Zadaci za vježbu

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Zadatak 1. *Gaussovom metodom eliminacije riješite sustav linearnih jednadžbi:*

$$\begin{cases} 2x & +y & +z & +w & =2 \\ x & +2y & +z & +w & =1 \\ x & +2y & +2z & +w & =1 \\ & -2y & -2z & -w & =0 \end{cases}.$$

Rj: Zapisujemo matricu sustava i odaberemo pivotni element s kojim poništavamo nenul elemente u tom stupcu:

$$\left(\begin{array}{cccc|c} 2 & 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 & 1 \\ 0 & -2 & -2 & \color{red}{-1} & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 2 & -1 & -1 & 0 & 2 \\ 1 & 0 & -1 & 0 & 1 \\ \color{red}{1} & 0 & 0 & 0 & 1 \\ 0 & -2 & -2 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & \color{red}{-1} & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & -2 & -2 & -1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 0 & \color{red}{-1} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right)$$

Slijedi $x = 1$, $y = 0$, $z = 0$, $w = 0$ je jedinstveno rješenje sustava.

Zadatak 2. *Pronađite inverz matrice (ako postoji):*

(a) $\begin{pmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{pmatrix}$ ***Rj:*** $\begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2 \end{pmatrix}$ ***Rj:*** $\begin{pmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}$ ***Rj:*** $\frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}$

Zadatak 3. *Zadane su matrice A, B, C:*

$$A = \begin{pmatrix} 3 & 1 & -1 \\ 2 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 2 & 4 \\ 1 & 4 & 2 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & -1 & 2 \\ 0 & 1 & -3 \\ 4 & 0 & -3 \\ 1 & -2 & -1 \end{pmatrix}.$$

Izračunajte $5A^{-1} + r(C)I_3 + BC$.

Zadatak 4. *Izračunajte*

$$A^2 - (\mathrm{tr}A)A + \det AI_2$$

ako je $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $\mathrm{tr}[a_{i,j}] = \sum_i^n a_{i,i}$, za $[a_{i,j}] \in M_n(\mathbb{R})$. A^2 je pokrata za množenje matrice samom sa sobom tj, $A^2 = AA$. Koliku vrijednost dobivate ako matricu A zamijenite s A^{-1} ili A^t?

Rj: Stavimo da je $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Tada je

$$\begin{aligned} A^2 - (\mathrm{tr}A)A + \det AI_2 &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} - (a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} + (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} + \begin{pmatrix} -a^2-ad & -ab-bd \\ -ac-cd & -ad-d^2 \end{pmatrix} + \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

Dani izraz poništava opće matrice $A \in M_2(\mathbb{R})$. Ako je potrebno inverz matrice reda 2 možete dobiti preko formule:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Zadatak 5. *Odredite rang matrice i determinantu*

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 3 & 3 & 3 & 0 \\ 4 & 4 & 4 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Rj: A je produkt dviju matrica B, C

$$A = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 3 & 3 & 3 & 0 \\ 4 & 4 & 4 & 4 \end{pmatrix}}_{B}\underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{C}$$

Znamo da je $\det(A) = \det(BC) = \det(B)\det C$. S obzirom da se radi o gornjetrokutasim i donjetrokutastim matricama, znamo da je determinanta produkt glavne dijagonale, tj. $\det(B) = 1 \cdot 2 \cdot 3 \cdot 4 = 24, \det(C) = 1$, čime je $\det(A) = 24$. Jer je $\det(A) \neq 0, A \in M_4(\mathbb{R})$ znamo da je $r(A) = 4$.

Zadatak 6. *Pronađite inverze matrica:*

(a) $A = \begin{pmatrix} 8 & 6 & 5 & 2 \\ 3 & 3 & 2 & 1 \\ 4 & 2 & 3 & 1 \\ 7 & 4 & 5 & 2 \end{pmatrix}$ ***Rj:*** $A^{-1} = \begin{pmatrix} 1 & -2 & -2 & 1 \\ 0 & 1 & 1 & -1 \\ -1 & 2 & 4 & -2 \\ -1 & 0 & -5 & 4 \end{pmatrix}$

(b) $B = \begin{pmatrix} -3 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & 2 & 4 \end{pmatrix}$ ***Rj:*** $B^{-1} = \begin{pmatrix} -2 & -6 & 1 \\ -3 & -11 & 2 \\ 2 & 7 & -1 \end{pmatrix}$

(c) $C = \begin{pmatrix} 8 & 3 & 5 \\ 3 & 3 & 2 \\ 4 & 2 & 3 \end{pmatrix}$ ***Rj:*** $C^{-1} = \frac{1}{7}\begin{pmatrix} 5 & 1 & -9 \\ -1 & 4 & -1 \\ -6 & -4 & 15 \end{pmatrix}$