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Another Approach to The Quintic Spline

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ABSTRACT. A set of end conditions for interpolatory quintic spline is derived by use of integration. These end conditions are only in terms of function values at the knots (data), and rise to $O(h^6)$ spline approximation.

1. INTRODUCTION

In the literature, most of the end conditions for the spline functions are obtained from conditions imposed on the spline fuctions, its finite differences, and/or its derivatives near the two endpoints. Recently, Behforooz [1] has established a new class of end conditions for cubic spline by use of integration. In this paper his idea is extended to the case of quintic spline, and four end conditions are derived by use of integration.

2. PRELIMINARY RESULTS

Let Q be a quintic spline on the interval $[a,b]$ with equally spaced knots

(2.1) $$x_i = a+ih;\ i=0,1,\cdots,k,$$

where $h=(b-a)/k$. Let Q interpolates the function $y=y(x)$ at the knots x_i; $i=0,1,\cdots,k$, i.e.

(2.2) $$Q(x_i)= y_i;\ i=0,1,\cdots,k,$$

where $y_i=y(x_i)$, and $y\in C^n[a,b]$; $n\geq 7$. To simplify the presentation we use the abbreviations $m_i=Q^{(1)}(x_i)$ and $M_i=Q^{(2)}(x_i)$. Then by use of Hermite's two point interpolation formula, $Q(x)$ can be written as:

(2.3) $$\begin{aligned} Q(x)=y_i+m_i(x-x_{i-1}) &+ Q[x_{i-1}, x_{i-1}, x_i](x-x_{i-1})^2 \\ &+ Q[x_{i-1}, x_{i-1}, x_i, x_i](x-x_{i-1})^2(x-x_i) \\ &+ Q[x_{i-1}, x_{i-1}, x_{i-1}, x_i, x_i](x-x_{i-1})^2(x-x_i)^2 \\ &+ Q[x_{i-1}, x_{i-1}, x_{i-1}, x_i, x_i, x_i](x-x_{i-1})^3(x-x_i)^2; \\ &x_{i-1}\leq x\leq x_i;\ i=1,2,\cdots,k, \end{aligned}$$

where

$$(2.4)\quad Q[x_{i-1}, x_{i-1}, x_i]=\frac{1}{h^2}\{(y_i- y_{i-1})-hm_{i-1}\},$$

$$(2.5)\quad Q[x_{i-1}, x_{i-1}, x_i, x_i]=\frac{1}{h^3}\{-2(y_i- y_{i-1})+h(m_i+m_{i-1})\},$$

$$(2.6)\quad Q[x_{i-1},x_{i-1},x_{i-1},x_i,x_i]=\frac{1}{2h^4}\{-6(y_i-y_{i-1})+2h(m_i+2m_{i-1}) +h^2M_{i-1}\}$$

$$(2.7)\quad Q[x_{i-1}, x_{i-1}, x_{i-1}, x_i, x_i, x_i]=\frac{1}{2h^5}\{12(y_i- y_{i-1})-6h(m_i+m_{i-1}) +h^2(M_i-M_{i-1})\},$$

are usual notations for divided differences. To determine the $k+1$ parameters m_i, the following consistency relations are used

$$(2.8)\quad m_{i-2}+ 26m_{i-1}+ 66m_i+ 26m_{i+1}+ m_{i+2}= = \frac{5}{h}(-y_{i-2} - 10y_{i-1}+ 10y_{i+1}+ y_{i+2}); \; i= 2,3,\cdots,k-2.$$

The equations (2.8) provide only $k-3$ linear equations in $k+1$ parameters m_i; $i=0,1,\cdots,k$. It follows that the conditions (2.2) are not sufficient to determine a unique quintic spline Q, and four additional linearly independent equations are needed for this purpose. In the next section, these four end conditions will be derived by use of integration. The following identities between M's, m's and y's will be needed to compute the parameters M_i; $i=0,1,\cdots,k$.

$$(2.9)\quad M_0=\frac{-1}{32h}(222m_0+454m_1+158m_2+6m_3)+\frac{10}{32h^2}(-47y_0+13y_1 +31y_2+3y_3),$$

$$(2.10)\quad M_1=\frac{1}{32h}(6m_0-66m_1-58m_2-2m_3)+\frac{10}{32h^2}(3y_0-18y_1-3y_2+35y_3 -16y_4-y_6),$$

$$(2.11)\quad M_i=\frac{1}{32h}(m_{i-2}+ 32m_{i-1}- 32m_{i+1}- m_{i+2}) +\frac{5}{32h^2}(y_{i-2}+16y_{i-1}-34y_i+16y_{i+1}+y_{i+2}); \quad i= 2,3,\cdots,k-2,$$

$$(2.12)\quad M_{k-1}=\frac{-1}{32h}(6m_k-66m_{k-1}-58m_{k-2}-2m_{k-3})-\frac{10}{32h^2}(3y_k-18y_{k-1} -3y_{k-2}+35y_{k-3}-16y_{k-4}-y_{k-5}),$$

$$(2.13)\quad M_k=\frac{1}{32h}(222m_k+454m_{k-1}+158m_{k-2}+6m_{k-3})-\frac{10}{32h^2}(-47y_k +13y_{k-1}+31y_{k-2}+ 3y_{k-3}).$$

REMARK: To derive the results of the next section, a large number of quintic spline identities from Behforooz and Papamichael [2] and [3] have been used. Some of the algebra involved in the derivation of theseresults is laborious but the proofs are otherwise elementary and for this reason they are omitted.

3. END CONDITIONS

For simplicity we explain only how the first end condition at near $x=a$ is obtained. The other three results can be obtained in a similar manner. From (2.3)—(2.7) we can find that

$$(3.1)\qquad \int_{x_{i-1}}^{x_i} Q(x)dx = \frac{h}{2}\left(y_{i-1}+y_i\right)+\frac{h^2}{10}\left(m_{i-1}-m_i\right)+\frac{h^3}{120}\left(M_{i-1}+M_i\right).$$

Let $P_i(x)$ denotes the quintic interpolating polynomial which matches the function $y=y(x)$ at six points $x_{i-1}, x_i,\cdots,x_{i+4}$. Then it can be shown that

$$(3.2)\qquad \int_{x_{i-1}}^{x_i} P_i(x)dx = \frac{h}{1440}\Big(475y_{i-1}+1427y_i-798y_{i+1} + 482y_{i+2}-173y_{i+3}+27y_{i+4}\Big).$$

By setting

$$\int_{x_{i-1}}^{x_i} Q(x)dx = \int_{x_{i-1}}^{x_i} P_i(x)dx,$$

and using the results (3.1) and (3.2), and eliminating M's by use of the quintic spline identities (see for example, Behforooz and Papamichael [2] and [3]), we will get the following two end conditions (3.3) at near endpoint $x=a$. In a similar manner, the other two end conditions, (3.4), can be derived at other endpoint $x=b$.

$$(3.3)\qquad 21m_i-113m_{i+1}-27m_{i+2}-m_{i+3}=\frac{1}{24h}\Big(-640y_i+5806y_{i+1} -7224y_{i+2}+2716y_{i+3}-904y_{i+4}+246y_{i+5}\Big);\quad i=0,1,$$

$$(3.4)\qquad 21m_i-113m_{i-1}-27m_{i-2}-m_{i-3}=\frac{-1}{24h}\Big(-640y_i+5806y_{i-1} -7224y_{i-2}+2716y_{i-3}-904y_{i-4}+246y_{i-5}\Big);\quad i=k,k-1.$$

The equations (3.3) and (3.4) together with (2.8) are used to compute the $k+1$ parameters m_i; $i=0,1,\cdots,k$. Then, by using the identities (2.9)—(2.13) the values of $k+1$ parameters M_i; $i=0,1,\cdots,k$, can be computed. When the values of parameters m's and M's are computed, then the formula (2.3) together with (2.4)—(2.7) can be used to compute $Q(x)$ at any point $x \in [a,b]$. Now we state the followin theorem without proof:

THEOREM 3.1. *Let Q be the quintic spline which interpolates the function* $y \in C^7[a,b]$ *at the equally spaced knots* (2.1) *with* $k \geq 7$, *and satisfies the end conditions* (3.3) *and* (3.4). *Then*

$$\|Q^{(r)} - y^{(r)}\| = O\left(h^{6-r}\right); \; r = 0,1,\cdots,5, \tag{3.5}$$

where $\|*\|$ *denotes the uniform norm (* L_∞ *function norm) on* $[a,b]$.

REMARK: The quintic spline established in this paper does not belonge to the class of $E(\alpha,\beta,\gamma)$ quintic spline which has beenm established by Behforooz and Papamichael [2].

REFERENCES

1. G. Behforooz, *End conditions for cubic spline interpolation derived from integration*, to appear in Appl. Maths and Comp.

2. G. Behforooz, and N. Papamichael, *End conditions for interpolatory quintic splines*, IMA J. of Numer. Anal. 1 (1981), 81—93.

3. G. Behforooz, and N. Papamichael, *Overconvergence properties of quintic spline*, to apear in J. of Comp. and Applied Maths.