8.3.2. Račun derivacije oblika za stacionarni Navier-Stokes.

Podsjetimo slaba formulacija za N-S glasi

$$(S)\begin{cases} \text{Pronađi } (u,p)\in u_0+V \text{ takve da} \\ \forall (v,q)\in V, \quad \int_\Omega \sigma(u,p):\varepsilon(v) + Du\,u\cdot v - q\,\mathrm{div}\,u = \int_{\Gamma_{out}} h\cdot v \end{cases}$$

gdje su
$$V = \left\{ (u,p)\in H^1(\Omega;\mathbb{R}^3)\times L^2(\Omega) \;:\; u|_{\Gamma_{in}\cup\Gamma} = 0\right\}$$

$$u_0+V = \left\{ (u,p)\in H^1(\Omega;\mathbb{R}^3)\times L^2(\Omega) : u|_{\Gamma_{in}} = u_0,\; u|_{\Gamma} = 0 \right\}.$$

Zadaća na domeni $\phi_\theta(\Omega)$ (koja je vraćena na Ω) uz notaciju

$$a(\theta; u,p; v,q) = \int_\Omega 2\mu\, S(D(u)\,D\phi_\theta^{-1}) : S(D(v)D\phi_\theta^{-1})\,J_\theta$$
$$- \int_\Omega p\, \mathrm{tr}(D(v)\,D\phi_\theta^{-1})\,J_\theta$$
$$+ \int_\Omega D(u)\,D\phi_\theta^{-1}\,u\cdot v\;J_\theta$$
$$- \int_\Omega q\,\mathrm{tr}(Du\,D\phi_\theta^{-1})\,J_\theta$$

$$b(\theta; v,q) = \int_{\Gamma_{out}} h_0\cdot v$$

glasi

$$\left.\begin{array}{l} \text{Pronađi } (u,p)\in u_0+V \text{ t.d.} \\ \forall (v,q)\in V \quad a(\theta; u,p; v,q) = b(\theta; v,q). \end{array}\right\}$$

Dano rješenje je očito $(u_\theta\circ\phi_\theta, p_\theta\circ\phi_\theta)$ gdje su (u_θ,p_θ) rješenja od (S) na domeni $\phi_\theta(\Omega)$.

Definiramo L:

$$F(\theta; u,p; v,q) = a(\theta; u,p; v,q) - b(\theta; v,q)$$

možemo pokazati da je $\quad F_0 : K(\eta,\delta) \subseteq W^{1,\infty}(\mathbb{R}^3;\mathbb{R}^3) \times H^1_0(\Omega;\mathbb{R}^3) \times L^2_0(\Omega)$
$$\to H^{-1}(\Omega;\mathbb{R}^3) \times L^2_0(\Omega)$$

gdje $\quad \underset{u \in H^1_0(\Omega;\mathbb{R}^3)}{F_0(\theta; u,p; v,q)} = F(\theta; u + \tilde{u}_0, p; v, q)$

$\tilde{u}_0$ rj. Poisse toka na cilindru.

osiguravamo da radimo na prostorima gdje smijemo primijeniti teorem o implicitnoj funkciji. Za detalje pogledajte referencu od zadnjeg puta:

J. A. Bello, E. Fernández-Cara, J. Lemoine, J. Simon:
"The differentiability of the drag with respect to the variations of a Lipschitz domain in a Navier-Stokes Flow"

Time je $(u_\theta \circ \phi_\theta - \tilde{u}_0,\ p_\theta \circ \phi_\theta)$ Fréchet diferencijabilno i opravdan s time sljedeći formalni račun.

$$\text{(MD)} \begin{cases} (\dot{u}_\theta, \dot{p}_\theta) \in V \\ D_{(u,p)} F(0, u,p; v,q)[\dot{u}_\theta, \dot{p}_\theta] = -D_\theta F(0, u,p; v,q)[\theta], \quad \forall (v,q) \in V \end{cases}$$

uočite F umjesto F_0, rigorozni postupak preskačem.

$$D_{(u,p)} F(0, u,p; v,q)[\tilde{u}, \tilde{p}] = \int_\Omega \sigma(\tilde{u},\tilde{p}) : \varepsilon(v) + D\tilde{u}\, u \cdot v + D u\, \tilde{u} \cdot v - \int_\Omega q \operatorname{div} \tilde{u}$$

$$D_\theta F(0, u,p; v,q)[\theta] = \int_\Omega -2\mu\, S(Du\, D\theta) : S(Dv) - 2\mu\, S(Du) : S(Dv\, D\theta) + \int_\Omega \operatorname{div}\theta\, 2\mu\, \varepsilon(u) : \varepsilon(v) + p \operatorname{tr}(Dv\, D\theta) - p \operatorname{div} v \operatorname{div}\theta$$

$$+\int_\Omega Du\,u\cdot v\,\mathrm{div}\,\theta - Du\,D\theta\,u\cdot v$$

$$+\int_\Omega q\,\mathrm{tr}(Du\,D\theta) - q\,\mathrm{div}\,u\,\mathrm{div}\,\theta$$

Istaknimo da eventualni doprinos od desne strane b zbog $\theta|_{\Gamma_{out}} = 0$ nestaje pa (MD) postaje

$$(\dot{u}_\theta, \dot{p}_\theta) \in V$$

$$\int_\Omega \sigma(\dot{u}_\theta, \dot{p}_\theta) : \varepsilon(v) + D\dot{u}_\theta\,u\cdot v + Du\,\dot{u}_\theta\cdot v - q\,\mathrm{div}\,\dot{u}_\theta =$$

$$\int_\Omega \underline{2\mu\,S(Du\,D\theta) : S(Dv)} + \underline{2\mu}\,S(Du) : S(Dv\,D\theta)$$

$$+\,\sigma(u,p) : S(Dv\,D\theta)$$

$$+\int_\Omega \underline{p}\,\mathrm{div}\,v\,\mathrm{div}\,\theta - \underline{2\mu\,\varepsilon(u) : \varepsilon(v)}\,\mathrm{div}\,\theta - \underline{p\,\mathrm{tr}(Dv\,D\theta)}$$

$$-\,\sigma(u,p) : \varepsilon(v)\,\mathrm{div}\,\theta$$

$$+\int_\Omega Du\,D\theta\,u\cdot v - Du\,u\cdot v\,\mathrm{div}\,\theta$$

$$+\int_\Omega q\,\mathrm{div}\,u\,\mathrm{div}\,\theta - \underline{q\,\mathrm{tr}(Du\,D\theta)} \quad + \sigma(v,q) : S(Du\,D\theta)$$

što možemo zapisati kao

$$\left.\begin{aligned}
&\int_\Omega \sigma(\dot{u}_\theta, \dot{p}_\theta) : \varepsilon(v) + D\dot{u}_\theta\,u\cdot v + Du\,\dot{u}_\theta\cdot v - q\,\mathrm{div}\,\dot{u}_\theta = \\
&\quad \int_\Omega \sigma(u,p) : (Dv\,D\theta) + \sigma(v,q) : (Du\,D\theta) + Du\,D\theta\,u\cdot v \\
&\quad + \int_\Omega \left(q\,\mathrm{div}\,u - Du\,u\cdot v - \sigma(u,p) : \varepsilon(v) \right) \mathrm{div}\,\theta\,.
\end{aligned}\right\} \text{(SMD)}$$

Preostaje izračunati derivaciju oblika

$$J(\phi_\theta(\Omega)) = \int_{\phi_\theta(\Omega)} 2\mu\, \varepsilon(u_\theta) : \varepsilon(u_\theta)$$

$$= \int_\Omega 2\mu\, \varepsilon(u_\theta)\circ\phi_\theta : \varepsilon(u_\theta)\circ\phi_\theta\, J_\theta$$

$$= \int_\Omega 2\mu\, S\left(D(u\circ\phi_\theta)D\phi_\theta^{-1}\right) : S\left(D(u\circ\phi_\theta)D\phi_\theta^{-1}\right) J_\theta$$

$$= \int_\Omega 2\mu\,\varepsilon(u):\varepsilon(u) - \int_\Omega 4\mu\, S(DuD\theta):\varepsilon(u)$$

$$+ \int_\Omega 2\mu\, \varepsilon(u):\varepsilon(u)\operatorname{div}\theta + \int_\Omega 4\mu\,\varepsilon(\dot{u}_\theta):\varepsilon(u) + o(\theta)$$

$$J'(\Omega;\theta) = -\int_\Omega 4\mu S(Du\, D\theta):\varepsilon(u) + \int_\Omega 2\mu\,\varepsilon(u):\varepsilon(u)\operatorname{div}\theta$$

$$+ \int_\Omega 4\mu\,\varepsilon(u):\varepsilon(\dot{u}_\theta)$$

Prisjetimo se adjungirane slabe formulacije:

$$(\mathrm{ASF})\left\{\begin{array}{l} \forall\ (\varphi,\psi)\in \mathbf{V} \\ \displaystyle\int_\Omega \sigma(v,q):\varepsilon(\varphi) + D\varphi\, u\cdot v + Du\,\varphi\cdot v - \psi\operatorname{div} v = -\int_\Omega 4\mu\,\varepsilon(u):\varepsilon(\varphi) \end{array}\right.$$

Stavimo $(\varphi,\psi) = (\dot{u}_\theta, \dot{p}_\theta)$ i ubacimo u $J'(\Omega;\theta)$

$$J'(\Omega;\theta) = \int_\Omega 2\mu\, \varepsilon(u):\varepsilon(u)\, \mathrm{div}\,\theta - 4\mu\, \varepsilon(u): S(Du D\theta)$$

$$- \int_\Omega 2\mu\, \varepsilon(v):\varepsilon(\dot{u}_\theta) - q\, \mathrm{div}(\dot{u}_\theta) + D\dot{u}_\theta\, u \cdot v + Du\, \dot{u}_\theta \cdot v - \dot{p}_\theta\, \mathrm{div}\, v$$

$$= \int_\Omega 2\mu\, \varepsilon(u):\varepsilon(u)\, \mathrm{div}\,\theta - 4\mu\, \varepsilon(u): S(Du D\theta)$$

$$- \int_\Omega \sigma(\dot{u}_\theta, \dot{p}_\theta):\varepsilon(v) + D\dot{u}_\theta\, u \cdot v + Du\, \dot{u}_\theta \cdot v - q\, \mathrm{div}(\dot{u}_\theta)$$

(SMD)

$$= \int_\Omega 2\mu\, \varepsilon(u):\varepsilon(u)\, \mathrm{div}\,\theta - 4\mu\, \varepsilon(u): S(Du D\theta)$$

$$- \int_\Omega \sigma(u,p):(Dv D\theta) + \sigma(v,q):(Du D\theta) + Du\, D\theta \cdot v$$

$$- \int_\Omega \left(q\, \mathrm{div}\, u - Du\, u \cdot v - \sigma(u,p): \varepsilon(v)\right) \mathrm{div}\,\theta$$

$$= \int_\Omega \left(\sigma(u,q):\varepsilon(u) + \sigma(u,p):\varepsilon(v) + Du\, u \cdot v\right) \mathrm{div}\,\theta$$

$$- \int_\Omega 4\mu\, \varepsilon(u): S(Du D\theta) - \sigma(u,p):(Dv D\theta) - \sigma(v,q):(Du D\theta) + Du\, D\theta \cdot v$$

$\blacksquare$

8.3.3. Newtonova metoda za tretiranje nelinearnosti kod N-S

Ideja: Neka je $f : X \to X'$, X Hilbertov prostor. Rješenje $f(x)=0$ tražimo kroz Newtonovu metodu (modificiranu, klasično je za $f: X \to X$) tako da iteracije gradimo kroz

$$x_{k+1} = x_k + \delta x_k$$

// $x_{k+1} = x_k - \left(f'(x_k)\right)^{-1} f(x_k)$ za $f : X \to X$.

gdje je δx_k rješenje od

$$f'(x_k)\,\delta x_k = -f(x_k) \quad (\text{jednakost vrijedi u dualnom prostoru } X')$$

U našem slučaju $X = H^1(\Omega;\mathbb{R}^d) \times L^2(\Omega)$, $X' = (H^1(\Omega;\mathbb{R}^d))' \times L^2(\Omega)$,

$f \equiv a(0; u,p; \cdot,\cdot) : X \to X'$, tj.

$$a(0; u,p; v,q) = \int_\Omega \sigma(u,p) : \varepsilon(v) + Du\,u\cdot v - q \operatorname{div} u$$

Laganim računom dobivamo da f' (u oznaci $D_{(u,p)} a(0; u,p; v,q)$)

$$D_{(u,p)} a(u,p;v,q)[\delta u, \delta p] = \int_\Omega \sigma(\delta u, \delta p) : \varepsilon(v) + D\delta u\, u\cdot v$$

$$+ \int_\Omega Du\,\delta u\cdot v - q \operatorname{div}(\delta u)$$

Iterativna metoda glasi:

1) (u^0, p^0) je rješenje Stokesove zadaće:

$$\begin{cases} -\operatorname{div}(\sigma(u^0,p^0)) = 0 & u\ \Omega \\ \operatorname{div} u^0 = 0 & u\ \Omega \\ u^0 = u_0 & na\ \Gamma_{in} \\ u^0 = 0 & na\ \Gamma \\ \sigma(u^0,p^0)n = h_0 & na\ \Gamma_{out} \end{cases}$$

2) Ponavljaj za $k \in \mathbb{N}$ dok se ne postigne konvergencija

(u^{k+1}, p^{k+1}) dobivamo preko

$$(u^{k+1}, p^{k+1}) = (u^k, p^k) + (\delta u^k, \delta p^k)$$

gdje je $(\delta u^k, \delta p^k)$ rješenje lineariziranog Navier-Stokesa

$$-\mathrm{div}(\sigma(\delta u^k, \delta p^k)) + D\delta u^k u^k + Du^k \delta u^k = \mathrm{div}(\sigma(u^k, p^k)) - Du^k u^k$$

$$\mathrm{div}(\delta u^k) = 0 \quad \text{u } \Omega$$

$$\delta u^k = 0 \quad \text{na } \Gamma_{in} \cup \Gamma$$

$$\sigma(\delta u^k, \delta p^k) n = 0 \quad \text{na } \Gamma_{out}$$