# 8.3. Optimizacija oblika za inkompresibilni Navier-Stokesov fluid u ograničenoj domeni

## 8.3.1. Iskaz problema

Neka je $D \subseteq \mathbb{R}^3$ cilindar oblika

$$D = \left\{ (x_1, x_2, x_3) : \ x_1^2 + x_2^2 \le R_0^2, \ 0 \le x_3 \le L \right\}, \quad R_0, L > 0.$$

Područje fluida označavamo s $\Omega$, a rub $\partial\Omega$, te je $\Omega \subset D$.


Ulazni tok fluida definiran je na granici

$$\Gamma_{in} = \partial\Omega \cap \left\{ (x_1, x_2, x_3) \in D : \ x_3 = 0 \right\}$$

Izlazni tok fluida definiran je na granici

$$\Gamma_{out} = \partial\Omega \cap \left\{ (x_1, x_2, x_3) \in D : \ x_3 = L \right\}$$

Radi jednostavnosti pretpostavljamo da je $\Gamma_{in} = \left\{ (x_1, x_2, 0) : x_1^2 + x_2^2 \le R^2 \right\}$

$\Gamma_{out} = \left\{ (x_1, x_2, L) : x_1^2 + x_2^2 \le R^2 \right\}$

Time je $\partial\Omega = \Gamma \cup \Gamma_{in} \cup \Gamma_{out}$ disjunktna unija, gdje je $\Gamma$ nepoznati dio ruba koji želimo optimizirati.

Konkretno, N-S sustav PDJ možemo zapisati kao

$$Du\, u$$

$$v\big|_{\Gamma_{in} \cap \Gamma} = 0$$

$$(NS) \begin{cases} -\mathrm{div}(\sigma(u,p)) + (u \cdot \nabla) u = 0 \ / \cdot v \\ \mathrm{div}\, u = 0 \ / \, q \end{cases} \ \text{u } \Omega$$

$$\begin{cases} u = u_0 = (0, 0, c(x_1^2 + x_2^2 - R^2)) & \text{na } \Gamma_{in} \\ u = 0 & \text{na } \Gamma \\ \sigma(u,p) n = h_0 = (2\mu c x_1, 2\mu c x_2, -p_1) & \text{na } \Gamma_{out} \end{cases}$$

$$\sigma(u,p) = -pI + 2\mu\varepsilon(u)$$

$$\varepsilon(u) = \frac{1}{2}(Du + Du^T)$$

$\mu$ označava viskoznost, $c > 0$, $p_1 \in \mathbb{R}$.

Interpretacija - ako je $\Gamma$ cilindar tj. $\Gamma = \{(x_1,x_2,x_3) : x_1^2+x_2^2=R^2\}$ dobivamo Poiseuillov tok te je $c = \dfrac{p_1-p_0}{4\mu L}$, $p_0$ tlak na $\Gamma_{in}$, $p_1$ tlak na $\Gamma_{out}$

Za traženi $\Gamma$ znamo i eksplicitno rješenje

$$u(x_1,x_2,x_3) = (0,0,c(x_1^2+x_2^2-R^2))$$
$$p(x_1,x_2,x_3) = 4\mu c(x_3-L)+p_1.$$

Neka je $V = \{(u,p) \in H^1(\Omega;\mathbb{R}^2)\times L^2(\Omega) \; : u|_{\Gamma_{in}\cup\Gamma} = 0\}$

$u_0 + V = \{(u,p)\in H^1(\Omega;\mathbb{R}^2)\times L^2(\Omega) \; : u|_{\Gamma_{in}} = u_0, \; u|_{\Gamma} = 0\}$

Slaba formulacija (NS) glasi:

Pronaći $(u,p)\in u_0+V$ t.d.

$$\left.\forall (v,q)\in V, \; \int_\Omega \sigma(u,p):\varepsilon(v) + Du\, u\cdot v - q \operatorname{div} u = \int_{\Gamma_{out}} h_0 v \, dS \right\} (S)$$

Ako je $\Omega$ regularna domena (različita od cilindra) rješenje postoji, recimo vidi 9. poglavlje od

G. P. Galdi: „An Introduction to the Mathematical Theory of the Navier-Stokes Equations"

ili

R. Teman: „Navier-Stokes Equations" 2. poglavlje.

Teorem 8.3.1

Neka $u_0 \in H^{3/2}(E;\mathbb{R}^3)$ i $h \in H^{1/2}(S;\mathbb{R}^3)$. Ako je $\mu$ dovoljno velik tada problem (NS) ima jedinstveno rješenje $(u,p)\in H^1(\Omega;\mathbb{R}^3)\times L^2(\Omega)$.

Ideja je minimizirati energiju fluida ili otpor fluida (eng. drag)

$$J(\Omega) = 2\mu \int_\Omega |\varepsilon(u)|^2 \, dx ,$$

Zapišimo Lagrangeov funkcional na standardni način kao dosada

$$L(\Omega, u, p; v, q) = 2\mu \int_\Omega \varepsilon(u) : \varepsilon(u) \, dx + \int_\Omega \sigma(u,p) : \varepsilon(v) + Du\, u \cdot v$$

$$- q \operatorname{div} u - \int_{\Gamma_N} h_0 \cdot v \, dS$$

To nam daje ideju kako zapisati adjungiranu zadaću.
Uočimo da $\frac{\partial L}{\partial (v,q)} = 0$ vraćaju na (S).

$$\frac{\partial L}{\partial (u,p)}(\Omega; u,p; v,q)[\varphi, \psi] = \int_\Omega 4\mu \, \varepsilon(u) : \varepsilon(\varphi)$$

$$+ \int_\Omega 2\mu \, \varepsilon(\varphi) : \varepsilon(v) - \psi \operatorname{div} v + D\varphi \, u \cdot v + Du \, \varphi \cdot v - q \operatorname{div} \varphi = 0$$

$$(\mathrm{ASF}) \begin{cases} \forall \; (\varphi, \psi) \in \mathbf{V} \\ \int_\Omega \sigma(v,q) : \varepsilon(\varphi) + D\varphi \, u \cdot v + Du \, \varphi \cdot v - \psi \operatorname{div} v = -\int_\Omega 4\mu \, \varepsilon(u) : \varepsilon(\varphi) \end{cases}$$

$$\Rightarrow \int_\Omega \operatorname{div}(\sigma(v,q)\varphi) - \operatorname{div}\sigma(v,q) \cdot \varphi + u \cdot D\varphi^T v + Du^T v \cdot \varphi$$

$$- \psi \operatorname{div} v = - \int_\Omega 4\mu \, \varepsilon(u) : \varepsilon(\varphi)$$

$$u \cdot D\varphi^T v = u \cdot (\operatorname{grad}(v \cdot \varphi) - Dv^T \varphi) = - Dv \, u \cdot \varphi + u \cdot \operatorname{grad}(v \cdot \varphi)$$

$$= -Dvu\,\psi + \operatorname{div}((v\cdot\varphi)u) - (v\cdot\varphi)\underbrace{\operatorname{div} u}_{=0}$$

$$\int_\Omega u\cdot D\varphi^T v = -\int_\Omega Dvu\cdot\varphi + \int_{\partial\Omega}(v\cdot\varphi)\,u\cdot n \quad \swarrow \varphi|_{\Gamma_D\cup\Gamma} = 0$$

$$= -\int_\Omega Dvu\cdot\varphi + \int_{\Gamma_N}(u\cdot n)v\cdot\varphi$$

$$\Rightarrow \int_\Omega\left(-\operatorname{div}\sigma(v,q) - Dvu + Du^T v\right)\cdot\varphi + \int_{\Gamma_N}\left(\sigma(v,q)n + (u\cdot n)v\right)\cdot\varphi$$

$$-\psi\operatorname{div} v = \int_\Omega 2\mu\Delta u - \int_{\Gamma_N} 4\mu\,\varepsilon(u)n\cdot\varphi\,, \quad \forall(\varphi,\psi)\in V$$

Dakle adjungirana zadaća glasi

$$\begin{cases} -\operatorname{div}\sigma(v,q) + Du^T v - Du\,v = 2\mu\Delta u & u\ \Omega \\ \operatorname{div} v = 0 & u\ \Omega \\ v = 0 & na\ \Gamma_{in}\cup\Gamma \\ \sigma(v,q)n + (u\cdot n)v = 0 & na\ \Gamma_{out} \end{cases}$$

Zapišimo slabu formulaciju od S na perturbiranoj domeni

$\phi_\theta(\Omega) = (Id+\theta)(\Omega)$. $\quad V_\theta = \{ (u,p)\in H^1(\Omega;\mathbb{R}^3)\times L^2(\Omega) : u|_{\phi(\Gamma)\cup\Gamma_D} = 0 \}$

Pronađi $(u_\theta, p_\theta) \in u_0 + V_\theta$

$$\int_{\phi_\theta(\Omega)} \sigma(u_\theta,p_\theta):\varepsilon(v) + Du\,u\cdot v - q\operatorname{div} u_\theta = \int_{\phi_\theta(\Gamma_{out}) = \Gamma_{out}} h\cdot v\,, \quad \forall(v,q)\in V_\theta$$

Nap. Dogovor je $\theta = 0$ na okolini od $\Gamma_{in}$ i $\Gamma_{out}$

kao i kod elastičnosti

Koristimo

$$e(u)\circ\phi_\theta = S\left(D(u\circ\phi_\theta)\, D\phi_\theta^{-1}\right)$$

$$Du\circ\phi_\theta = D(u\circ\phi_\theta)\, D\phi_\theta^{-1}$$

Zapišimo sve na referentnoj domeni

$$\int_\Omega \left(2\mu\, \varepsilon(u_\theta)\circ\phi_\theta : \varepsilon(v)\circ\phi_\theta - p_\theta\circ\phi_\theta\, \mathrm{div}(v)\circ\phi_\theta\right) J_\theta$$

$$+\left(Du_\theta\circ\phi_\theta\, u_\theta\circ\phi_\theta\cdot v\circ\phi_\theta - q\circ\phi_\theta\, \mathrm{div}(u_\theta)\circ\phi_\theta\right) J_\theta = \int_{\Gamma_N} h\cdot v$$

$$\Rightarrow \int_\Omega 2\mu\, S\left(D(u_\theta\circ\phi_\theta)\, D\phi_\theta^{-1}\right) : S\left(D(v\circ\phi_\theta)\, D\phi_\theta^{-1}\right) J_\theta$$

$$+ \int_\Omega -p_\theta\circ\phi_\theta\, \mathrm{tr}\left(D(v\circ\phi_\theta)\, D\phi_\theta^{-1}\right) J_\theta$$

$$+ \int_\Omega D(u_\theta\circ\phi_\theta)\, D\phi_\theta^{-1}\, u_\theta\circ\phi_\theta\cdot v\circ\phi_\theta\, J_\theta$$

$$- \int_\Omega q\circ\phi_\theta\, \mathrm{tr}\left(D(u_\theta\circ\phi_\theta)\, D\phi_\theta^{-1}\right) J_\theta = \int_{\Gamma_N} h\cdot v$$

Znači $(u_\theta\circ\phi_\theta, p_\theta\circ\phi_\theta)$ zadovoljava gornju jednadžbu.

Može se pokazati da je $\theta \mapsto (u_\theta\circ\phi_\theta, p_\theta\circ\phi_\theta)$ Fréchet diferencijabilna u $0$, ali dokaz preskačem.

Ideja je primjeniti istu proceduru kao i u

J. A. Bello, E. Fernández-Cara, J. Lemoine, J. Simon:

"The differentiability of the drag with respect to the variations of a Lipschitz domain in a Navier-Stokes Flow"

Konkretno, može se pokazati koristeći teorem o implicitnoj funkciji.