - W_D)^{-1}\,T(\lambda, H_D)^{1/2}.$$

Pri tome je

$$\begin{aligned} &T(\lambda, A_D) \;:\; W_2^{-1}(D) \mapsto \dot{W}_2^1(D), \\ &\|T(\lambda, A_D)\|_{\mathsf{L}(W_2^{-1}, W_2^1)} \le \frac{1}{\sqrt{\overline{M}\underline{M}}}\,\frac{1}{\lambda+\lambda_0}\left(\lambda + \lambda_0 + \overline{M}\right)^{1/2}, \end{aligned}$$

gdje je $\mathsf{L}(W_2^{-1}, W_2^1)$ linerani prostor ogranicenih operatora sa $W_2^{-1}(D)$ u $\dot{W}_2^1(D)$, te je A zatvoren na $\mathcal{D}(A_D) = T(\lambda, A_D)\,W_2^{-1}(D)$. Povrh toga operator $T(\lambda, A_D)$ je bijekcija izmedju prostora $W_2^{-1}(D)$ i $\dot{W}_2^1(D)$. U prostoru $L_2(D)$ je operator $T(\lambda, A_D)$ kompaktan.

Temeljni rezultat iz ovog i prethodnog odjeljka se moze izreci u jednom.

DEFINITION 1.3.1 *Let A, H be defined on $\mathcal{D}(A), \mathcal{D}(H)$, respectively, in a Hilbert space L. Let there exist two bilinear forms $a(\cdot,\cdot)$ and $h(\cdot,\cdot)$ on $\mathcal{D}\times\mathcal{D}$ where $\mathcal{D}$ is dense in L. Let us assume that the following two assumptions are valid:*

(a) $\mathcal{D}(A) \subset \mathcal{D}$, $a(v,u) = (v|Au)$ for $v, u \in \mathcal{D}(A)$ and $\mathcal{D}(H) \subset \mathcal{D}$, $h(v,u) = (v|Hu)$ for $v, u \in \mathcal{D}(H)$.

(b) There exists a positive number $\gamma < 1$ such that

$$\gamma\,h(u,u) \;\le\; a(u,u) \;\le\; h(u,u).$$

Tehen we say that A, H is a pair of intervening operators and $a(\cdot,\cdot), h(\cdot,\cdot)$ is a pair of intervening bilinear forms.

[th6.3] THEOREM **1.3.4 (on pairs of intervening operators)** *Let A, H be a pair of intervening operators, each staisfying HY-conditions. Then $W = R(\lambda, H)^{1/2}(H - A)R(\lambda, H)^{1/2}$ has the norm $\|W\| \le 1-\gamma$ and*

[exp3.4] (1.10) $$T(\lambda, A) \;=\; T(\lambda, H)^{1/2}\,(I - W)^{-1}\,R(\lambda, H)^{1/2}$$

for any positive λ. If $H^{-1/2}$ exists then (1.10) is valid for $\lambda = 0$ as well.

[Th3.2] THEOREM **1.3.5** *The operator A_{0D} has a single minimal eigenvalue corresponding to an eigenfuction having one sign on D.*

PROOF: If the minimal eigenvalue is not single then there exist a set $S \subset D$ and $Z = S^c$ such that $L_2(S), L_2(Z)$ are non-trivial and reduce the resolvent $T(\lambda, A_{0D})$. We assume now that $S \subset D$ is such that $L_2(S)$ reduces $T(\lambda, A_{0D})$ and there exists a simple eigenvalue $\mu = 1/(\lambda+\lambda_0)$ of the restriction $T(\lambda, A_{0D})|L_2(S)$. Then $L_2(D \setminus \overline{S})$ also reduces $T(\lambda, A_{0D})$ and its restriction to this subspace has generaly multiple eigenvalue μ. Hence, we conclude

$$-\sum_{ij=1}^{d} \partial_i a_{ij}(\boldsymbol{x}) \partial_j \, \phi_k(\boldsymbol{x}) \;=\; 0, \quad k \in \mathbb{N}_0, \quad \boldsymbol{x} \in S^c,$$

so taht a_{ij} on S^c can be changed without any effect on this sequance of equalities.

The diffusion tenzor a is changed into $\tilde{a} = a1\!\!1_S + \delta_{ij} 1\!\!1_T$, i.e. it is the Laplacean on the set Z. Let $\tilde{A}_D$ be opearator define by the diffusion tenzor $\tilde{a}$ and Dirichlet boundary condition on ∂D. We have $T(\lambda, A_{0D})1\!\!1_S = T(\lambda, \tilde{A}_D)1\!\!1_S$ so that $T(\lambda, \tilde{A}_D)$ is also reduced by the pair $L_2(S), L_2(Z)$. The eigenfunctions of $T(\lambda, \tilde{A}_D)1\!\!1_Z$ are denoted by $\psi_k, k \in \mathbb{N}_0$ and we have

[exp5.1]
$$-\sum_{ij=1}^{d} \partial_i \tilde{a}_{ij}(\boldsymbol{x}) \partial_j \, \psi_k(\boldsymbol{x}) \;=\; 0, \quad k \in \mathbb{N}_0, \quad \boldsymbol{x} \in S. \tag{1.11}$$

Procedure of transforming $a \mapsto \tilde{a}$ by changing a on Z can be applied once more to $\tilde{a} \mapsto \{\delta_{ij}\}_{11}^{dd}$ by changing $\tilde{a}$ on S. The reslting differential operator is $H = -\Delta$ on D and (1.11) must be valid. However, the equalities (1.11) are not possiblke for H_D. **QED**

Sada promatramo diferencijalni operator $A = -\sum_{ij} \partial_i a_{ij} \partial_j + \sum_i b_i' \partial_i + \sum_i (\partial_i b_i'' \cdot) + c$, te pripadni zatvoreni A_D na $L_2(D)$. Njegova rezolventa je kompaktan operator sa spektrom $Sp(A_D) = \{\lambda_k \,:\, k \in \mathbb{N}_0\}$, $\lambda_0 > 0$.

[lem6.1] LEMMA **1.3.4** *Let L be some of $L_p(D, \mu)$, $D \subseteq \mathbb{R}^d$ and A be a linear operator in L satisfying HY-conditions. Then:*

(i) There exists $T(\lambda, A)^\alpha$ in L such that $D(\alpha) = T(\lambda, A)^\alpha L$ is dense in L and there exists $(\lambda I + A)^\alpha$ on $D(\alpha)$ such that $(\lambda I + A)^\alpha T(\lambda, A)^\alpha = I$.

(ii) If $T(\lambda, A) \geq 0$ there exists a unique $T(\lambda, A)^\alpha \geq 0$.

(iii) If L is a Hilbert space then the unique operator $T(\lambda, A)^\alpha$ of (ii) is positive definite.

(iv) Let $T(\lambda, A) \geq 0$ in L and $\langle\, v | T(\lambda, A) u \,\rangle = 0$ for some $\lambda \in (0, \infty)$ and a pair $v \in L^\dagger, u \in L$, $v \geq 0, u \geq 0$. Then $\langle\, v | T(\lambda, A)^\alpha u \,\rangle = 0$ for all $\alpha > 0$ and all $\Re\lambda > 0$.

PROOF: The assertions (i)-(iii) follow from Lewmma 1.1.3. To prove (iv) we use the representation

$$T(\lambda, A) \;=\; \sum_{k=0}^{\infty} (\mu - \lambda)^k \, T(\mu, A)^{k+1},$$

and conclude $\langle\, v | T(\lambda, A) u \,\rangle = 0 \Rightarrow \langle\, v | T(\mu, A)^m u \,\rangle = 0$ for all $m \in \mathbb{N}$ and all $\mu > \lambda$. By utilizing the analyticity in $\Re\lambda > 0$ we get $\langle\, v | T(\lambda, A)^m u \,\rangle = 0$ for all $m \in \mathbb{N}$ and all $\Re\lambda > 0$. The remaining part of assertion follows from (1.2). **QED**

[corLa10.4] PROPOSITION **1.3.2** *For the differential operator $A_0(\boldsymbol{x})$ we have $T(\lambda, A_D) > 0$ in $L_2(D)$ for all $\lambda \geq 0$. For the differential operator $A(\boldsymbol{x})$ there exists $\lambda_0 > 0$ such that $T(\lambda, A_D) > 0$ in $L_2(D)$ for all $\lambda > \lambda_0$.*

PROOF: It is sufficient to consider the case of $A_0(\boldsymbol{x})$. Let us assume that there exist two continuous non-trivial functions $v \geq 0, u \geq 0$ with supports in D such that $(v | T(\lambda, A_D) u) = 0$. By the previous lemma we have $(v | T(\lambda, A_D)^m u) = 0$ for all $m \in \mathbb{N}$. Let us rewrite the obtained equalities by using eigenfunctions of A_D, $\phi_k, k \in \mathbb{N}_0$. We have

$$(v | T(\lambda, A_D)^m u) \;=\; \sum_{k=0}^{\infty} \frac{1}{(\lambda + \lambda_k)^m} \, (\phi_k | v) \, (\phi_k | u) \;=\; 0.$$

From
$$\sum_{k=0}^{\infty} \frac{1}{(\lambda+\lambda_k)^m}\,(\phi_k|v)\,(\phi_k|u) \;=\; \frac{1}{(\lambda+\lambda_0)^m}\sum_{k=0}^{\infty} \frac{(\lambda+\lambda_0)^m}{(\lambda+\lambda_k)^m}\,(\phi_k|v)\,(\phi_k|u),$$
we get
$$\Big|\sum_{k=1}^{\infty} \frac{(\lambda+\lambda_0)^m}{(\lambda+\lambda_k)^m}\,(\phi_k|v)\,(\phi_k|u)\Big| \;<\; (\phi_0|v)\,(\phi_0|u)$$
for sufficiently large m. This means $(v|T(\lambda, A_D)^m u) > 0$. **QED**

1.4 SVOJSTVA INVERZNOG OPERATORA

1.

Kako se ponasa rjesenje za velike $|\boldsymbol{x}|$? Evo nekih rezultata.

Promatra se $A_0(\boldsymbol{x})$. Ako postoji izmjeriva funkcija $(\boldsymbol{x},\boldsymbol{y}) \mapsto F(\boldsymbol{x},\boldsymbol{y})$ na $\mathbb{R}^d\times\mathbb{R}^d$ takva de je $T(\lambda, A_0)$ integralni operator s tom jezgrom onda se ona zove fundamentalno rjesenje differencijalnog operatore $\lambda I + A_0(\boldsymbol{x})$, te se formalno pise:

[sv4.1] (1.12) $$(\lambda I + A_0(\boldsymbol{x}))\,F(\boldsymbol{x},\boldsymbol{y}) \;=\; \delta(\boldsymbol{x}-\boldsymbol{y}).$$

Posluzimo se posljedicom Aronsonovih nejednakosti, koje su izlozene u R. F. Bass, *Diffusions and Elliptic operators*, Springer, N. Y. 1977.

THEOREM **1.4.1** *Postoje pozitivni brojevi* $\underline{c}(\underline{M},\overline{M}), \overline{c}(\underline{M},\overline{M})$, *pozitivni brojevi* $\sigma_1^2(\underline{M},\overline{M}), \sigma_2^2(\underline{M},\overline{M})$, *pridruzeni diferencijalni operatori* $H_1(\boldsymbol{x}), H_2(\boldsymbol{x})$ *i fundamentalno rjesenje* F *na* $\mathbb{R}^d\times\mathbb{R}^d$ *problema (1.12) takvo da vrijedi:*
$$\underline{c}(\underline{M},\overline{M})\,t(\lambda,H,\boldsymbol{x}-\boldsymbol{y}) \;\le\; F(\boldsymbol{x},\boldsymbol{y}) \;\le\; \overline{c}(\underline{M},\overline{M})\,t(\lambda,H,\boldsymbol{x}-\boldsymbol{y}), \qquad \boldsymbol{x}\neq\boldsymbol{y}.$$

Oznacimo s $W_2^1(S,\mu)$ Hilbertov prostor W_2^1-funkcija na S s mjerom $\exp(\sigma|\boldsymbol{x}|)\,d\boldsymbol{x}$.

LEMMA **1.4.1 (Asimptotsko ponasanje rjesenja)** *Let* $D\subset\mathbb{R}^d$ *be a bounded domain,* $u\in W_2^1(D^c)$ *and* $\big(A(\boldsymbol{x})+\lambda\big)u(\boldsymbol{x}) = f(\boldsymbol{x})$ *on the set* D^c. *For each* $\lambda>0$ *there exists* $\sigma>0$ *with the following properties: If* $\boldsymbol{x}\mapsto \exp(\sigma|\boldsymbol{x}|/2)f(\boldsymbol{x})\in L_2(D^c)$ *then*
$$\boldsymbol{x} \;\mapsto\; u(\boldsymbol{x}) \;\in\; W_2^1(D^c,\mu).$$

Proof: In this proof the balls $B_R(\mathbf{0}), B_{R+1}(\mathbf{0})$ are denoted by B_1, B_2, respectively. It is assumed $D\subset B_1$. The measure μ, defined as the Lebesgue measure multiplied by the function $\exp(\sigma|\cdot|)d\boldsymbol{x}$, determines the Hilbert space of functions on $\mathbb{R}^d$ as usually. This space is denoted by $L_2(\mathbb{R}^d,\mu)$. The corresponding subspace of functions restricited to a subset S is denoted by $L_2(S,\mu)$. A cut-off function of the class $C^\infty(\mathbb{R}^d)$, with values in $[0,1]$, and $\phi|B_1 = 0, \phi|B_2^c = 1$ is denoted by ϕ.

Let us multiply the equality $\big(A(\boldsymbol{x})+\lambda\big)u(\boldsymbol{x}) = f(\boldsymbol{x})$ on B_1^c with the function $\boldsymbol{x}\mapsto \exp(\sigma\boldsymbol{x})\phi(\boldsymbol{x})u(\boldsymbol{x})$ and integrate over $\mathbb{R}^d$. The rsult is $\langle \exp(\sigma|\cdot|)\phi u|Au\rangle + \lambda\langle \exp(\sigma|\cdot|)\phi u|u\rangle = (\exp(\sigma|\cdot|)\phi f|u)$. By using the ellipticity the obtained equality is easily transformed to the following inequality:
$$\begin{aligned}\underline{M}\,\|\nabla u\|^2_{L_2(B_2^c,\mu)} + \lambda\,\|\sqrt{\phi}\,u\|^2_{L_2(B_1^c,\mu)} &\le \overline{M}\,\sigma\,\|u\|_{L_2(B_2^c,\mu)}\,\|\nabla u\|_{L_2(B_2^c,\mu)}\\ +\overline{M}\,(\sigma + \sqrt{d}\,\max_i\|\partial_i\phi\|_\infty)\,\|u\|_{L_2(B_2\setminus B_1,\mu)}\,\|\nabla u\|_{L_2(B_2\setminus B_1,\mu)} &+ \|\sqrt{\phi}f\|_{L_2(B_1^c,\mu)}\,\|\sqrt{\phi}\,u\|_{L_2(B_1^c,\mu)}.\end{aligned}$$

By utilizing the inequality $ab \le (1/2)\big((1/\varepsilon)a^2+\varepsilon b^2\big)$, to the first and last terms of right hand side, with $\varepsilon = \underline{M}/(\sigma\overline{M})$ and $\varepsilon=\lambda$, respectively, we get:
$$\begin{aligned}&\frac{1}{2}\underline{M}\,\|\nabla u\|^2_{L_2(B_2^c,\mu)} + \left(\frac{\lambda}{2} - \frac{(\sigma\overline{M})^2}{2\underline{M}}\right)\|u\|^2_{L_2(B_2^c,\mu)}\\ &\le c\,\|u\|_{L_2(B_2\setminus B_1,\mu))}\,\|\nabla u\|_{L_2(B_2\setminus B_1,\mu)} + (2\lambda)^{-1}\,\|\sqrt{\phi}\,f\|^2_{L_2(B_1^c,\mu)}\\ &\le c\,\exp\big(\sigma(R+1)\big)\|u\|_{L_2(B_2\setminus B_1)}\,\|\nabla u\|_{L_2(B_2\setminus B_1)} + (2\lambda)^{-1}\,\|f\|^2_{L_2(B_1^c,\mu)},\end{aligned}$$
where $c = \overline{M}(\sigma + \sqrt{d}\,\|\nabla\phi\|_\infty)$. **QED**

4.

[prop3.1] PROPOSITION **1.4.1** *U svakom $L_p(\mathbb{R}^d), p \in [1,\infty]$, postoji operator $T(\lambda, A)$, $T(\lambda, A) > 0$ i $\|T(\lambda, A)\|_p = 1/\lambda$.*

PROOF: Treba dokazati samo za $p = 1$ i $p = \infty$. Idemo s $p = 1$. Dovoljno je promatrati $f \geq 0, f \in L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$. Uvedemo kugle $B_1 \subset B_2$ s radijusima $R, 2R$ i funkciju rezanja ϕ_R klase $C^{(\infty)}$, koja je 1 na prvoj, nula izvan druge, te ima vrijednosti u $[0,1]$. Na primjer $\phi_R(\boldsymbol{x})) = \phi_1(\boldsymbol{x})/R)$ gdje je ϕ_1 takva za $R = 1$. Za nju je

$$\lambda\,(\phi_R|u) \;+\; a(\phi_R, u) \;=\; \langle\; \phi_R \,|\, f\,\rangle,$$

odakle slijedi u limesu

$$\|u\|_1 \;=\; \frac{1}{\lambda}\,\|f\|_1 \;-\; \lim_R \,a(\phi_R, u).$$

Sada

$$a(\phi_R, u) \;=\; \sum_{ij} \int_{B_2\setminus B_1} \partial_i\phi_R(\boldsymbol{x})\, a_{ij}(\boldsymbol{x})\, \partial_j u(\boldsymbol{x}) d\boldsymbol{x}.$$

Oznaka je $\rho = \exp(\sigma/2|\cdot|)$. Iz Lemme o asimptotskom ponasanju slijedi da $\rho^{1/2}\partial_j u \in L_2$, tako da imamo

$$|a(\phi_R, u)| \;\leq\; \overline{M}\,\|\nabla\phi_R\|_{L_2(S,1/\mu)}\,\|\nabla u\|_{L_2(S,\mu)} \;\leq\; \overline{M}\,\|\nabla\phi_R\|_\infty\, \exp(-\sigma R)\,|S|\,\|u\|_2,$$

a pri tome su $S_R = B_2 \setminus B_1$. Faktori s funkcijama tezi nuli za $R \to \infty$.

Sada promatramo $p = \infty$. Dualni operator mora imati istu normu, t.j. $\|T(\lambda, A_n)^\dagger\|_\infty = 1/\lambda$. Kako je $T(\lambda, A)$ integralni sa simetricnom jezgrom dualni se podudara s izvornim, t.j. $\|T(\lambda, A_n)\|_\infty = 1/\lambda$. **QED**

5.

LEMMA **1.4.2** *Neka je $u \in W_2^1(D)$ generalizirano rjesenje jednadzbe $\lambda u(\boldsymbol{x}) + A(\boldsymbol{x})u(\boldsymbol{x}) = 0, \lambda \geq 0$, te neka postoje brojevi $\beta_\pm$ takvi da je $\beta_- \leq u|\partial u \leq \beta_+$. Tada funkcija u na D ima vrijednosti u intervalu $[\beta_-, \beta_+]$.*

PROOF: Dovoljno je dokazati slucaj $u(\boldsymbol{x}) \leq \beta_+$. Krenimo od suprotnog, $u(\boldsymbol{x}) > \beta_+$ za neki $\boldsymbol{x} \in D$. Neka je $k > \beta_+$. Definirmao funkciju

$$u(k)_+ \;=\; \max\{u - k, 0\}.$$

Slijedi da $u(k)_+$ ima nosac u D, t.j. $u(k)_+ \in \dot{W}_2^1(D)$. Generalizirana jednadzba za nutarnje tocke je

$$\lambda(v, u) \;+\; a(v, u) \;=\; 0, \quad v \in \dot{W}_2^1(D),$$

pa uvrstion $v = u(k)_+$. Sada imamo:

$$\begin{aligned} (u(k)_+|u) \;&=\; (u(k)_+|u - k) \;+\; k(u(k)_+|1) \;=\; (u(k)_+|u(k)_+) \;+\; k(u(k)_+|1), \\ a(u(k)_+, u) \;&=\; a(u(k)_+, u(k)_+), \end{aligned}$$

tako da je jednadzba oblika

$$\lambda\,\|u(k)_+\|_2^2 \;+\; \lambda\,k\,(u(k)_+|1) \;+\; a(u(k)_+, u(k)_+) \;=\; 0.$$

Odavde slijedi $u(k)_+ = 0$. **QED**

PROPOSITION **1.4.2** *Operator $T(\lambda, A_D)$ je ogranicen u svakom $L_p(D)$ s normom ne vecom od $1/\lambda$, i $T(\lambda, A_D) \geq 0$*

PROOF: Dovoljno je promatrati $T(\lambda, A_D)f$ sa $f \geq 0, f \in L_\infty(D)$. Za takav f je funckija $w = T(\lambda, A)1\!\!1_D f - T(\lambda, A_D)$ definirana na D te ima svojstvo $A(\boldsymbol{x})w(\boldsymbol{x}) = 0$ na D u generaliziranom smilslu, t.j. $\lambda(v|w) + a(v, w) = 0$ za svaki $v \in \dot{W}_2^1(D)$. Osim toga je $w|\partial D = T(\lambda, A)f|\partial D$, pa zakljucujemo da w ima ogranicene vrijednosti na ∂D, koje oznacavamo s $w_\pm$. Iz prethodne lemme slijedi $w_- \leq w(\boldsymbol{x}) \leq w_+, \boldsymbol{x} \in D$. Iduci zakljucak je $0 \leq T(\lambda, A_D) \leq 1\!\!1_D T(\lambda, A)1\!\!1_D$. Tvrdnja slijedi is Propozicije 1.4.1. Ako f nije ogranicen onda promatramo $f_N = \min\{f, N\}$ i dobijemo iz dokazanog $w_N = T(\lambda, A_D)f_N$ sa svojstvom $\|w_N\|_p \leq \lambda^{-1}\|f_N\|_p \leq \lambda^{-1}\|f\|_p$. **QED**

The derived boundedness of $T(\lambda, A_D)$ in $L_1(D)$, i.e. $\|T(\lambda, A_D)\|_1 \leq 1/\lambda$ cannot be utilized for $\lambda = 0$. The boundedness for $\lambda = 0$ must be proved separately. In the present proof we utlize the boundedness of $T(\lambda, A_D)$ for $\lambda > 0$ by using an expansion of A_D^{-1} in terms of powers $T(\lambda, A_D)^m$.

[prop6.3] PROPOSITION **1.4.3 (on traces of a pair of intervening operators)** *Let A, H be a pair of intervening operators, each staisfying HY-conditions. If $\mathrm{Tr}\,(H^{-m})$ exisats for some $m \in \mathbb{N}$ then*

$$\mathrm{Tr}\,(H^{-m}) \;\leq\; \mathrm{Tr}\,(A^{-m}) \;\leq\; \gamma^{-m}\,\mathrm{Tr}\,(H^{-m}). \tag{1.13}$$

PROOF: The following sequence of inclusions is valid: $\mathrm{Tr}\,(T(\lambda, H)^m) < \infty \Rightarrow \mathrm{Tr}\,(T(\lambda, H)^{m-1}T(\lambda, A)) = \mathrm{Tr}\,(T(\lambda, H)^m(I - W)^{-1}) \leq \gamma^{-1}\mathrm{Tr}\,(T(\lambda, H)^m)$, where we used (1.10). Now $\mathrm{Tr}\,(T(\lambda, H)^{m-2}T(\lambda, A)^2) = \mathrm{Tr}\,(T(\lambda, H)^{m-1}(I - W)^{-1}T(\lambda, H)(I - W)^{-1}) \leq \gamma^{-2}\mathrm{Tr}\,(T(\lambda, H)^m) \Rightarrow \mathrm{Tr}\,(T(\lambda, A)^m) \leq \gamma^{-m}\mathrm{Tr}\,(T(\lambda, H)^m)$. **QED**

LEMMA **1.4.3** *There exists $c(D)$ such that:*

$$\|A_D^{-1}\|_\infty \;=\; \|A_D^{-1}\|_1 \;<\; c(D).$$

PROOF: By using the equality

$$A_D^{-1} \;=\; \sum_{r=1}^{m} \lambda^{r-1}\, T(\lambda, A_D)^r \;+\; \lambda^m\, T(\lambda, A_D)^m\, A_D^{-1} \tag{1.14}$$

one gets the following estimate:

$$\|A_D^{-1}\| \;\leq\; \sum_{r=1}^{m} \|T(\lambda, A_D)\| \Big(\lambda^{r-1}\, \|T(\lambda, A_D)^{r-1}\|\Big) \\ + \lambda^m\, \|T(\lambda, A_D)^m\, A_D^{-1}\|.$$

The quantity in parantheses can be estimated by 1, so that

$$\|A_D^{-1}\| \;\leq\; \frac{m}{\lambda} \;+\; \lambda^m\, \|T(\lambda, A_D)^m\, A_D^{-1}\|.$$

Here λ is arbitrary and we choose $\lambda = 1$. It remains to get an estimate of the last term of right hand side. It is sufficient to consider only the L_1-norm.

$$\|T(\lambda, A_D)^m\, A_D^{-1}\|_1 \;= \\ \sup\Big\{\big|\langle\, v|T(\lambda, A_D)^m\, A_D^{-1}u\,\rangle\big| \;:\; \|v\|_\infty = 1,\; \|u\|_1 = 1\Big\}.$$

By using the eigenvectors ϕ_k and the corresponding eigenvalues λ_k of A_D we have

$$\langle\, v|T(\lambda, A_D)^m\, A_D^{-1}u\,\rangle \;=\; \sum_k (\lambda + \lambda_k)^{-m}\lambda_k^{-1}\, (v\,|\,\phi_k)\,(u\,|\,\phi_k).$$

Hence, by using CSB inequality we get

$$\big|\langle\, v|T(\lambda, A_D)^m\, A_D^{-1}u\,\rangle\big| \;<\; (v|A_D^{-(2m+1)}\, v)^{1/2}\, (u|A_D^{-1}\, u)^{1/2} \\ <\; \lambda_1^{-1/2}\, |D|^{1/2}\, (v|A_D^{-(2m+1)}\, v)^{1/2} \;<\; \lambda_1^{-1/2}\, |D|^{1/2}\, (|v||A_D^{-(2m+1)}\, |v|)^{1/2} \\ <\; \lambda_1^{-1/2}\, |D|^{1/2}\, (|v||A^{-(2m+1)}\, |v|)^{1/2}.$$

For the last term on the right hand side we have

$$(|v||A^{-(2m+1)}\,|v|) \;<\; |D|\,\|v\|_1^2\,\mathrm{Tr}\,(\mathbb{1}_D\,A^{-(2m+1)}\,\mathbb{1}_D),$$

being finite for $m > (d-2)/4$. Therefore, we have the assertion with $c(D) = m + \lambda_1^{-1/2}|D|\mathrm{Tr}\,(\mathbb{1}_D\,A^{-(2m+1)}\,\mathbb{1}_D)^{1/2}$. **QED**