8. Optimizacija oblika u primjeni

8.1. Linearna elastičnost

Linearna elastičnost - podsjetnik

Neka je $\Omega \subseteq \mathbb{R}^d$, $d=2,3$ ograničena domena (tijelo) ispunjeno homogenim, izotropičnim elastičnim materijalom. To znači da su deformacije tijela Ω uslijed vanjskog naprezanja trenutačne te da se tijelo Ω vraća u ravnotežno stanje čim naprezanje prestane.

Gibanje takvog tijela opisuje se funkcijom deformacije $\phi: \overline{\Omega} \to \mathbb{R}^d$, pri čemu se svaka točka $x \in \Omega$ pomiče u položaj $\phi(x)$.
Ekvivalentno može se promatrati pripadnu funkciju pomaka $u = \phi - Id$.

Kako bismo izmjerili nastalu deformaciju, odnosno naprezanje unutar Ω, pratimo Cauchy-Greenov tenzor deformacije (oznaka $C(\phi)$) i Green-Saint-Venantov tenzor naprezanja (oznaka $E(\phi)$) definirani kao:

$$C(\phi) = \nabla\phi^T \nabla\phi = (I + \nabla u^T + \nabla u + \nabla u^T \nabla u)$$

(desni) Cauchy-Greenov tenzor deformacije mjeri deformaciju krivulje nacrtane na Ω uzrokovanu preslikavanjem ϕ, fizikalno daje kvadrat promjene udaljenosti uslijed deformacije

$$E(\phi) = \frac{1}{2}(C(\phi) - I)$$

Green-Saint-Venantov tenzor naprezanja kvantificira koliko je deformacija različita od pomaka krutog tijela (poznati kao i Lagrangeov konačni tenzor naprezanja)

Teorija linearne elastičnosti temelji se na sljedeće dvije fundamentalne aproksimacije:

- aproksimacija malih deformacija: Green-Saint-Venantov tenzor naprezanja može se aproksimirati lineariziranim tenzorom deformacije

$$e(u) := \frac{\nabla u^{\tau} + \nabla u}{2}, \quad \text{tj.} \quad E(\phi) \approx e(u)$$

- linearnost ponašanja materijala: konstitutivni zakon materijala definira vezu između Cauchyjeva tenzora naprezanja σ i deformacije. Uz razumne fizikalne pretpostavke vrijedi

$$\sigma = 2\mu E + \lambda \operatorname{tr}(E) + o(E)$$

gdje su λ, μ Laméovi koeficijenti materijala.
Članovi višeg reda $o(E)$ u linearnoj elastičnosti zanemarujemo.
Za detalje izvoda pogledajte poglavlja 3.6 i 3.7 u

P. G. Ciarlet, Mathematical Elasticity, vol I: Three Dimensional Elasticity, North Holland Publishing Company (1988)

Neka je Ω učvršćena na dijelu Γ_D svoje granice $\partial\Omega$, izložena sili f, te rubnim opterećenjem g na $\Gamma_N = (\partial\Omega \setminus \Gamma_D)$ jednadžbe ravnoteže glase:

$$\begin{cases} -\operatorname{div}(\sigma(u)) = f & \text{u } \Omega \\ u = 0 & \text{na } \Gamma_D \\ \sigma(u)\, n = g & \text{na } \Gamma_N \end{cases}$$

Matematičke pretpostavke:

- Ω ograničen otvoren skup s Lipschitzovim rubom, Γ_D ima pozitivnu $(d-1)$ Hausdorffovu mjeru (u suprotnom treba ravnotežni odnos između f i g)
- sila na tijelo $f \in H^{-1}(\Omega; \mathbb{R}^d)$, $g \in H^{-1/2}(\partial\Omega; \mathbb{R}^d)$.

Primjenom Kornove nejednakosti zajedno s Lax-Milgramovom lemom, može se pokazati da gornja zadaća ima jedinstveno rješenje $u \in H^1_{\Gamma_D}(\Omega;\mathbb{R}^d)$ gdje je

$$H^1_{\Gamma_D}(\Omega;\mathbb{R}^d) = \{ v \in H^1(\Omega;\mathbb{R}^d),\ v = 0 \text{ na } \Gamma_D \}$$

Rezultati vezani za regularnost postoje, ali su dosta teži za pokazati. Na primjer rezultat iz Ciarlet: Math. Elast. Teorem 6.3.6

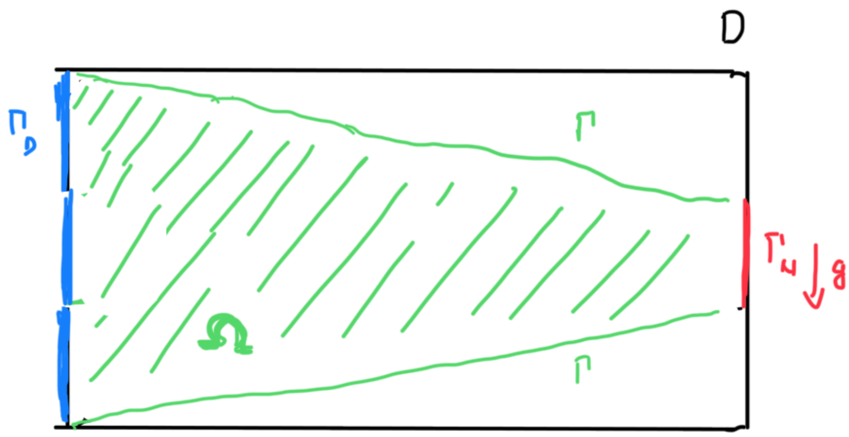
Teorem
Neka je $\Gamma_D = \partial\Omega$. Neka je $m \in \mathbb{N}_0$, $p \in \langle 1, +\infty\rangle$, $p \geq \frac{6}{5+2m}$, $d=3$. Neka je $\partial\Omega$ klase C^{m+2} i $f \in W^{m,p}(\Omega;\mathbb{R}^3)$. Tada je rješenje od (LEla) u $W^{m+2,p}(\Omega;\mathbb{R}^3)$ i postoji konstanta $C = C(\Omega) > 0$

$$\|u\|_{W^{m+2,p}(\Omega;\mathbb{R}^3)} \leq C \|f\|_{W^{m,p}(\Omega;\mathbb{R}^3)}.$$

∎

Sličan rezultat postoji i ako je $\Gamma_N = \partial\Omega$, zanimljivo ako postoji više različitih tipova rubnih uvjeta, klasični rezultati regularnosti ne vrijede bez dodatnih pretpostavki. Preciznije, regularnost u blizini prijelaza ne mora nužno vrijediti.

8.2. Optimizacija oblika u linearnoj elastičnosti



$$
(S)\quad \begin{cases} -\operatorname{div}(\sigma(u)) = f & u\ \Omega \\ u = 0 & na\ \Gamma_D \\ \sigma(u)n = g & na\ \Gamma_N \\ \sigma(u)n = 0 & na\ \Gamma \end{cases} \qquad \begin{matrix} f \in H^1(\mathbb{R}^d;\mathbb{R}^d) \\ g \in H^2(\mathbb{R}^d;\mathbb{R}^d) \end{matrix}
$$

Dopustivi skup domena

$$
\mathcal{O} = \left\{ \Omega \subseteq D,\ \text{ograničena i Lipschitz},\ \Gamma_D \cup \Gamma_N \subset \partial\Omega \right\}
$$

Iako biramo varijacije iz $W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)$ pretpostavljamo da svaki $\theta \in W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)$ zadovolja $\theta = 0$ na otvorenoj okolini oko $\Gamma_D \cup \Gamma_N$.

Tipično tražimo derivaciju oblika sljedećih funkcionala:

$$
C(\Omega) = \int_\Omega Ae(u_\Omega) : e(u_\Omega)\, dx = \int_\Omega f \cdot u_\Omega\, dx + \int_{\Gamma_N} g \cdot u_\Omega\, ds \quad (\text{compliance})
$$

tj. tražimo oblik tijela kojim dobivamo tijelo koje je kruće (manje fleksibilno). Fizikalno to je rad vanjskih sila na tijelo.

Drugi tip funkcional oblika je

$$
D(\Omega) = \left(\int_\Omega k\, |u_\Omega - u_0|^\alpha\, dx \right)^{\frac{1}{\alpha}}, \quad \alpha \geq 2,\ k \in L^\infty(\mathbb{R}^d),\ u_0 \in H^1(D, \mathbb{R}^d)
$$

Prvo se fokusiramo na funkcional C.

Definiramo Lagrangeovu funkciju

$$
L(\Omega, u, p) = C(\Omega) + \int_\Omega Ae(u) : e(p) - \int_\Omega f \cdot p - \int_{\Gamma_N} g \cdot p
$$

Perturbacija identitete radimo s obzirom na parametar $t \in [0, \tau]$ gdje je $\tau > 0$ dovoljno mali, $\theta \in W^{2,\infty}(\mathbb{R}^d;\mathbb{R}^d)$

$$
\phi_t = Id + t\theta.
$$

$$G(t,u,p) = L(\phi_t(\Omega), u, p)$$

$$= \int_{\phi_t(\Omega)} f\cdot u + \int_{\phi_t(\Gamma_N)} g\cdot u + \int_{\phi_t(\Omega)} A e(u) : e(p)$$

$$- \int_{\phi_t(\Omega)} f\cdot p - \int_{\phi_t(\Gamma_N)} g\cdot p$$

HP Prop. 5.4.2.

zapis u originalnoj domeni

$$= \int_\Omega f\circ\phi_t \cdot u\circ\phi_t J_t + \int_{\Gamma_N} g\circ\phi_t \cdot u\circ\phi_t J_t |\nabla\phi_t^{-\tau} n|$$

$$+ \int_\Omega \left(2\mu\, e(u)\circ\phi_t : e(p)\circ\phi_t + \lambda \operatorname{div}(u)\circ\phi_t \operatorname{div}(p)\circ\phi_t\right) J_t$$

$$- \int_\Omega f\circ\phi_t \cdot p\circ\phi_t J_t - \int_{\Gamma_N} g\circ\phi_t \cdot p\circ\phi_t J_t |\nabla\phi_t^{-\tau} n|$$

$$= \int_\Omega f\circ\phi_t \cdot (u\circ\phi_t - p\circ\phi_t) J_t$$

$$+ \int_{\Gamma_N} g\circ\phi_t \cdot (u\circ\phi_t - p\circ\phi_t) J_t |\nabla\phi_t^{-\tau} n|$$

$S(A) = \frac{1}{2}(A + A^\tau)$

$$+ \int_\Omega 2\mu\, S(D(u\circ\phi_t)\nabla\phi_t^{-1}) : S(D(p\circ\phi_t)\nabla\phi_t^{-1}) J_t$$

$$+ \int_\Omega \lambda \operatorname{tr}(D(u\circ\phi_t)\nabla\phi_t^{-1}) \operatorname{tr}(D(p\circ\phi_t)\nabla\phi_t^{-1}) J_\tau$$

$$Du^\tau = [\operatorname{grad} u_1, \dots, \operatorname{grad} u_n] \,/ \circ\phi_t$$

Prop 7.3.5

$$Du^\tau\circ\phi_t = [\operatorname{grad} u_1\circ\phi_t, \dots, \operatorname{grad} u_n\circ\phi_t]$$

$$[\nabla\phi_t^{-\tau}\operatorname{grad}(u_1\circ\phi_t), \dots, \nabla\phi_t^{-\tau}\operatorname{grad}(u_n\circ\phi_t)] = \nabla\phi_t^{-\tau} D(u\circ\phi_t)^\tau$$

$$\Rightarrow e(u)\circ\phi_t = S(D(u\circ\phi_t)\nabla\phi_t^{-1})$$

Time dobivamo da je

$$\tilde{G}(t,u,p) = G(t, u\circ\phi_t^{-1}, p\circ\phi_t^{-1})$$

$$= \int_\Omega f\circ\phi_t\cdot(u-p)J_t + \int_{\Gamma_N} g\circ\phi_t\cdot(u-p)J_t|\nabla\phi_t^{-\tau}n|$$

$$+ \int_\Omega 2\mu\, S(D(u)\nabla\phi_t^{-1}) : S(D(p)\nabla\phi_t^{-1})\, J_t$$

$$+ \int_\Omega \lambda\, tr(D(u)\nabla\phi_t^{-1})\, tr(D(p)\nabla\phi_t^{-1})\, J_t$$

Uočimo da $\quad d\tilde{G}(0,\bar{u},\bar{p}; 0,\varphi) = 0, \quad \forall\varphi\in H^1_{\Gamma_D}(\Omega)$

$$\Leftrightarrow \quad (S)\begin{cases} \bar{u}\in H^1_{\Gamma_D}(\Omega) \\ \int_\Omega Ae(\bar{u}) : e(\varphi) = \int_\Omega f\cdot\varphi + \int_{\Gamma_N} g\cdot\varphi \end{cases}$$

$$d\tilde{G}(0,\bar{u},\bar{p}; \varphi, 0) = 0, \quad \forall\varphi\in H^1 \in H^1_{\Gamma_D}(\Omega)$$

$$\Leftrightarrow \quad (A)\begin{cases} \bar{p}\in H^1_{\Gamma_D}(\Omega) \\ \int_\Omega Ae(\bar{p}) : e(\varphi) = -\int_\Omega f\cdot\varphi - \int_{\Gamma_N} g\cdot\varphi \end{cases}$$

Odnosno $\bar{p} = -\bar{u}$ (za $C(\Omega)$!)

Perturbirana zadaća $u^t\in H^1_\Gamma(\Omega)$ glasi $\left(d\tilde{G}(t,u^t,p;0,\varphi)=0\right)$

$$(PS)\begin{cases} u^t\in H^1_{\Gamma_D}(\Omega) \\ \int_\Omega \left[2\mu\, S(D(u^t)\nabla\phi_t^{-1}) : S(D(\varphi)\nabla\phi_t^{-1}) + \lambda\, tr(D(u)\nabla\phi_t^{-1})\, tr(D(\varphi)\nabla\phi_t^{-1})\right] J_t = \\ \int_\Omega f\circ\phi_t\cdot\varphi\, J_t + \int_{\Gamma_N} g\circ\phi_t\cdot\varphi\, J_t\, |\nabla\phi_t^{-\tau}n| \end{cases}$$

Teorem 8.1 (analiza osjetljivost $t \to u^t$)

Postoji $\tau > 0$ t.d.

$$\|u^t\|_{H^1_{\Gamma_D}} \le C \qquad \forall t \in [0,\tau]$$

Nadalje,

$$\|u^t - \bar{u}\|_{H^1_{\Gamma_D}} \xrightarrow{t \downarrow 0} 0 .$$

Dokaz:

$$a_t(u,\varphi) = \int_\Omega 2\mu\, S(D(u)\nabla\phi_t^{-1}) : S(D(\varphi)\nabla\phi_t^{-1}) + \lambda\, tr(D(u)\nabla\phi_t^{-1}) \cdot tr(D(\varphi)\nabla\phi_t^{-1})$$

$$b_t(\varphi) = \int_\Omega f\circ\phi_t \cdot \varphi J_t + \int_{\Gamma_N} g\circ\phi_t \cdot \varphi\, J_t\, |\nabla\phi_t^{-T} n|$$

Kornova nejednakost daje $u \in H^1_0(D;\mathbb{R}^d)$, $\Omega \subset\subset D$

$$c\|u\|^2_{H^1(D)} \le \int_\Omega 2\mu\, \varepsilon(u):\varepsilon(u) + \lambda\, div(u)\, div(u)$$

preciznije c ovisi samo o domeni D. netrivijalo za dokazati

Time postoji $c_{\tau,D}$ takav da $t \in [0,\tau]$, $u \in H^1_{\phi_t\Gamma_D}(\phi_t(\Omega))$

$$c_{\tau,D}\|u\|^2_{H^1(\phi_t(\Omega))} \le \int_{\phi_t(\Omega)} 2\mu\, \varepsilon(u):\varepsilon(u) + \lambda\, div(u)\, div(u)$$

$$\hookrightarrow \quad c_{\tau,D}\int_\Omega J_t\left(|Du\nabla\phi_t^{-1}|^2 + |u|^2\right) \le a_t(u,u), \quad \forall u \in H^1_{\Gamma_D}(\Omega)$$

Zbog neprekidnosti od $t \mapsto J_t$, $\nabla\phi_t^{-1}$ slijedi koercitivnost

Kako tretirati $t \mapsto |\nabla\phi_t^{-T} n|$ $\quad$ n vanjska jedinična normala

Lema 8.2

Neka je Ω otvoren, ograničen skup klase C^1. Preslikavanje

$$\theta \longmapsto \det(I+\nabla\theta)\,\|(I+\nabla\theta)^{-\tau} n\|$$

iz $W^{2,\infty}(\mathbb{R}^d;\mathbb{R}^d)$ u $C(\partial\Omega)$ je Fréchet diferencijal u 0 i vrijedi

$$D_\theta\Big(\det(I+\nabla\theta)\,\|(I+\nabla\theta)^{-\tau} n\|\Big)(0)[\psi] = \operatorname{div}_\Gamma \psi$$

$$\operatorname{div}_\Gamma \psi = \operatorname{div}\psi - \nabla\theta n\cdot n$$

Dokaz:

Označimo s $v(t) = (I + t\nabla\theta)^{-\tau} n$

$w(t) = \|v(t)\|$

$$w'(t) = \boxed{\frac{v(t)}{\|v(t)\|}}\, v'(t)$$

KAKO PREPOZNATI O ČEMU SE RADI?

Neka je dan Ω implicitno preko funkcije nivo skup

$$G^{-1}(\{0\}) = \partial\Omega, \qquad G^{-1}(\langle-\infty,0\rangle) = \Omega .$$

Ako je Ω klase C^1 tada postoji G klase C^1 sa traženim svojstvom. Kako definirati normalu n?

$$n(x) = \frac{\operatorname{grad} G(x)}{\|\operatorname{grad} G(x)\|} \quad \text{za } x\in\partial\Omega$$

(uočimo ovo daje i proširenje od normale na okolinu od $\partial\Omega$).

Perturbirani domen $\phi_t(\Omega)$ možemo opisati i kroz funkciju nivo skupa tj.

$$G\circ\phi_t^{-1}\;{}^{-1}(\{0\}) = \partial\phi_t(\Omega), \quad G\circ\phi_t^{-1}\;{}^{-1}(\langle-\infty,0\rangle) = \phi_t(\Omega).$$

Stavimo $D(G) = D(G\circ\phi_t^{-1}\circ\phi_t)$

$$= D(G\circ\phi_t^{-1})\circ\phi_t\,\nabla\phi_t \quad /^{\tau}$$

$$\operatorname{grad} G = \nabla\phi_t^{\tau}\operatorname{grad}(G\circ\phi_t^{-1})\circ\phi_t$$

$$n\,\|\mathrm{grad}\,G\| = \nabla\phi_t^{\tau}\,\mathrm{grad}(G\circ\phi_t^{-1})\circ\phi_t$$

$$\Rightarrow \quad \mathrm{grad}(G\circ\phi_t^{-1})\circ\phi_t = \nabla\phi_t^{-\tau}\,n\,\|\mathrm{grad}\,G\| \quad \Big/ \circ\phi_t^{-1}$$

$$n_t = \frac{\mathrm{grad}(G\circ\phi_t^{-1})}{\|\mathrm{grad}(G\circ\phi_t^{-1})\|} = \frac{(\nabla\phi_t^{-\tau}n)\circ\phi_t^{-1}}{\|(\nabla\phi_t^{-\tau}n)\circ\phi_t^{-1}\|} = \frac{v(t)}{\|v(t)\|}\circ\phi_t^{-1}$$

$$\Rightarrow \quad \frac{v(t)}{\|v(t)\|} = n_t\circ\phi_t$$ „perturbira normalu vraćamo na $\partial\Omega$"

$$\lim_{t\searrow 0}\frac{v(t)}{\|v(t)\|} = n$$

$$v(t) = (I + t\nabla\theta)^{-\tau}n$$

$$w'(0) = n\cdot v'(0) \qquad v'(0) = -\nabla\theta^{\tau}n$$

$$w'(0) = -\nabla\theta^{\tau}n\cdot n = -\nabla\theta\, n\cdot n$$

Time smo pokazali da je

$$D_\theta\left(\det(I+\nabla\theta)\|\nabla\phi_t^{-\tau}n\|\right)(\theta)[\psi] = \mathrm{div}\,\psi - \nabla\psi\, n\cdot n$$
$$= \mathrm{div}_\Gamma\,\psi$$

$\blacksquare$

Time smo pokazali da je $b_t(\psi)$ ograničen linearni operator

$$C\|u^t\|_{H^1(\Omega)}^{\cancel{2}} \le a_t(u^t,u^t) = b_t(u^t) \le M\cancel{\|u^t\|}_{H^1(\Omega)}$$

$u^t \xrightarrow{t\to 0} \bar{u}$ u $H^1(\Omega)$ na podnizu zbog ograničenosti.

$$a_t(u^t,\varphi) = b_t(\varphi)$$
$$a_0(\bar u,\varphi) = b_0(\varphi)$$

$$a_t(u^t,\varphi) - a_0(u^t,\varphi) + a_0(u^t,\varphi) = b_t(\varphi) - b_0(\varphi) + b_0(\varphi)$$
$$a_t(u^t,\varphi) - a_0(u^t,\varphi) + a_0(u^t-\bar u,\varphi) + \cancel{a_0(\bar u,\varphi)} = b_t(\varphi) - b_0(\varphi) + \cancel{b_0(\varphi)}$$

Stavimo $\varphi = u^t - \bar u$ i iskoristimo Kornovu nejednakost

$$c\|u^t-\bar u\|_{H^1} \le a_0(u^t-\bar u, u^t-\bar u) = -\big(a_t(u^t, u^t-\bar u) - a_0(u^t, u^t-\bar u)\big) + \big(b_t(u^t-\bar u) - b_0(u^t-\bar u)\big)$$

$$\lesssim \|a_t - a_0\|_{L^\infty}\|u^t\|_{L^2}\|u^t-\bar u\|_{L^2} + \|b_t - b_0\|_{L^2}\|u^t-\bar u\|_{L^2}$$

$\longrightarrow$ formalno, ali može se napraviti do zadnjih detalja.

Time $u^t \to \bar u$ u H^1 $\blacksquare$

Uvjet (H1) od Tm Correa-Seeger je zadovoljen.

(H2) je direktna provjera tj. $t \mapsto \tilde G(t,u,p)$ je F-diferencijabilna na $[0,\bar\tau]$

$$\partial_t \tilde G(t,u,p) = \int_\Omega A\,S(Du\nabla\phi_t^{-1}\nabla\theta\nabla\phi_t^{-1}) : S(Dp\nabla\phi_t^{-1})\,J_t$$
$$+ \int_\Omega A\,S(Du\nabla\phi_t^{-1}) : S(Dp\nabla\phi_t^{-1}\nabla\theta\nabla\phi_t^{-1})\,J_t$$
$$+ \int_\Omega A\,S(Du\nabla\phi_t^{-1}) : S(Dp\nabla\phi_t^{-1})\,J_t \operatorname{tr}(\nabla\phi_t^{-1}\nabla\theta)$$
$$+ \int_\Omega Df\circ\phi_t\,\theta\cdot(u-p)\,J_t + f\circ\phi_t\cdot(u-p)\,J_t \operatorname{tr}(\nabla\phi_t^{-1}\nabla\theta)$$

$$+ \int_{\Gamma_N} Dg \circ \phi_t \, \theta \cdot (u-p) J_t |\nabla \phi_t^{-\tau} n| + g \circ \phi_t \cdot (u-p) J_t \operatorname{tr}(\nabla \phi_t^{-1} \nabla \theta) |\nabla \phi_t^{-\tau} n|$$

$$+ \int_{\Gamma_N} g \circ \phi_t \cdot (u-p) J_{t_0} |- \nabla \phi_t^{-\tau} \nabla \theta^{\tau} \nabla \phi_t^{-\tau} n|$$

Uz Teorem 8.1. slijedi da smijemo primjeniti tm. C-S.

Konkretno

$$C'(\Omega; \theta) = \partial_t \tilde{G}(0, \bar{u}, -\bar{u})$$

$$= \int_\Omega A S(D\bar{u} \nabla \theta) : S(D\bar{u}) + A S(D\bar{u}) : S(D\bar{u} \nabla \theta)$$

$$+ 2 \int_\Omega Df \theta \cdot u + f \cdot u \operatorname{div} \theta$$

$$+ 2 \int_{\Gamma_N} Dg \theta \cdot u + g \cdot u \operatorname{div}_\Gamma (\theta) \quad //$$