Primjena teorema Correa-Seeger na zadaću (1.1) iz 7.4.1.

$$\begin{cases} -\Delta u + u = f & u\ \Omega \\ u = 0 & \text{na } \partial\Omega \end{cases} \qquad J(\Omega) = \frac{1}{2}\int_\Omega |u_\Omega - u_0|^2 dx$$

$$\tilde{G}(t,\varphi,\psi) = \frac{1}{2}\int_\Omega |\varphi - u_0 \circ \phi_t|^2 J_t \, dx$$

$$+ \int_\Omega A_t \nabla\varphi \cdot \nabla\psi + J_t(\varphi\psi - f\circ\phi_t \, \psi) dx$$

gdje je $J_t = \det \nabla\phi_t$, $\phi_t = Id + t\theta$, $\theta \in W^{1,\infty}$

$A_t = J_t \nabla\phi_t^{-1} \nabla\phi_t^{-T}$

$d\tilde{G}(t, u_t, p_t; 0, \psi) = 0$

$\Updownarrow$

(S) $$\begin{cases} u^t \in H^1(\Omega) \\ \int_\Omega A_t \nabla u^t \cdot \nabla\psi + J_t u^t \psi - f\circ\phi_t \, \psi \, dx = 0 \end{cases}$$

$d\tilde{G}(t, u_t, p_t; \varphi, 0) = 0$

$\Updownarrow$

(A) $$\begin{cases} p^t \in H^1(\Omega) \\ \int_\Omega A_t \nabla p^t \cdot \nabla\varphi + J_t p^t \varphi + (u^t - u_0 \circ\phi_t)\varphi \, dx = 0 \end{cases}$$

$(u^t, p^t) \in H^1(\Omega) \times H^1(\Omega)$ je sedlasta točka za

$(u,p) \mapsto \tilde{G}(t,u,p)$.

$$(u^t, p^t) \in S(t) \Rightarrow \begin{cases} u^t \in X(t) \\ p^t \in X(t) \end{cases}$$

$\tilde{G} : [0,\tau] \times H^1(\Omega) \times H^1(\Omega)$, $X = Y = H^1(\Omega)$

Lema prije tm. C-S

$X(t) = \{u^t\}$, $Y(t) = \{p^t\}$, $\forall t \in [0,\tau]$ jer

zbog neprekidnosti $t \mapsto A_t$ oko nule slijedi da je

$$\alpha|\xi|^2 \leq A_t \xi \cdot \xi \leq \beta|\xi|^2, \quad \alpha > 0, \ \beta > 1.$$

Time je pretpostavka (H1) zadovoljena.

(H2) je zadovoljen jer je

$$\widetilde{G}(t,u,p) : [0,\tau] \times H^1(\Omega) \times H^1(\Omega)$$

$t \mapsto \widetilde{G}(t,u,p)$ Fréchet diferencijabilna na okolini nule.

$$\phi_{t+s} = Id + (t+s)\theta$$

$$\widetilde{G}(t+s,u,p) = \frac{1}{2}\int_\Omega |u - u_0 \circ \phi_{t+s}| \, J_{t+s} \, dx$$

$$+ \int_\Omega A_{t+s} \nabla u \cdot \nabla p + J_{t+s}\, u p - f \circ \phi_{t+s} \, J_{t+s}\, p$$

Propozicija 7.3.3. (tehnički rezultat o diferencijabilnosti za $\theta \neq 0$)

$f \in W^{1,p}(\mathbb{R}^d)$. Tada je

a) $\theta \mapsto f \circ (Id + \theta)$ Fréchet diferencijabilno na okolini nule od $W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ u $L^p(\mathbb{R}^d)$ i vrijedi

$$D_\theta(f \circ (Id+\theta))(\bar{\theta})[\psi] = \nabla f \circ (Id + \bar{\theta}) \cdot \psi$$

b) $\theta \mapsto \det(I + \nabla\theta)$ Fréchet diferencij. na okolini nule od $W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ u $L^\infty(\mathbb{R}^d)$ i vrijedi

$$D_\theta(\det(I+\nabla\theta))(\bar{\theta})[\psi] = \det(I+\nabla\bar{\theta})\, tr\left((I+\nabla\bar{\theta})^{-1}\nabla\psi\right)$$

c) $\theta \mapsto (I + \nabla\theta)^{-1}$ Fréchet diferencijabilno na okolini nule od $W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ u $L^\infty(\mathbb{R}^d)$ i vrijedi

$$D_\theta((I+\nabla\theta)^{-1})(\bar{\theta})[\psi] = -(I+\nabla\bar{\theta})^{-1}\nabla\psi\,(I+\nabla\bar{\theta})^{-1}$$

Dokaz:

a)

$$f\circ(\mathrm{Id}+\theta+\psi) = f\circ\left(\mathrm{Id}+\psi\circ(\mathrm{Id}+\theta)^{-1}\right)\circ(\mathrm{Id}+\theta)$$

po oduzimanju $f\circ(\mathrm{Id}+\theta)$ dobivamo

$$f\circ(\mathrm{Id}+\theta+\psi) - f\circ(\mathrm{Id}+\theta) = \left\{ f\circ\left(\mathrm{Id}+\psi\circ(\mathrm{Id}+\theta)^{-1}\right) - f\right\}\circ(\mathrm{Id}+\theta)$$

Prop. 7.3.5

$\psi\circ(\mathrm{Id}+\theta)^{-1}\in W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)$

$$= \left\{\nabla f\cdot\psi\circ(\mathrm{Id}+\theta)^{-1} + o(\psi)\right\}\circ(\mathrm{Id}+\theta)$$

$$= \nabla f\circ(\mathrm{Id}+\theta)\cdot\psi + o(\psi)$$

b) $$\det(I+\nabla\theta+\nabla\psi) = \det\left((I+\nabla\theta)(I+(I+\nabla\theta)^{-1}\nabla\psi)\right)$$

$$= \det(I+\nabla\theta)\det(I+(I+\nabla\theta)^{-1}\nabla\psi)$$

Prop. 7.3.1. b)

$$= \det(I+\nabla\theta)\left(1+\mathrm{tr}((I+\nabla\theta)^{-1}\nabla\psi)+o(\psi)\right)$$

$$= \det(I+\nabla\theta)+\det(I+\nabla\theta)\,\mathrm{tr}((I+\nabla\theta)^{-1}\nabla\psi)+o(\psi)$$

c)

$$(I+\nabla\theta+\nabla\psi)^{-1} = \left((I+\nabla\theta)(I+(I+\nabla\theta)^{-1}\nabla\psi)\right)^{-1}$$

$$= (I+(I+\nabla\theta)^{-1}\nabla\psi)^{-1}(I+\nabla\theta)^{-1}$$

Prop. 7.3.2. b)

$$= \left(I-(I+\nabla\theta)^{-1}\nabla\psi+o(\psi)\right)(I+\nabla\theta)^{-1}$$

$$= (I+\nabla\theta)^{-1} - (I+\nabla\theta)^{-1}\nabla\psi(I+\nabla\theta)^{-1}+o(\psi)$$

Koristeći Prop. 7.3.3. možemo zaključiti da je $t\mapsto\tilde{G}(t,u,p)$ F-diferencijabilno na $[0,\tau]$ te je

$$\partial_t\tilde{G}(t,u,p) = \int_\Omega (u-u_0\circ\phi_t)^2 J_t\,\mathrm{tr}(\nabla\phi_t^{-1}\nabla\theta)\,dx$$

$$+\int_\Omega -(u-u_0\circ\phi_t)\nabla u_0\circ\phi_t\cdot\theta\, J_t + J_t\,\mathrm{tr}(\nabla\phi_t^{-1}\nabla\theta)\nabla u\cdot\nabla p\,dx$$

$$+\int_\Omega J_t\,\mathrm{tr}(\nabla\phi_t^{-1}\nabla\theta)\nabla\phi_t^{-1}\nabla\phi_t^{-T}\nabla u\cdot\nabla p\,dx$$

$$+\int_\Omega J_t\left(-\nabla\phi_t^{-1}\nabla\psi\,\nabla\phi_t^{-1}\right)\nabla\phi_t^{-T}\nabla u\cdot\nabla p\,dx$$

$$+\int_\Omega J_t \nabla\phi_t^{-1}\left(-\nabla\phi_t^{-\tau}\nabla\Psi^{\tau}\nabla\phi_t^{-\tau}\right)\nabla u\cdot\nabla p\,dx$$

$$+\int_\Omega -\nabla f\circ\phi_t\cdot\Theta\, p\,dx \quad //$$

(H3 - H4) pretpostavke

(S) povlači da u^t zadovoljava

$$\int_\Omega A_t\nabla u^t\cdot\nabla u^t + J_t u^t u^t = \int_\Omega f\circ\phi_t u^t$$

$$\Rightarrow \quad \alpha\|u^t\|^2_{H^1(\Omega)} \le \|f\|_{L^2}\|u^t\|_{L^2} \Rightarrow \alpha\|u^t\|_{H^1}\le\|f\|_{L^2}$$

(A) povlači da p^t zadovoljava

$$\alpha\|p^t\|_{H^1(\Omega)} \le \|u^t - u_0\circ\phi_t\|_{L^2}$$

Jer je u^t ograničen za $t\in[0,\tau]$.

Prema Banach-Alaoglu znamo da postoje podniz t_{n_k} t.d. $u^{t_{n_k}}\rightharpoonup\bar u$, $p^{t_{n_k}}\rightharpoonup\bar p$. Zaključuje da (puštajući $t_{n_k}\to 0$) u (S) i (A):

$$\int_\Omega \nabla\bar u\cdot\nabla\Psi + \bar u\Psi = \int_\Omega f\Psi$$

$$\int_\Omega \nabla\bar p\cdot\nabla\Psi + \bar p\Psi = -\int_\Omega(\bar u - u_0)\Psi$$

(5) $$\int_\Omega A_t \nabla u^t \cdot \nabla \psi + J_t u^t \psi = \int_\Omega f \circ \phi_t J_t \psi$$

$$\left.\begin{aligned} \Rightarrow \ & \int_\Omega (A_t - I)\nabla u^t \cdot \nabla \psi + \nabla u^t \cdot \nabla \psi + (J_t - 1) u^t \psi + u^t \psi \\ & = \int_\Omega (f \circ \phi_t J_t - f)\psi + \int_\Omega f \psi \\ & \int_\Omega \nabla \bar{u} \cdot \nabla \psi + u \psi = \int_\Omega f \psi \end{aligned}\right\} -$$

$$\Rightarrow \int_\Omega (A_t - I)\nabla u^t \cdot \nabla \psi + \nabla(u^t - \bar{u}) \cdot \nabla \psi + (J_t - 1) u^t \psi + (u^t - u)\psi$$

$$= \int_\Omega (f \circ \phi_t J_t - f)(u^t - \bar{u})$$

Stavimo $\psi = u^t - \bar{u}$.

$$\Rightarrow \quad \| u^t - \bar{u} \|_{H^1(\Omega)} = \int_\Omega (I - A_t)\nabla u^t \cdot \nabla(u^t - \bar{u}) + (1 - J_t) u^t \psi$$
$$+ (f \circ \phi_t J_t - f)(u^t - \bar{u})\, dx$$

$$\Rightarrow \quad \| u^t - \bar{u} \|_{H^1(\Omega)} \leq \| A_t - I \|_{L^\infty} \| \nabla u^t \|_{L^2} \| \nabla(u^t - \bar{u}) \|_{L^2}$$
$$+ \| f \circ \phi_t J_t - f \|_{L^2} \| u^t - \bar{u} \|_{L^2}$$

Jer je u^t ograničen slijedi

$$\| u^t - \bar{u} \|_{H^1(\Omega)} \leq c \left(\| A_t - I \|_{L^\infty} + \| f \circ \phi_t J_t - f \|_{L^2} \right) \xrightarrow{t \to 0} 0$$

$\Rightarrow \quad u^t \to \bar{u}$ u $H^1(\Omega)$.

$$(A)\quad \int_\Omega A_t \nabla p^t \cdot \nabla \varphi + J_t p^t \varphi + (u^t - u_0 \circ \phi_t)\varphi J_t = 0$$

$$\Leftrightarrow \quad \left.\begin{aligned} &\int_\Omega (A_t - I)\nabla p^t \cdot \nabla \varphi + \nabla p^t \cdot \nabla \varphi + (J_t - 1) p^t \varphi + p^t \varphi \\ &\quad + (u^t J_t - \bar{u})\varphi + \bar{u}\varphi - (u_0 \circ \phi_t J_t - u_0)\varphi - u_0 \varphi = 0 \\ &\int_\Omega \nabla \bar{p} \cdot \nabla \varphi + p\varphi + (\bar{u} - u_0)\varphi = 0 \end{aligned}\right\} -$$

$$\Rightarrow \int_\Omega (A_t - I)\nabla p^t \cdot \nabla \varphi + \nabla (p^t - \bar{p}) \cdot \nabla \varphi + (J_t - 1) p^t \varphi$$

$$+ (p^t - \bar{p})\varphi + (u^t J_t - \bar{u})\varphi - (u_0 \circ \phi_t J_t - u_0)\varphi = 0$$

Stavimo $\varphi = p^t - \bar{p}$.

$$\| p^t - \bar{p} \|_{H^1}^2 = \int_\Omega (I - A_t)\nabla p^t \cdot \nabla (p^t - \bar{p}) + (1 - J_t) p^t (p^t - \bar{p})$$

$$+ (u^t J_t - \bar{u})(p^t - \bar{p}) - (u_0 \circ \phi_t J_t - u_0)(p^t - p)\, dx$$

$$\leq \left(\| A_t - I \|_{L^\infty} + \| J_t - 1 \|_{L^\infty} \right) \| \nabla p^t \|_{L^2} \| \nabla (p^t - \bar{p}) \|_{L^2}$$

$$+ \| u^t J_t - \bar{u} \|_{L^2} \| p^t - \bar{p} \|_{L^2} + \| u_0 \circ \phi_t J_t - u_0 \|_{L^2} \| p^t - \bar{p} \|_{L^2}$$

Analogno zaključujemo da $p^t \to \bar{p}$ u $H^1(\Omega)$.

Zbog jake konvergencije i činjenice da je $\partial_t G(t, u, p)$ neprekid za (u, p) vrijede (H3-H4) ii) pretpostavke.

Prema tm. C-S slijedi da je

$$J'(\Omega,\theta) = \partial_t \tilde{G}(0,\bar{u},\bar{p})$$

$$= \int_\Omega \frac{1}{2}(\bar{u}-u_0)^2 \operatorname{div}\theta - (\bar{u}-u_0)\nabla u_0\cdot\theta$$

$$+ \int_\Omega (\operatorname{div}\theta I - \nabla\theta - \nabla\theta^T)\nabla\bar{u}\cdot\nabla\bar{p} + \bar{u}\bar{p}\operatorname{div}\theta$$

$$+ \int_\Omega -\operatorname{div}(f\theta)\bar{p}\, dx$$

gdje je $\bar{u}\in H^1(\Omega)$ rješenje od

$$\int_\Omega \nabla\bar{u}\cdot\nabla\varphi + \bar{u}\varphi = \int_\Omega f\varphi, \quad \forall\varphi\in H^1(\Omega),$$

a $\bar{p}\in H^1(\Omega)$ rješenje od

$$\int_\Omega \nabla\bar{p}\cdot\nabla\varphi + u\varphi = -\int_\Omega (\bar{u}-u_0)\varphi, \quad \forall\varphi\in H^1(\Omega).$$

Napomena. U praksi dovoljno je da $p^t \rightharpoonup \bar{p}$ u $H^1(\Omega)$, dakle tvrdnja se može pokazati i tada.