$$f'(x) = \sum_{n=0}^{+\infty} (-1)^n x^{2n} \Big/ \int dx$$

$$\Rightarrow \quad f(x) = c + \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{za} \quad |x| < 1$$

$$f(0) = tg^{-1}(0) = c \quad \Rightarrow \quad c = 0$$

pa dobivamo da je $\displaystyle f(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ za $|x| < 1$

ovo je radijus konvergencije

e) $f(x) = x^3 + 2x^2 - 5$ oko točke $c = 1$.

$$f(x) = ((x-1)+1)^3 + 2((x-1)+1)^2 - 5$$

$$= (x-1)^3 + 3(x-1)^2 + 3(x-1) + 1 + 2(x-1)^2 + 4(x-1) + 2 - 5$$

$$= (x-1)^3 + 5(x-1)^2 + 7(x-1) - 2,$$ razvoj vrijedi $\forall x \in \mathbb{R}$ $\Leftrightarrow$ $r = +\infty$ je radijus konvergencije.

Teorem Neka je $n \in \mathbb{N}$, $I \subseteq \mathbb{R}$ otvoren interval, $0 \in I$, $f \in C^{n+1}(I)$.
Tada $\forall x \in I$ postoji $c_x \in \langle -|x|, |x| \rangle$ t.d.

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k}_{T_n(x)} + \underbrace{\frac{f^{(n+1)}(c_x)}{(n+1)!} x^{n+1}}_{R_n(x)}$$

Zad Koristeći Taylorovu aproksimaciju izračunajte zadane vrijednosti funkcije f do na grešku 10^{-3}. Koristeći kalkulator izračunajte razliku između dobivene aproksimacije i stvarne vrijednosti:

a) $f(x) = \cos x$, $\cos(0.1)$

b) $f(x) = ch\, x$, $ch(0.3)$

c) $f(x) = \ln(1+x)$, $\ln(1.2)$

d) $f(x) = \operatorname{arctg} x$, $\operatorname{arctg} 0.1$

Rj: a)
$$\left.\begin{array}{l} f(x) = \cos x \\ f'(x) = -\sin x \\ f''(x) = -\cos x \\ f'''(x) = \sin x \end{array}\right\} \Rightarrow \left.\begin{array}{l} f(0) = 1 \\ f'(0) = 0 \\ f''(0) = -1 \\ f'''(0) = 0 \end{array}\right\} \Rightarrow T(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots$$
$f^{(4)}(x) = f(x)$

! Želimo da je apsolutna vrijednost razlike između Taylorovog polinoma i f-je f manja od $\varepsilon = 10^{-3}$ i to na segmentu $[-0.1, 0.1]$.

Tražimo vrijednost $\cos(0.1)$

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + \frac{f^{(n+1)}(c_x)}{(n+1)!} x^{n+1}$$

Stavimo $n = 3$

$$f(x) = 1 - \frac{x^2}{2} + R_3(x)$$

$|\cos(c_x)| \leq 1$

$$\Rightarrow \left| f(x) - \left(1 - \frac{x^2}{2}\right) \right| = |R_3(x)| = \frac{|f^{(4)}(c_x)|}{4!} |x|^4$$

ako je $|x| < 0.1$

$$\leq \frac{1}{4!} |x|^4 \leq \frac{0.1^4}{4!} = 4.17 \cdot 10^{-4} < \varepsilon$$

ocjena greške

$$f(0.1) = \cos 0.1 \approx 0.995004$$

$$T_3(x) = T_2(x) = 1 - \frac{x^2}{2}$$

$$T_3(0.1) = 1 - \frac{0.1^2}{2} = 0.995$$

$$|f(0.1) - T_3(0.1)| \approx 4 \cdot 10^{-6} < 4.17 \cdot 10^{-3} < 10^{-3}$$

Istaknimo da je gornja ocjena uniformna i vrijedi za $x \in [-0.1, 0.1]$.
Općenito pokazali smo da je $|R_3(x)| \leq \frac{1}{24} |x|^4$

tj.

$$\left| f(x) - \left(1 - \frac{x^2}{2}\right) \right| \leq \frac{1}{...} |x|^4$$

tj.

$$\left| f(x) - \left(1 - \frac{x^2}{2}\right) \right| \le \frac{1}{24}|x|^4$$

Nap Korektno je koristiti: $\left| f(x) - \big(\underbrace{1 - \tfrac{x^2}{2}}_{T_2(x)}\big) \right| = |R_2(x)| \le \frac{1}{6}|x|^3$.

b) $f(x) = \operatorname{ch} x$, vrijednost $\operatorname{ch}(0.3)$ na točnost od barem 10^{-3}.

$f'(x) = \operatorname{sh} x$

$f''(x) = f(x) = \operatorname{ch} x$

$\Rightarrow T(x) = 1 + \frac{1}{2}x^2 + \frac{1}{4}x^4 + \dots$

$$\Big| \operatorname{ch} x - \big(\underbrace{1 - \tfrac{1}{2}x^2}_{T_2(x)}\big) \Big| = |R_2(x)| = \left| \frac{\overbrace{f^{(3)}(c_x)}^{=\operatorname{sh}(c_x)}}{3!} x^3 \right|$$

$\operatorname{sh}(c_x)$ gdje je $c_x \in \langle -0.3, 0.3 \rangle$ za $|x| < 0.3$

$$\le \frac{|\operatorname{sh}(c_x)|}{3!}|x|^3 \quad \text{za } |c_x| < 0.3$$

$$\le \frac{0.3}{3!}|x|^3 \Rightarrow |R_2(x)| < \frac{0.3}{3!}|x|^3$$

$|\operatorname{sh}(c_x)| < \operatorname{sh}(0.3) \approx 0.3$

Uočimo da je $|R_2(0.3)| < \frac{0.3}{3!}0.3^3 \approx 0.00135 \not< \varepsilon = 10^{-3}$

Potrebna je bolja ocjena!

$$\Big| \operatorname{ch} x - \big(\underbrace{1 - \tfrac{1}{2}x^2}_{T_3(x) = T_2(x)}\big) \Big| = |R_3(x)| = \left| \frac{\overbrace{f^{(4)}(c_x)}^{=\operatorname{ch}(c_x)}}{4!} x^4 \right| \le \frac{5}{3\,4!}|x|^4$$

$$|\operatorname{ch}(c_x)| < \operatorname{ch}(0.3) \le \operatorname{ch}(\ln 3) = \frac{e^{\ln 3} + e^{-\ln 3}}{2} = \frac{3 + \frac{1}{3}}{2} = \frac{10}{6} = \frac{5}{3}$$

jer je f-ja parna i rastuća na $\langle 0, +\infty \rangle$.

Tehnički: $\operatorname{ch}(0.3)$ možemo izraziti u kalkulatoru čime je zadatak odmah gotov, ali ako tražimo aproksimaciju od $\operatorname{ch}(0.3)$ nema smisla koristiti „točnu" vrijednost u

... na $\langle 0, +\infty \rangle$.

... aproksimacija od $ch(0.3)$ nema smisla koristiti „točnu" vrijednost u ocjeni greške

$$|R_3(0.3)| < \frac{5}{3 \cdot 4!} 0.3^4 \approx 0.0006 < \varepsilon$$

$$T_3(0.3) = 1 - \frac{1}{2} 0.3^2 = 0.955$$

čime je vrijednost od $ch(0.3)$ u segmentu $[0.955 - 6 \cdot 10^{-4}, 0.955 + 6 \cdot 10^{-4}]$

Prava vrijednost je $\cos(0.3) \approxeq 0.9553364$

čime je stvarna greška $|\cos(0.3) - T_3(0.3)| \approx 0.0003 < |R_3(0.3)| < \varepsilon$.

c) $f(x) = \ln(1+x)$, $\quad \ln(1.2) = ?$

$$f'(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{+\infty} (-x)^n \Big/ \int dx \qquad \text{za } |x| < 1$$

$$\Rightarrow f(x) = c + \sum_{n=0}^{+\infty} (-1)^n \frac{x^{n+1}}{n+1} \Big/ \quad \text{za } x = 0$$

$$f(0) = \ln(1) = 0 = c \quad \Rightarrow \quad f(x) = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{x^n}{n}$$

$$= \underbrace{x - \frac{x^2}{2} + \frac{x^3}{3}}_{T_3(x)} - \frac{x^4}{4} + \dots$$

$$f'(x) = \frac{1}{1+x}$$

$$f''(x) = \frac{-1}{(1+x)^2}$$

$$f'''(x) = \frac{2}{(1+x)^3}$$

$$\boxed{f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}}$$

Time je

$$|f(x) - T_n(x)| = |R_n(x)| = \left| \frac{\frac{\cancel{n!}}{(1+cx)^{n+1}}}{\cancel{(n+1)!}_{(n+1)}} \cdot x^{n+1} \right|$$

$$= \left| \frac{1}{(1+cx)^{n+1}(n+1)} x^{n+1} \right|$$

Preslikavanje $cx \longmapsto \frac{1}{(1+cx)^{n+1}}$ je strogo padajuće na $[-0.2, 0.2]$ i nenegativno čime je

$$\left| \frac{1}{(1+cx)^{n+1}} \right| \leq \frac{1}{0.2^{n+1}}$$

$$\left|\frac{1}{(1+c_x)^{n+1}}\right| \leq \frac{1}{0.8^{n+1}}$$

$$|R_3(x)| \leq \frac{1}{0.8^4 \cdot 4} 0.2^4 \approx 0.00098 < 10^{-3} = \varepsilon$$

čime je $T_3(0.2) = 0.2 - \frac{0.2^2}{2} + \frac{0.2^3}{3} \approx 0.18267$

dobra aproksimacija od $\ln(1.2)$.

Zaista $|\ln(1.2) - T_3(0.2)| \approx 0.0003 < 0.00098 < 10^{-3} = \varepsilon$.

d) $f(x) = \operatorname{arctg} x$, $\operatorname{arctg} 0.1$

$$f'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{+\infty} (-x^2)^n = \sum_{n=0}^{+\infty} (-1)^n x^{2n} \Big/ \int dx$$

$$\Rightarrow \quad f(x) = c + \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$\Rightarrow$ $c = 0$ jer je $f(0) = \operatorname{arctg} 0 = 0$.

$$f(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

$$f''(x) = \frac{-2x}{(1+x^2)^2}, \quad f'''(x) = \frac{-2(1+x^2)^2 + 8x^2(1+x^2)}{(1+x^2)^4} = -\frac{2}{(1+x^2)^4} + \frac{8x^2}{(1+x^2)^3}$$

$$|f'''(c_x)| \leq 2 + 8c_x^2 \leq 2 + 8 \cdot 0.1^2 = 2.08$$

$$\Rightarrow \quad |f(x) - T_2(x)| \leq |R_2(x)| = \left|\frac{f'''(c_x)}{3!} x^3\right| \leq \frac{2.08}{3!} |x|^3$$

procjena greške

$$|\operatorname{arctg} 0.1 - 0.1| \leq \frac{0.1^3}{3!} < 3.47 \cdot 10^{-4} < \varepsilon$$

$\Rightarrow$ Zaista $\operatorname{arctg} 0.1 \approx 0.09967$ $\Rightarrow$ $|\operatorname{arctg} 0.1 - 0.1| \approx 3.3 \cdot 10^{-4}$

$\Rightarrow$ zaista $\text{arctg}\, 0.1 \approx 0.09967$ $\Rightarrow$ $|\text{arctg}\, 0.1 - 0.1| \approx 3{,}3 \cdot 10^{-4}$

prava greška,