8.3.2. Račun derivacije oblika za stacionarni Navier-Stokes.

Podsjetimo slaba formulacija za N-S glasi

$$(S)\begin{cases} \text{Pronađi } (u,p)\in u_0+V \text{ takve da} \\ \forall (v,q)\in V, \quad \int_\Omega \sigma(u,p):\varepsilon(v) + Du\,u\cdot v - q\,\mathrm{div}\,u = \int_{\Gamma_{out}} h\cdot v \end{cases}$$

gdje su $\quad V = \left\{ (u,p)\in H^1(\Omega;\mathbb{R}^3)\times L^2(\Omega) \;:\; u|_{\Gamma_{in}\cup\Gamma} = 0 \right\}$

$$u_0+V = \left\{ (u,p)\in H^1(\Omega;\mathbb{R}^3)\times L^2(\Omega) : u|_{\Gamma_{in}} = u_0,\ u|_{\Gamma} = 0 \right\}.$$

Zadaća na domeni $\phi_\theta(\Omega)$ (koja je vraćena na Ω) uz notaciju

$$\begin{aligned} a(\theta; u,p; v,q) = &\int_\Omega 2\mu\, S(D(u)\,D\phi_\theta^{-1}) : S(D(v)D\phi_\theta^{-1})\, J_\theta \\ &- \int_\Omega p\, \mathrm{tr}(D(v)\, D\phi_\theta^{-1})\, J_\theta \\ &+ \int_\Omega D(u)\, D\phi_\theta^{-1}\, u\cdot v\ J_\theta \\ &- \int_\Omega q\, \mathrm{tr}(Du\, D\phi_\theta^{-1})\, J_\theta \end{aligned}$$

$$b(\theta; v,q) = \int_{\Gamma_{out}} h_0\cdot v$$

glasi

$$\left.\begin{aligned} &\text{Pronađi } (u,p)\in u_0+V \text{ t.d.} \\ &\forall (v,q)\in V \quad a(\theta; u,p; v,q) = b(\theta; v,q). \end{aligned}\right\}$$

Dano rješenje je očito $(u_\theta\circ\phi_\theta, p_\theta\circ\phi_\theta)$ gdje su (u_θ,p_θ) rješenja od (S) na domeni $\phi_\theta(\Omega)$.

Definiramo L:

$$F(\theta; u,p; v,q) = a(\theta; u,p; v,q) - b(\theta; v, q)$$

možemo pokazati da je $F_0 : K(\eta, \delta) \subseteq W^{1,\infty}(\mathbb{R}^3; \mathbb{R}^3) \times H_0^1(\Omega; \mathbb{R}^3) \times L_0^2(\Omega)$
$$\to H^{-1}(\Omega; \mathbb{R}^3) \times L_0^2(\Omega)$$

gdje $\underset{u \in H_0^1(\Omega; \mathbb{R}^3)}{F_0(\theta; u,p; v,q)} = F(\theta; u + \tilde{u}_0, p; v, q)$ $\tilde{u}_0$ rj. Poisse toka na cilindru.

osigurava da radimo na prostorima gdje smijemo primijeniti teorem o implicitnoj funkciji. Za detalje pogledajte referencu od zadnjeg puta:

J. A. Bello, E. Fernández-Cara, J. Lemoine, J. Simon:
"The differentiability of the drag with respect to the variations of a Lipschitz domain in a Navier-Stokes Flow"

Time je $(u_\theta \circ \phi_\theta - \tilde{u}_0, p_\theta \circ \phi_\theta)$ Fréchet diferencijabilno i opravdan s tim sljedeći formalni račun.

$$\text{(MD)} \left\{ \begin{array}{l} (\dot{u}_\theta, \dot{p}_\theta) \in V \\ D_{(u,p)} F(0, u,p; v,q)[\dot{u}_\theta, \dot{p}_\theta] = -D_\theta F(0, u,p; v,q)[\theta], \ \forall (v,q) \in V \end{array} \right.$$

uočite F umjesto F_0, rigorozni postupak preskačem.

$$D_{(u,p)} F(0, u,p; v,q)[\tilde{u}, \tilde{p}] = \int_\Omega \sigma(\tilde{u}, \tilde{p}) : \varepsilon(v) + D\tilde{u}\, u \cdot v + Du\, \tilde{u} \cdot v - \int_\Omega q \operatorname{div} \tilde{u}$$

$$D_\theta F(0, u,p; v,q)[\theta] = \int_\Omega -2\mu\, S(Du D\theta) : S(Dv) - 2\mu\, S(Du) : S(Dv D\theta) + \int_\Omega \operatorname{div}\theta\, 2\mu\, \varepsilon(u) : \varepsilon(v) + p \operatorname{tr}(Dv D\theta) - p \operatorname{div} v \operatorname{div}\theta$$

$$+\int_\Omega Du\,u\cdot v\,\mathrm{div}\,\theta - Du\,D\theta\,u\cdot v$$

$$+\int_\Omega q\,\mathrm{tr}(Du\,D\theta) - q\,\underbrace{\mathrm{div}\,u}_{=0}\,\mathrm{div}\,\theta$$

Istaknimo da eventualni doprinos od desne strane b zbog $\theta|_{\Gamma_{out}} = 0$ nestaje pa (MD) postaje

$$(\dot u_\theta, \dot p_\theta) \in V$$

$$\int_\Omega \sigma(\dot u_\theta, \dot p_\theta) : \varepsilon(v) + D\dot u_\theta\,u\cdot v + Du\,\dot u_\theta\cdot v - q\,\mathrm{div}\,\dot u_\theta =$$

$$\int_\Omega \underline{2\mu\, S(Du\,D\theta) : S(Dv)} + \underline{2\mu\, S(Du)} : S(Dv\,D\theta)$$

$$+\sigma(u,p) : S(Dv\,D\theta)$$

$$+\int_\Omega \underline{p\,\mathrm{div}\,v}\,\mathrm{div}\,\theta - \underline{2\mu\,\varepsilon(u):\varepsilon(v)}\,\mathrm{div}\,\theta - \underline{p\,\mathrm{tr}(Dv\,D\theta)}$$

$$-\sigma(u,p) : \varepsilon(v)\,\mathrm{div}\,\theta$$

$$+\int_\Omega Du\,D\theta\,u\cdot v - Du\,u\cdot v\,\mathrm{div}\,\theta$$

$$+\int_\Omega q\,\underbrace{\mathrm{div}\,u}_{=0}\,\mathrm{div}\,\theta - \underline{q\,\mathrm{tr}(Du\,D\theta)} \quad +\sigma(v,q) : S(Du\,D\theta)$$

što možemo zapisati kao

$$\left.\begin{aligned}
&\int_\Omega \sigma(\dot u_\theta, \dot p_\theta) : \varepsilon(v) + D\dot u_\theta\,u\cdot v + Du\,\dot u_\theta\cdot v - q\,\mathrm{div}\,\dot u_\theta = \\
&\quad\int_\Omega \sigma(u,p) : (Dv\,D\theta) + \sigma(v,q) : (Du\,D\theta) + Du\,D\theta\,u\cdot v \\
&\qquad - \int_\Omega \big(Du\,u\cdot v + \sigma(u,p) : \varepsilon(v)\big)\,\mathrm{div}\,\theta\,.
\end{aligned}\right\} \text{(SMD)}$$

Preostaje izračunati derivaciju oblika

$$J(\phi_\theta(\Omega)) = \int_{\phi_\theta(\Omega)} 2\mu\, \varepsilon(u_\theta) : \varepsilon(u_\theta)$$

$$= \int_\Omega 2\mu\, \varepsilon(u_\theta)\circ\phi_\theta : \varepsilon(u_\theta)\circ\phi_\theta\, J_\theta$$

$$= \int_\Omega 2\mu\, S\left(D(u\circ\phi_\theta)\,D\phi_\theta^{-1}\right) : S\left(D(u\circ\phi_\theta)\,D\phi_\theta^{-1}\right) J_\theta$$

$$= \int_\Omega 2\mu\,\varepsilon(u):\varepsilon(u) - \int_\Omega 4\mu\, S(Du\,D\theta):\varepsilon(u)$$

$$+ \int_\Omega 2\mu\,\varepsilon(u):\varepsilon(u)\,\mathrm{div}\,\theta + \int_\Omega 4\mu\,\varepsilon(\dot u_\theta):\varepsilon(u) + o(\theta)$$

$$J'(\Omega;\theta) = -\int_\Omega 4\mu\, S(Du\,D\theta):\varepsilon(u) + \int_\Omega 2\mu\,\varepsilon(u):\varepsilon(u)\,\mathrm{div}\,\theta$$

$$+ \int_\Omega 4\mu\,\varepsilon(u):\varepsilon(\dot u_\theta)$$

Prisjetimo se adjungirane slabe formulacije:

$$(\mathrm{ASF})\left\{\begin{array}{l} \forall\ (\varphi,\psi)\in \mathbb{V} \\[2ex] \displaystyle\int_\Omega \sigma(v,q):\varepsilon(\varphi) + D\varphi\, u\cdot v + Du\,\varphi\cdot v - \psi\,\mathrm{div}\,v = -\int_\Omega 4\mu\,\varepsilon(u):\varepsilon(\varphi) \end{array}\right.$$

Stavimo $(\varphi,\psi) = (\dot u_\theta, \dot p_\theta)$ i ubacimo u $J'(\Omega;\theta)$

$$J'(\Omega;\theta) = \int_\Omega 2\mu\, \varepsilon(u):\varepsilon(u)\,\mathrm{div}\,\theta - 4\mu\, \varepsilon(u): S(DuD\theta)$$

$$- \int_\Omega 2\mu\, \varepsilon(v):\varepsilon(\dot{u}_\theta) - q\,\mathrm{div}(\dot{u}_\theta) + D\dot{u}_\theta u\cdot v + Du\,\dot{u}_\theta\cdot v - \dot{p}_\theta\,\mathrm{div}\, v$$

$$= \int_\Omega 2\mu\, \varepsilon(u):\varepsilon(u)\,\mathrm{div}\,\theta - 4\mu\, \varepsilon(u): S(DuD\theta)$$

$$- \int_\Omega \sigma(\dot{u}_\theta, \dot{p}_\theta):\varepsilon(v) + D\dot{u}_\theta u\cdot v + Du\,\dot{u}_\theta\cdot v - q\,\mathrm{div}(\dot{u}_\theta)$$

(SMD)

$$= \int_\Omega 2\mu\, \varepsilon(u):\varepsilon(u)\,\mathrm{div}\,\theta - 4\mu\, \varepsilon(u):(DuD\theta)$$

$$- \int_\Omega \sigma(u,p):(DvD\theta) + \sigma(v,q):(DuD\theta) + Du\,D\theta u\cdot v$$

$$+ \int_\Omega \left(Du\,u\cdot v + \sigma(u,p):\varepsilon(v)\right)\mathrm{div}\,\theta$$

gdje je (v,q) rj. od (ASt) i (u,p) rješenje od (S).

8.3.3. Newtonova metoda za tretiranje nelinearnosti kod N-S

Ideja: Neka je $f: X \to X'$, X Hilbertov prostor. Rješenje $f(x)=0$ tražimo kroz Newtonovu metodu (modificiranu, klasično je za $f: X \to X$) tako da iteracije gradimo kroz

$$x_{k+1} = x_k + \delta x_k$$

$// \; x_{k+1} = x_k - (f'(x_k))^{-1} f(x_k)$ za $f: X \to X$.

gdje je δx_k rješenje od

$$f'(x_k)\,\delta x_k = -f(x_k) \quad (\text{jednakost vrijedi u dualnom prostoru } X')$$

U našem slučaju $X = H^1(\Omega;\mathbb{R}^d) \times L^2(\Omega)$, $\quad X' = (H^1(\Omega;\mathbb{R}^d))' \times L^2(\Omega)$,

$$f \doteq a(0;u,p;\cdot,\cdot) - b(0;\cdot,\cdot)$$

$$f(u,p;v,q) = \int_\Omega \sigma(u,p) : \varepsilon(v) + Du\,u\cdot v - q \operatorname{div} u - \int_{\Gamma_{out}} h_0 \cdot v$$

Laganim računom dolazimo da f' (u oznaci $D_{(u,p)} a(0;u,p;v,q)$)

$$D_{(u,p)} f(u,p;v,q)[\delta u, \delta p] = \int_\Omega \sigma(\delta u, \delta p) : \varepsilon(v) + D\delta u\, u \cdot v$$

$$+ \int_\Omega Du\,\delta u \cdot v - q \operatorname{div}(\delta u)$$

Iterativna metoda glasi:

1) (u^0,p^0) je rješenje Stokesove zadaće:

$$\begin{cases} -\operatorname{div}(\sigma(u^0,p^0)) = 0 & \text{u } \Omega \\ \operatorname{div} u^0 = 0 & \text{u } \Omega \\ u^0 = u_0 & \text{na } \Gamma_{in} \\ u^0 = 0 & \text{na } \Gamma \\ \sigma(u^0,p^0)\,n = h_0 & \text{na } \Gamma_{out} \end{cases}$$

2) Ponavljaj za $k \in \mathbb{N}$ dok se ne postigne konvergencija

(u^{k+1},p^{k+1}) dobivamo preko

$$(u^{k+1}, p^{k+1}) = (u^k, p^k) + (\delta u^k, \delta p^k)$$

gdje je $(\delta u^k, \delta p^k)$ rješenje lineariziranog Navier-Stokesa

$$\begin{cases} -\mathrm{div}(\sigma(\delta u^k, \delta p^k)) + D\delta u^k u^k + Du^k \delta u^k = \mathrm{div}(\sigma(u^k, p^k)) - Du^k u^k \\ \mathrm{div}(\delta u^k) = 0 \quad \text{u } \Omega \\ \delta u^k = 0 \quad \text{na } \Gamma_{in} \cup \Gamma \\ \sigma(\delta u^k, \delta p^k) n = h_0 \quad \text{na } \Gamma_{out} \end{cases}$$

8.3.3. Odabir konačnih elemenata u rješavanju inkompresibilnih modela fluida

Konačni elementi V^h/Q^h moraju zadovoljavati takozvani inf-sup uvjet ili diskretni Babuška-Brezzi ili diskretni Ladyzhenskaya-Babuška-Brezzi (LBB) uvjet tj.

$$\inf_{q^h \in Q^h \setminus \{0\}} \sup_{v^h \in V^h \setminus \{0\}} \frac{b^h(v^h, q^h)}{\|v^h\|_{V^h} \|q^h\|_{Q^h}} \geq \beta^h > 0$$

ili ekvivalentno

$$\sup_{v^h \in V^h \setminus \{0\}} \frac{b^h(v^h, q^h)}{\|v^h\|_{V^h}} \geq \beta^h \|q^h\|_{Q^h} \quad \forall q^h \in Q^h$$

gdje je b^h bilinearna forma, $b^h : V^h \times Q^h \to \mathbb{R}$

$$b^h(v^h, q^h) = -\sum_{K \in \mathcal{T}^h} \int_K q^h \operatorname{div} v^h$$

$\mathcal{T}^h$ - triangulacija

$$= -\int_\Omega q^h \operatorname{div} v^h$$

ako radim s konformnim konačnim elementima

$V^h \subset V$, $Q^h \subset Q$

$V = H^1(\Omega)$

$Q = L^2_0(\Omega) = \{ p \in L^2 : \int p = 0 \}$

Zato se tipično koriste $\mathbb{P}2$ Lagrangeovi elementi za brzinu v te $\mathbb{P}1$ Lagrangeovi elementi za tlak, za koje se može pokazati da vrijedi gornji uvjet.

Ako se koristi mješovita formulacija Stokes tj.

$$a(u, v) + b(v, p) + b(u, q) = F(v) \quad , \quad \forall (v, q) \in V^h \times Q^h$$

gdje je $a(u,v) = \int_\Omega 2\mu\, \varepsilon(u) : \varepsilon(v)$

$b(v,p) = \int_\Omega p \operatorname{div} v$

$-2\mu \operatorname{div}(\varepsilon(u)) + \nabla p = f$

$\operatorname{div} u = 0$

$+$ r.u.

kako bi osigurali da je tlak jedinstven, dobra je prakse dodati penalizaciju $-\varepsilon \int_\Omega pq$ tj. Stokes problem rješavamo kao

$$a(u,v) + b(v,p) + b(u,q) - \varepsilon \int_\Omega pq$$

Pritom ε biramo dovoljno „malim" kako ne bi utjecao na „fizikalni" problem, u našem slučaju $\varepsilon = 10^{-6}$.

Alternativno može se dodati uvjet $\int_\Omega p\,dx = 0$ koji je numerički teže računati

U slučaju da su $\mathbb{P}2$ elementi „preveliki", tipa u 3D primjeru poželjno je koristiti stabilizaciju tako da koristimo

$V^h = Q^h = \mathbb{P}1$ uz stabilizaciju,

Konkretno dodaje se bilinear član

$$d(p^h, q^h) = \delta \sum_{K \in \mathcal{T}^h} h_K^2 \int_K \operatorname{grad} p^h \cdot \operatorname{grad} q^h$$

tj. formulacija na lijevoj strani postaje

$$a(u,v) + b(v,p) + b(u,q) - \underbrace{\varepsilon \int_\Omega pq}_{\text{jedinst. tlak kroz penaliz.}} + \underbrace{\delta \int_\Omega \operatorname{grad} p \cdot \operatorname{grad} q}_{\text{stabilizacija}}$$

Za više detalja pogledajte

L. P. Franca, T. J. Hughes, R. Stenberg

" Stabilized finite element methods for the Stokes problem "
Teknillinen korkeakoulu (1991)

8.3.4. Proširena Lagrangeova metoda

Tipično postavljamo neki uvjet na volumen ili perimetar u obliku $G(\Omega) = 0$.

$$\min_{\Omega \in \mathcal{O},\, G(\Omega)=0} J(\Omega)$$

Time problem uvjetne optimizacije mijenjamo za

$$\min_{\Omega \in \mathcal{O}} J(\Omega) + \lambda G(\Omega) + \frac{a}{2} G(\Omega)^2 =: L(\Omega; \lambda, a)$$

Parametar $a > 0$ predstavlja penalizirajući faktor, dok je λ aproksimacija Lagrangeovog multiplikatora.

Pravila generiranja (λ^{k+1}, a^{k+1}) za dani $(\Omega^k, \lambda^k, a^k)$

$$\lambda^{k+1} = \lambda^k + a^k G(\Omega^k)$$

$$b^{k+1} = \begin{cases} \alpha b^k, & n < n_0 \\ b^k, & \text{inače} \end{cases}, \quad \text{gdje je } \alpha > 1.$$

Ideja je dobiti sve bolju aproksimaciju Lagrangeovog multiplikatora.

Uočite da proširena Lagrangeova metoda ne traži $b^k \xrightarrow{k \to \infty} +\infty$ što ju čini boljom od naivne kvadratne penalizacijske metode (L za $\lambda = 0$).