Local stability of optimal designs in transmission problems on a ball

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joint work with Marko Vrdoljak

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Analysis, PDEs and Applications
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General shape optimization problem

A shape optimization problem is usually written as the minimization of an shape functional J of the shape $\omega \subset D$:

$$\min_{\omega \in \mathcal{U}} J(\omega),$$

where $\mathcal{U}$ is a admissible set.

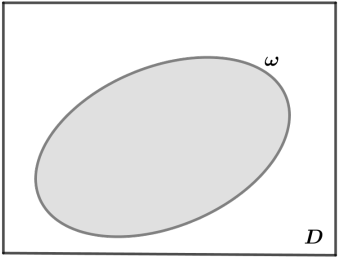


Figure: D that "hold-all" shapes ω

For most shape optimization problems, shape functional J depends on a shape ω via a state function u_ω, a solution to a PDE defined on ω.

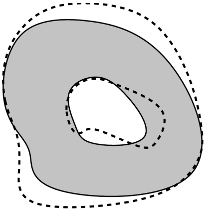
Shape sensitivity "measures" how does the functional behaves under suitable perturbations of the shape.

Shape derivative

Perturbation of the set ω is given with

$$(\mathrm{Id} + \theta)\omega$$

where $\theta \in \mathrm{W}^{k,\infty}(\mathbb{R}^d;\mathbb{R}^d)$, $k \in \mathbb{N}$.
We denote $\Phi_\theta = \mathrm{Id} + \theta$.



Definition 1 (Shape derivative)

Let $J = J(\omega)$ be a shape functional. J is said to be shape differentiable at ω if the map

$$\mathcal{J} : \theta \mapsto J((\mathit{Id} + \theta)\omega)$$

is well defined in a zero-neighbourhood of $\mathrm{W}^{k,\infty}(\mathbb{R}^d;\mathbb{R}^d)$ and Fréchet differentiable at zero.

$$J'(\omega;\theta) := D\mathcal{J}(0)[\theta]$$

is called the **shape derivative** *of J at ω in direction θ.*

Second order shape derivative

Definition 2 (Second order shape differentiability)

Let $k \in \mathbb{N}$. A shape functional J is said to be twice shape differentiable at ω if the map

$$\theta \mapsto D\mathcal{J}(\theta)[\cdot] \in (\mathrm{W}^{k,\infty}(\mathbb{R}^d;\mathbb{R}^d))'$$

is well defined in a zero-neighbourhood of $\mathrm{W}^{k,\infty}(\mathbb{R}^d;\mathbb{R}^d)$ and Fréchet differentiable at zero. Directional derivative at zero in the first variation θ and the second variation ψ is denoted with $J''(\omega;\theta,\psi)$, meaning

$$J''(\omega;\theta,\psi) = \lim_{t\to 0} \frac{1}{t}\left(D\mathcal{J}(t\psi)[\theta] - D\mathcal{J}(0)[\theta]\right).$$

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Example 1 (Second order shape derivative of the volume)

The first order shape derivative of the volume $\mathrm{vol}(\omega) = \int_\omega \mathrm{d}x$ is

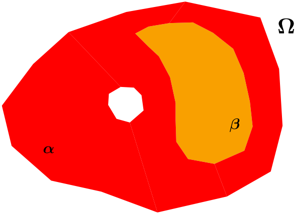
$$\mathrm{vol}'(\omega;\theta) = \int_\omega \mathrm{div}(\theta)\,\mathrm{d}x = \int_{\partial\omega} \theta\cdot\boldsymbol{n}\,\mathrm{d}S$$

and the second order shape derivative

$$\mathrm{vol}''(\omega;\theta,\psi) = \int_\omega -\nabla\theta : \nabla\psi^T + \mathrm{div}(\theta)\mathrm{div}(\psi)\,\mathrm{d}x = \int_{\partial\omega} \mathrm{H}\,\theta\cdot\boldsymbol{n}\psi\cdot\boldsymbol{n}\,\mathrm{d}S$$

where H is the mean curvature of $\partial\omega$.

Fill $\Omega \subset \mathbb{R}^d$ with two isotropic materials with conductivity $0 < \alpha < \beta$. We denote with Ω_α the domain where the conductivity is α and with $\chi := \chi_{\Omega_\alpha}$ the characteristic function of Ω_α.



$$\mathbf{a} = \alpha\chi + \beta(1-\chi),$$

$$\int_\Omega \chi \, \mathrm{d}x = q_\alpha \qquad \text{fixed amounts of materials}$$

State equation:

$$\text{(S)} \qquad \begin{cases} -\text{div}(\mathbf{a}\,\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Objective functional is

$$J(\Omega_\alpha) = \int_\Omega f u \, \mathrm{d}x.$$

Statement of the problem

Optimal design problem:

$$
(P) \qquad \begin{cases} \displaystyle\int_\Omega f u \, \mathrm{d}x \to \max \\ \text{s.t.} \quad \chi \in \mathrm{L}^\infty(\Omega; \{0,1\}), \quad \displaystyle\int_\Omega \chi \, \mathrm{d}x = q_\alpha, \\ u \text{ solves (S) with } \mathbf{a} = \chi\alpha + (1-\chi)\beta. \end{cases}
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If solution χ exists for (P) we call it *classical solution*.

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Important: For general optimal design problems the classical solutions usually do not exist.

Classical solutions

Assumptions:

- $\Omega \subset \mathbb{R}^d$ is ball or annulus,
- right-hand side f is radial function.

With this assumptions one can construct classical solutions.

Example: $f = 1$

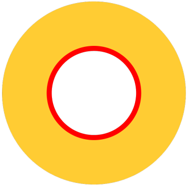
Ball:

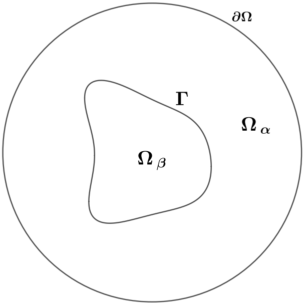
beta-alpha

Annulus:

alpha-beta-alpha
$(q_\alpha > q_{\text{crit}})$

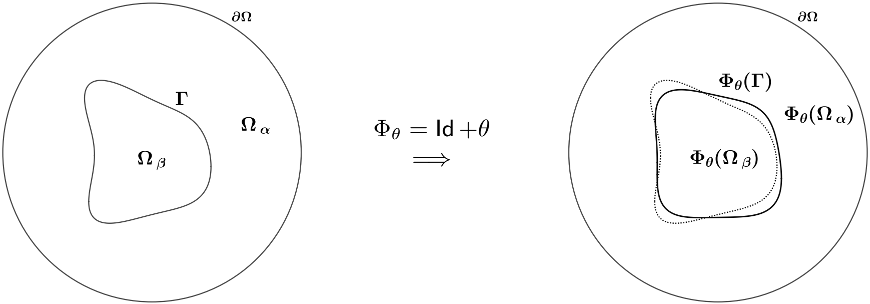
alpha-beta
$(q_\alpha < q_{\text{crit}})$





Variational problem:

$$(1)\qquad \begin{cases} \text{find } u \in \mathrm{H}^1_0(\Omega) \text{ such that for any } \varphi \in \mathrm{H}^1_0(\Omega) \\ \alpha \displaystyle\int_{\Omega_\alpha} \nabla u \cdot \nabla \varphi \,\mathrm{d}x + \beta \int_{\Omega_\beta} \nabla u \cdot \nabla \varphi \,\mathrm{d}x = \int_{\Omega} f \varphi \,\mathrm{d}x \end{cases}$$



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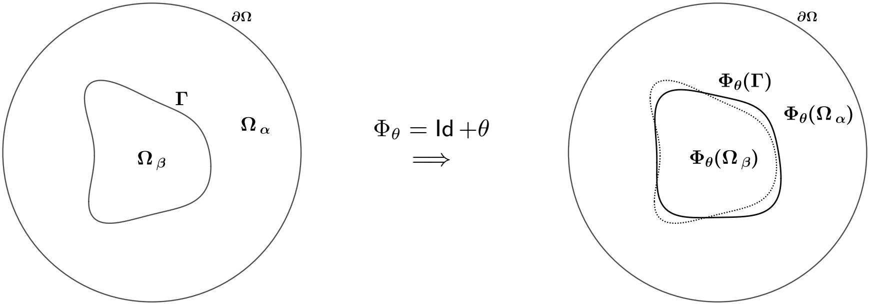
(1_θ) $$\begin{cases} \text{find } u(\theta) \in \mathrm{H}^1_0(\Omega) \text{ such that for any } \varphi \in \mathrm{H}^1_0(\Omega) \\ \alpha \displaystyle\int_{\Phi_\theta(\Omega_\alpha)} \nabla u(\theta) \cdot \nabla \varphi \,\mathrm{d}x + \beta \int_{\Phi_\theta(\Omega_\beta)} \nabla u(\theta) \cdot \nabla \varphi \,\mathrm{d}x = \int_{\Phi_\theta(\Omega)} f \varphi \,\mathrm{d}x \end{cases}$$



Variational problem:

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$$\begin{cases} \text{find } u \in \mathrm{H}_0^1(\Omega) \text{ such that for any } \varphi \in \mathrm{H}_0^1(\Omega) \\ \alpha \int\limits_{\Omega_\alpha} \nabla u \cdot \nabla \varphi \,\mathrm{d}x + \beta \int\limits_{\Omega_\beta} \nabla u \cdot \nabla \varphi \,\mathrm{d}x = \int\limits_{\Omega} f \varphi \,\mathrm{d}x \end{cases}$$

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Notation: $u = u(0)$.

By change of variables (1_θ) becomes:

$$\int_\Omega \mathbf{a}\, P_\theta \nabla(u(\theta)\circ\Phi_\theta)\cdot\nabla\varphi\,\mathrm{d}x = \int_\Omega p_\theta f\circ\Phi_\theta\varphi\,\mathrm{d}x, \quad \varphi\in \mathrm{H}_0^1(\Omega).$$

where $p_\theta = \det\nabla\Phi_\theta$ and $P_\theta = p_\theta\nabla\Phi_\theta^{-\tau}\nabla\Phi_\theta^{-1}$. For a ball $K(0;\delta)$ of $\mathrm{W}^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)$ we define the map $F: K(0,\delta)\times\mathrm{H}_0^1(\Omega)\to\mathrm{H}^{-1}(\Omega)$

$$\langle F(\theta,z),\varphi\rangle = \int_\Omega \mathbf{a}\, P_\theta\nabla z\cdot\nabla\varphi\,\mathrm{d}x - \int_\Omega p_\theta f\circ\Phi_\theta\varphi\,\mathrm{d}x$$

and apply Implicit function theorem to show that

$$\theta\mapsto u(\theta)\circ\Phi_\theta$$

is Fréchet differentiable at zero and directional derivative at zero in direction θ, denoted with $\dot{u}(\theta)\in\mathrm{H}_0^1(\Omega)$ satisfies for all $\varphi\in\mathrm{H}_0^1(\Omega)$:

$$\left\{\begin{array}{ll} \displaystyle\int_\Omega \mathbf{a}\,\nabla\dot{u}(\theta)\cdot\nabla\varphi\,\mathrm{d}x & = \displaystyle\int_\Omega \mathbf{a}(\nabla\theta+\nabla\theta^{\mathcal{T}}-\mathrm{div}(\theta)I)\nabla u\cdot\nabla\varphi\mathrm{d}x \\ & \quad + \displaystyle\int_\Omega \mathrm{div}(f\theta)\varphi\,\mathrm{d}x. \end{array}\right.$$

$$J(\Phi_\theta(\Omega_\alpha)) = \int\limits_{\Phi_\theta(\Omega)} f u(\theta)\,\mathrm{d}x = \int\limits_{\Omega} p_\theta (f \circ \Phi_\theta)(u(\theta) \circ \Phi_\theta)\,\mathrm{d}x$$

$$\approx 1 + \mathrm{div}\theta$$

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$$\approx u + \dot{u}(\theta)$$

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&= \int\limits_{\Omega} f u + f \dot{u}(\theta) + (\nabla f \cdot \theta + f \operatorname{div}(\theta)) u \,\mathrm{d}x + o(\|\theta\|_{\mathrm{W}^{1,\infty}})
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Since $\dot{u}(\theta)\in \mathrm{H}^1_0(\Omega)$,

$$
\begin{aligned}
\int_\Omega f\dot{u}(\theta)\,\mathrm{d}x &= \int_\Omega \mathbf{a}\,\nabla u\cdot\nabla\dot{u}(\theta)\,\mathrm{d}x \\
&= \int_\Omega \mathbf{a}(\nabla\theta + \nabla\theta^\tau - \mathrm{div}(\theta)I)\nabla u\cdot\nabla u + \mathrm{div}(f\theta)u\,\mathrm{d}x.
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&= \int\limits_{\Omega} \mathbf{a}(\nabla\theta + \nabla\theta^\tau - \mathrm{div}(\theta)I)\nabla u\cdot\nabla u + \mathrm{div}(f\theta)u\,\mathrm{d}x.
\end{aligned}
$$

Therefore,

$$
J'(\Omega_\alpha;\theta) = \int\limits_{\Omega} \mathbf{a}(\nabla\theta + \nabla\theta^\tau - \mathrm{div}(\theta)I)\nabla u\cdot\nabla u + 2\mathrm{div}(f\theta)u\,\mathrm{d}x.
$$

Assumption 1

$\Omega \subset \mathbb{R}^d$ is fixed, open set. $\Omega_\alpha, \Omega_\beta$ are Lipschitz domain such that the interface $\Gamma = \partial\Omega_\alpha \cap \partial\Omega_\beta$ belongs to Ω and $\Omega = \Omega_\alpha \dot{\cup} \Omega_\beta \dot{\cup} \Gamma$.

In terms of boundary integrals and for $\theta \in \mathrm{W}^{1,\infty}(\mathbb{R}^d; \mathbb{R}^d)$ such that $\operatorname{supp}\theta \subset \Omega$:

$$J'(\Omega_\alpha; \theta) = \int_\Gamma \theta \cdot \boldsymbol{n} \left[\!\left[2\,\mathbf{a} \left| \frac{\partial u}{\partial \boldsymbol{n}} \right|^2 - \mathbf{a}\, |\nabla u|^2 \right]\!\right] \mathrm{d}S$$

where $[\![w]\!]$ stands for trace of $w_{|\Omega_\alpha} - w_{|\Omega_\beta}$ on Γ and $\boldsymbol{n}$ is outer unit normal to Ω_α.

Theorem (First structure theorem)

If J is shape differentiable at some C^{k+1} bounded set ω, then there exist a continuous linear form l_1^J on $C^k(\partial\omega)$ such that

$$J'(\omega; \theta) = l_1^J(\theta_{|\partial\omega} \cdot \boldsymbol{n}), \quad \theta \in C^\infty(\mathbb{R}^d; \mathbb{R}^d).$$

Note that

$$l_1^J(\theta_{|\Gamma} \cdot \boldsymbol{n}) = \int_\Gamma \theta \cdot \boldsymbol{n} [\![\Lambda]\!] \,\mathrm{d}S$$

where $\Lambda = 2\,\mathbf{a} \left| \frac{\partial u}{\partial \boldsymbol{n}} \right|^2 - \mathbf{a}\, |\nabla u|^2$.

Theorem (Second structure theorem)

If J is twice shape differentiable at some C^{k+2} bounded set ω, then there exist a continuous bilinear form l_2^J on $C^k(\partial\omega) \times C^k(\partial\omega)$ such that for any $\theta, \psi \in C^\infty(\mathbb{R}^d; \mathbb{R}^d)$

$$J''(\omega; \theta, \psi) = l_2^J(\theta_{|\partial\omega} \cdot \boldsymbol{n}, \psi_{|\partial\omega} \cdot \boldsymbol{n}) + l_1^J(Z_{\theta,\psi}),$$

where $Z_{\theta,\psi} = D_\tau \boldsymbol{n} \theta_\tau \cdot \psi_\tau - \nabla_\tau(\theta \cdot \boldsymbol{n}) \cdot \psi_\tau - \nabla_\tau(\psi \cdot \boldsymbol{n}) \cdot \phi_\tau$.

$$J''(\Omega_\alpha; \theta, \psi) = \int_\Gamma (\theta \cdot \boldsymbol{n})(\psi \cdot \boldsymbol{n}) \mathrm{H}[\![\Lambda]\!] + \frac{\partial}{\partial \boldsymbol{n}} [\![\Lambda]\!] \, \mathrm{d}S + \int_\Gamma Z_{\theta,\psi} [\![\Lambda]\!] \, \mathrm{d}S$$
$$+ 2 \int_\Gamma (\theta \cdot \boldsymbol{n}) \left[\!\!\left[\left(\mathbf{a} \frac{\partial u}{\partial \boldsymbol{n}} \frac{\partial w(\psi)}{\partial \boldsymbol{n}} - 2 \frac{\partial u}{\partial \boldsymbol{n}} \nabla_\tau u \cdot \nabla_\tau (\psi \cdot \boldsymbol{n}_\alpha) - \nabla_\tau u \cdot \nabla_\tau w(\psi) \right) \right]\!\!\right] \mathrm{d}S$$

and $w(\psi) \in \mathrm{L}^2(\Omega)$ is local derivative of u determined by

$$\begin{cases} \operatorname{div}(\nabla w(\psi)) = 0 & \text{in } \Omega_\alpha \cup \Omega_\beta \\ [\![w(\psi)]\!] = \frac{\alpha - \beta}{\beta} (\nabla u_\alpha \cdot \boldsymbol{n}) \psi & \text{on } \Gamma \\ \left[\!\!\left[\mathbf{a} \frac{\partial w(\psi)}{\partial \boldsymbol{n}} \right]\!\!\right] = (\alpha - \beta) \operatorname{div}_\tau (\psi \nabla_\tau u) & \text{on } \Gamma \\ w(\psi) = 0 & \text{on } \partial\Omega. \end{cases}$$

Spherically symmetric problem

Let $\Omega \subset \mathbb{R}^2$ be a ball and f a radial function.
Due to the symmetry, it is expected that an optimal Ω_α is also spherically symmetric implying that the corresponding state u is radial: $u = u(r)$ and

$$\nabla u = u' \boldsymbol{e}_r \quad \left(\frac{\partial u}{\partial \boldsymbol{n}} = \pm u', \quad \nabla_\tau u = 0 \right)$$

Flux $\mathbf{a} \nabla$ has the form $\sigma \boldsymbol{e}_r$, with $\sigma = \mathbf{a}\, u'$ and the state problem in two-dimensions is

$$-\frac{1}{r}(r\sigma)' = f.$$

Our aim is to maximize the energy functional $J(\Omega_\alpha)$ but with constraint on the volume of the first phase $\mathrm{vol}(\Omega_\alpha)$ i.e. we maximize Lagrange function

$$L = J - \lambda \mathrm{vol}$$

First order optimality condition

$$[\![\Lambda]\!] = \left[\!\!\left[2\,\mathbf{a} \left| \frac{\partial u}{\partial \boldsymbol{n}} \right|^2 - \mathbf{a}\,|\nabla u|^2 \right]\!\!\right] = \left[\!\!\left[\mathbf{a} \left| \frac{\partial u}{\partial \boldsymbol{n}} \right|^2 - \mathbf{a}\,|\nabla_\tau u|^2 \right]\!\!\right] = \left[\!\!\left[\frac{\sigma^2}{\mathbf{a}} \right]\!\!\right]$$

We have

a) interface Γ is a sphere $\partial B(0, \hat{r})$

$$L' = J' - \lambda \mathrm{vol}' = 0 \qquad \iff \qquad \frac{\beta - \alpha}{\alpha\beta} \sigma(\hat{r})^2 = \lambda,$$

b) interface $\Gamma = \cup_{i=1}^{k} \partial B(0, r_i)$ is a union of k spheres

$$L' = J' - \lambda \mathrm{vol}' = 0 \qquad \iff \qquad \frac{\beta - \alpha}{\alpha\beta} \sigma(r_i)^2 = \lambda.$$

If Ω is a ball this leaves only few candidates for local maxima, since σ is uniquely determined by

$$-\frac{1}{r}(r\sigma)' = f.$$

Interface is a single sphere

We use the Fourier analysis:

Theorem

For $\Omega = B(0,R)$ let the interface between phases be a sphere, i.e. Ω_α is a ball $B(0,\hat{r})$, with $0 < \hat{r} < R$. For $\lambda = \frac{\beta-\alpha}{\alpha\beta}\sigma(\hat{r})^2$ the first order necessary optimality condition is satisfied and for

$$(J'' - \lambda \mathrm{vol}''_\beta)(\Omega_\alpha;\theta,\theta) = (l_2^J - \lambda l_2^{\mathrm{vol}})(\theta_{|\Gamma}\cdot \boldsymbol{n}, \theta_{|\Gamma}\cdot \boldsymbol{n})$$

the following holds

$$\begin{aligned}(l_2^J - \lambda l_2^{\mathrm{vol}})(\phi,\phi) = \; & 2\pi\sigma(\hat{r})\frac{\beta-\alpha}{\alpha\beta}\Big[\hat{r}\sigma'(\hat{r})\left(\frac{\alpha_0^2}{2} + \sum_{k=1}^{\infty}(\alpha_k^2+\beta_k^2)\right) \\ & + (\alpha-\beta)\sigma(\hat{r})\sum \frac{k(\hat{r}^{2k}+R^{2k})}{(\beta-\alpha)\hat{r}^{2k} + (\alpha+\beta)R^{2k}}(\alpha_k^2+\beta_k^2)\Big],\end{aligned}$$

where α_k and β_k stand for Fourier coefficients of the function $\phi = \theta_{|\Gamma}\cdot\boldsymbol{n}$.

For example $f(r) = 6 - 6r, \Omega = K(0,1)$ and $\Omega_\alpha = K(0, \hat{r})$ we know from Theorem:

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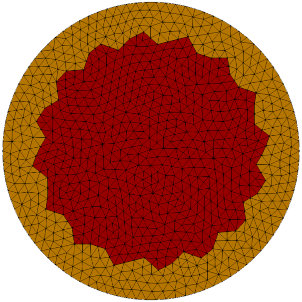

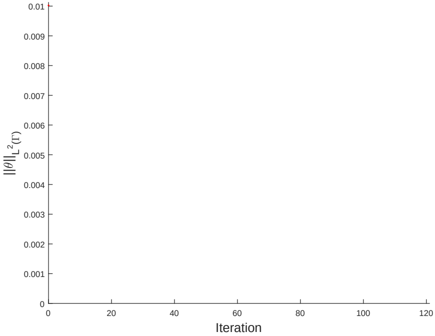

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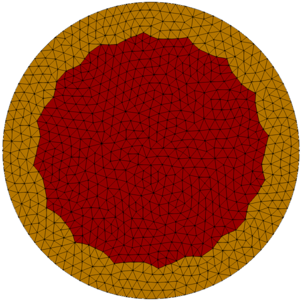

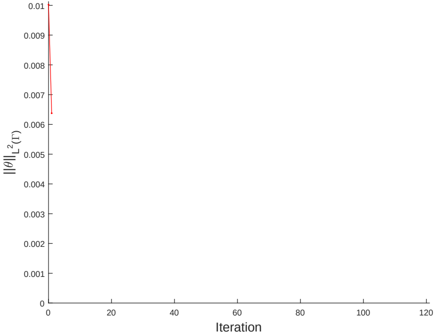

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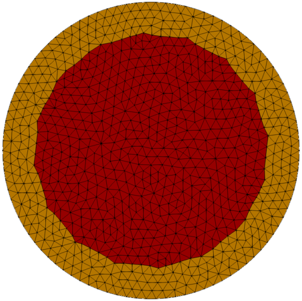

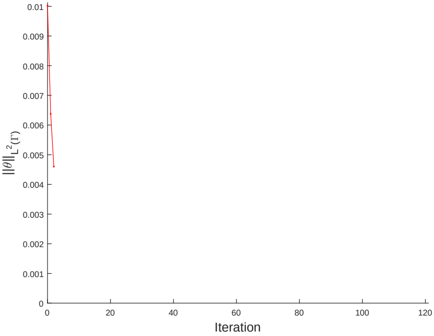

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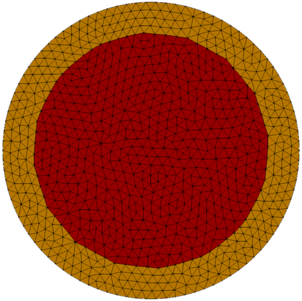

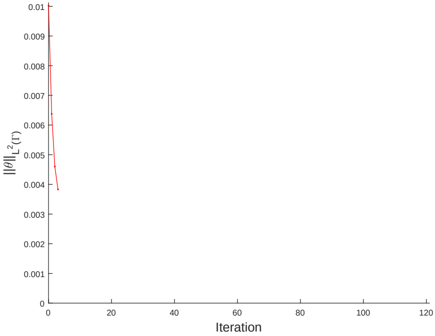

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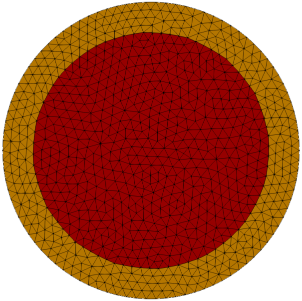

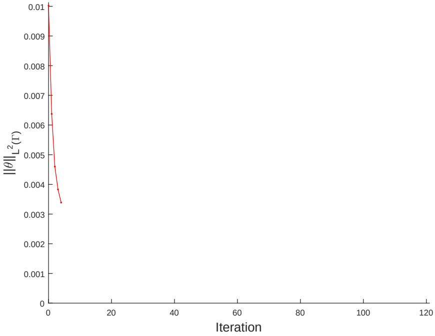

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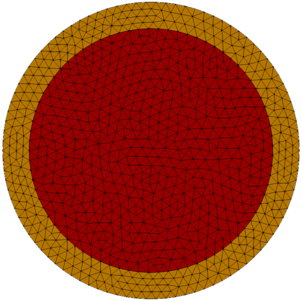

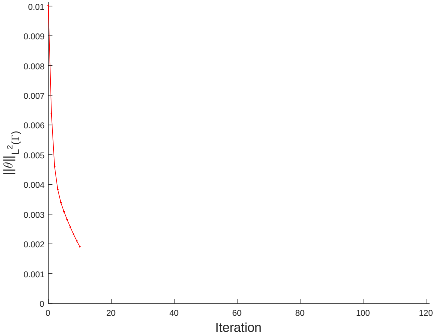

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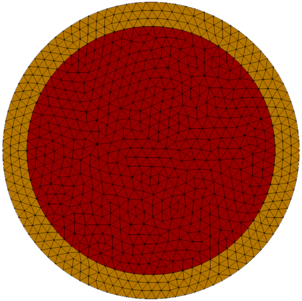

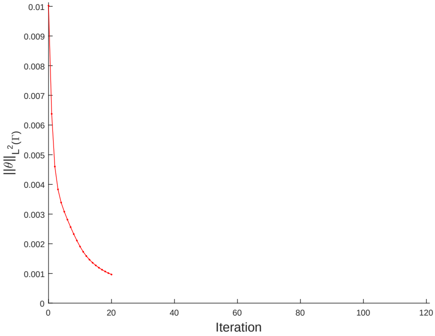

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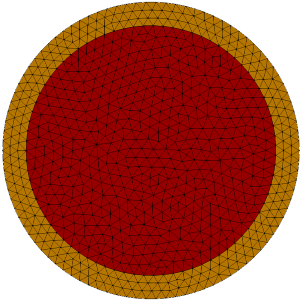

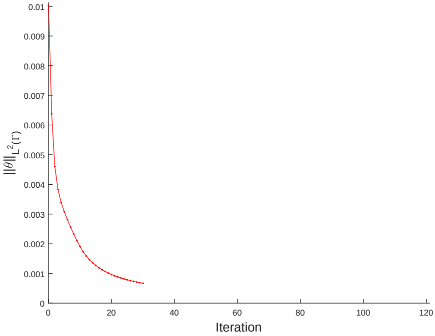

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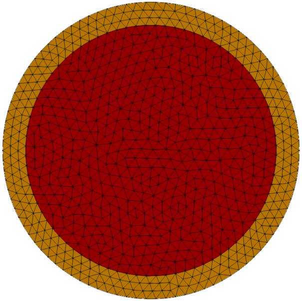

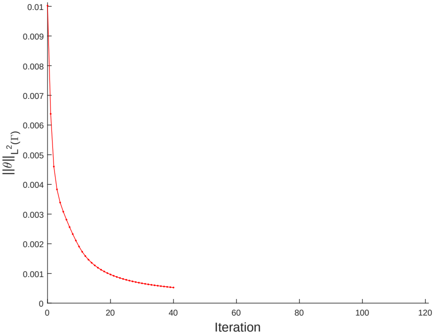

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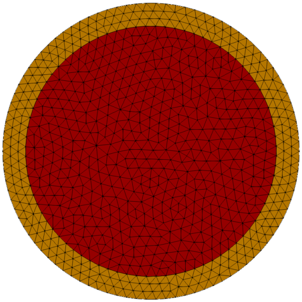

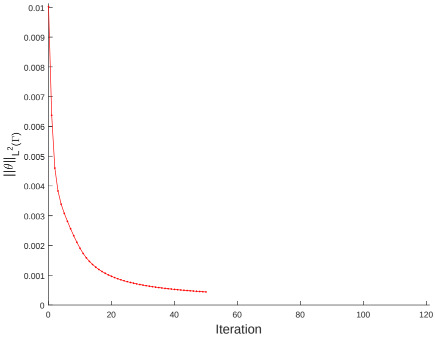

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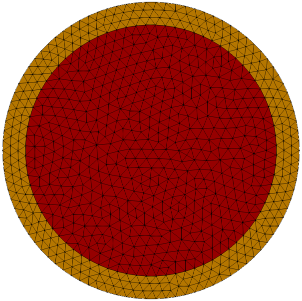

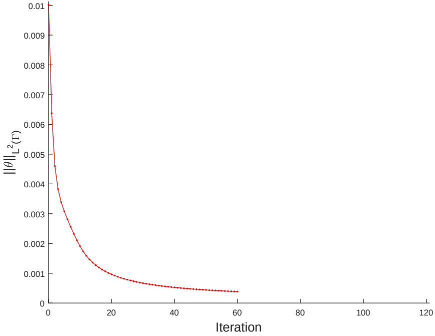

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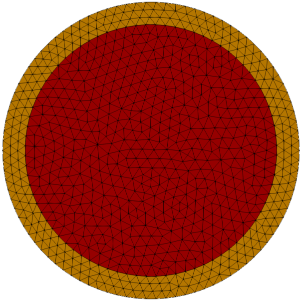
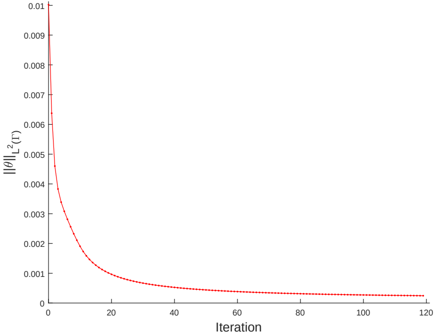
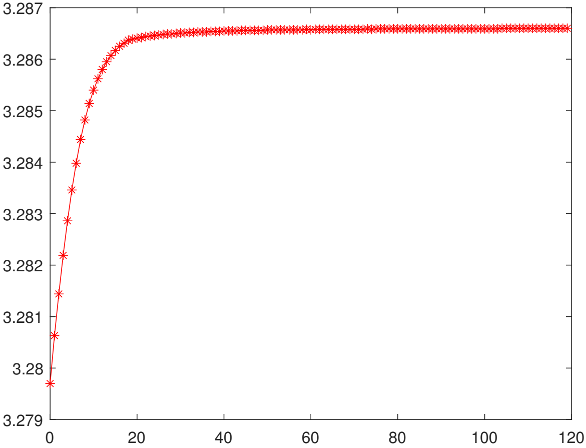




Lagrange function $L = J - \lambda \text{vol}$



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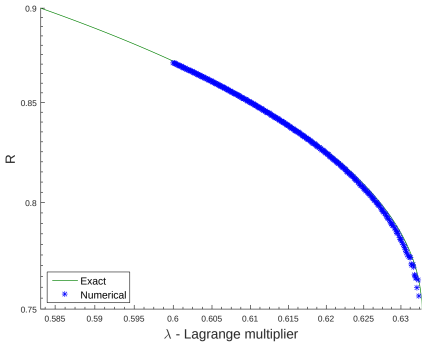
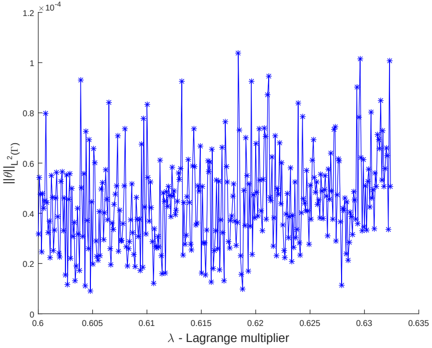
Numerical testing suggest the appearance of local maxima.





Future work:

- $$(l_2^J - \lambda l_2^{\mathrm{vol}})(\phi, \phi) \leq -c\|\phi\|^2_{\mathrm{H}^{1/2}(\Gamma)}$$

  is not enough to prove local maximum!
  The idea is to show the following inequality ($\eta > 0$)

  $$|j''(t) - j''(0)| \leq \omega(\|\phi\|_X)\|\phi\|^2_{\mathrm{H}^{1/2}(\Gamma)}, \quad \forall \|\phi\|_X < \eta$$

  where $j : t \mapsto J(\Omega_{\alpha,t})$ and $\Omega_{\alpha,t}$ is defined through its boundary

  $$\partial\Omega_{\alpha,t} = \{x + t\phi(x)\boldsymbol{n}(x),\ x \in \Gamma\} \cap (\partial\Omega_\alpha \setminus \Gamma)\,.$$

  $X \subset W^{1,\infty}(\Gamma)$ is a Banach space.

- Minimization is more natural with energy functional. By adding perimeter we could do analysis for local minima.

Thank You!