Red potencija

Neka je dan niz funkcija $(f_n)_n$ takve da
$f_n : I \subseteq \mathbb{C} \to \mathbb{C}$
Red $\sum_{n=0}^{\infty} f_n$ zovemo redom funkcija

Ako je $f_n(x) = a_n(x-c)^n$, $c \in I$ red

$$\sum_{n=0}^{\infty} a_n(x-c)^n$$

nazivamo redom potencija oko točke c.

Definiramo skup

$$K = \{ z \in \mathbb{C} : \text{red } \sum_{n=0}^{\infty} a_n(x-c)^n \text{ konvergira u točki } z \} \subseteq \mathbb{C}$$

Tada je radijus konvergencije reda $\sum_{n=0}^{\infty} a_n(x-c)^n$

$$r := \sup \{ |z-c| : z \in K \}$$

Ako je $|z-c| < r$ tada red konvergira apsolutno, dok ako je $|z-c| > r$ red potencija divergira.

Teorem Neka je $\sum_{n=0}^{\infty} a_n x^n$ red potencija, $\varrho = \limsup_n \sqrt[n]{|a_n|}$.
Tada je radijus konvergencije $r = 1/\varrho$ $\left(\text{uz } \frac{1}{0} = \infty, \frac{1}{\infty} = 0\right)$
Dodatno, ako postoji $\varrho_1 = \lim_n \sqrt[n]{|a_n|}$, tada je $r = \frac{1}{\varrho_1}$
Dodatno, ako postoji $\varrho_2 = \lim_n \left| \frac{a_{n+1}}{a_n} \right|$, tada je $r = \frac{1}{\varrho_2}$.

Zad Izračunajte radijus konvergencije redova:

a) $\sum_{n=0}^{\infty} x^n$

Rj: $a_n = 1$.

$$\lim_n \sqrt[n]{|a_n|} = 1 \Rightarrow \varrho = \limsup_n \sqrt[n]{|a_n|} = 1$$

$r = \frac{1}{\rho} = 1$ je radijus konvergencije reda.

b) $\sum_{n=1}^{\infty} n^n x^n \Rightarrow a_n = n^n$

$$\rho = \lim_n \sqrt[n]{n^n} = \lim_n n = +\infty$$

$\Rightarrow r = \frac{1}{\rho} = 0$ je radijus konverja reda.

c) $\sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow a_n = \frac{1}{n!}$

$$\rho = \lim_n \left| \frac{a_{n+1}}{a_n} \right| = \lim_n \frac{\frac{1}{\cancel{(n+1)!}}^{(n+1)}}{\frac{1}{\cancel{n!}}} = \lim_n \frac{1}{1+n} = 0$$

$\Rightarrow r = \frac{1}{\rho} = +\infty$ je radijus konvergencije reda.

d) $\sum_{n=1}^{\infty} \frac{\ln n}{2^n} x^n \Rightarrow a_n = \frac{\ln n}{2^n}$

$$\rho = \lim_n \sqrt[n]{|a_n|} = \lim_n \left(\frac{\ln n}{2^n} \right)^{\frac{1}{n}} = \lim_n \frac{\sqrt[n]{\ln n}}{2} = \frac{1}{2}$$

$n \to \infty$ $\nearrow$ 1

Znamo da je $\lim_n \sqrt[n]{\ln n} = 1$.

Zaista, $1 \leq \ln n \leq n$, $\forall n \geq 3$

te postoje limesi $\lim_n \sqrt[n]{1} = \lim_n \sqrt[n]{n} = 1$

i jednaki su, slijedi da je $\lim_n \sqrt[n]{\ln n} = 1$

tzv. teorem o sendviču .

$r = \frac{1}{\rho} = 2$ je radijus konvergencije reda.

Alternativno rješenje

gledam za $n > 1$

$$\rho = \lim_n \left|\frac{a_{n+1}}{a_n}\right| = \lim_n \frac{\frac{\ln(n+1)}{\cancel{2^{n+1}}}\,{}^{2}}{\frac{\ln n}{\cancel{2^n}}} = \lim_n \frac{\ln(n+1)}{2\ln n}$$

$$\overset{L'H}{=} \lim_n \frac{\frac{1}{n+1}}{2\,\frac{1}{n}} = \frac{1}{2}\lim_n \frac{n}{n+1} = \frac{1}{2}\lim_n \frac{1}{1+\frac{1}{n}} = \frac{1}{2}$$

$r = \frac{1}{\rho} = 2$ je traženi radijus konvergencije.

e) $\displaystyle\sum_{n=1}^{\infty} \frac{(3n)!}{n^{3n}}\, x^n \Rightarrow a_n = \frac{(3n)!}{n^{3n}}$

$$\rho = \lim_n \left|\frac{a_{n+1}}{a_n}\right| = \lim_n \frac{\frac{(3n+3)!}{(n+1)^{3n+3}}}{\frac{(3n)!}{n^{3n}}} = \lim_n \frac{\frac{(3n+3)(3n+2)(3n+1)\cancel{(3n)!}}{(n+1)^{3n}\,(n+1)^3}}{\frac{\cancel{(3n)!}}{n^{3n}}}$$

$$= \lim_n \underbrace{\frac{3n+3}{n+1}}_{\downarrow\, 3} \cdot \underbrace{\frac{3n+2}{n+1}}_{\downarrow\, 3} \cdot \underbrace{\frac{3n+1}{n+1}}_{\downarrow\, 3} \cdot \Bigg(\frac{1}{\underbrace{\left(1+\frac{1}{n}\right)^n}_{\to\, e}}\Bigg)^3 = \frac{3^3}{e^3}$$

$r = \frac{1}{\rho} = \frac{e^3}{27}$ je radijus konvergencije reda //

f) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\, x^n \qquad a_n = \frac{(-1)^n}{n}$

$$\rho = \lim_n \sqrt[n]{|a_n|} = \lim_n \sqrt[n]{\frac{1}{n}} = \lim_n \frac{1}{\underbrace{\sqrt[n]{n}}_{\xrightarrow{n\to\infty} 1}} = 1$$

$r = \frac{1}{\rho} = 1$ je traženi radijus konvergencije.

g) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n^2)}{n^n}\, x^n \Rightarrow a_n = \frac{\cos(n^2)}{n^n}$

$$0 \le \frac{|\cos(n^2)|}{n^n} \le \frac{1}{n^n} \quad / \sqrt[n]{\cdot}$$

$$\Rightarrow \quad 0 \le \sqrt[n]{|a_n|} \le \frac{1}{n} \xrightarrow{n\to\infty} 0$$

po tm. o sendviču je $\lim\limits_n \sqrt[n]{|a_n|} = 0$

$r = \frac{1}{\rho} = +\infty$ je traženi radijus konvergencije.

h) $\sum\limits_{n=1}^{\infty} (1+(-1)^n)\, x^n \quad \Rightarrow \quad a_n = \begin{cases} 2, & n \text{ parno} \\ 0, & n \text{ neparno} \end{cases}$

$$\rho = \limsup_n \sqrt[n]{|a_n|}$$

$$0 \le a_n \le 2$$
$$\Rightarrow \quad |a_n| \le 2$$
$$\Rightarrow \quad \sqrt[n]{|a_n|} \le 2^{\frac{1}{n}}$$
$$\Rightarrow \quad \limsup_n \sqrt[n]{|a_n|} \le \liminf_n 2^{\frac{1}{n}} = \lim_n 2^{\frac{1}{n}} = 1$$

S druge strane $\lim\limits_n \sqrt[n]{|a_{2n}|} = \lim\limits_n \sqrt[n]{2} = 1$ pa je

$\limsup\limits_n \sqrt[n]{|a_n|} = 1.$

$r = \frac{1}{\rho} = 1$ je traženi radijus konvergencije

Napomena: Gornji dokaz je za općenitu situaciju.

$$\left.\begin{aligned} \lim_n \sqrt[n]{|a_{2n}|} &= 1 \\ \lim_n \sqrt[n]{|a_{2n-1}|} &= 0 \end{aligned}\right\} \Rightarrow \limsup_n \sqrt[n]{|a_n|} = \max\{0,1\} = 1$$

i) $\sum\limits_{n=1}^{\infty} (1+(-2)^n)\, x^n \qquad a_n = 1+(-2)^n$

$$\rho = \lim_n \left|\frac{a_{n+1}}{a_n}\right| = \lim_n \left|\frac{1+(-2)^{n+1}}{1+(-2)^n}\right| = \lim_n \left|\frac{\overset{\nearrow 0}{\frac{1}{(-2)^{n+1}}} + 1}{\underset{\downarrow}{\frac{1}{(-2)^{n+1}}} - \frac{1}{2}}\right|$$

$$= \left| \frac{1}{-\frac{1}{2}} \right| = 2$$

$r = \frac{1}{\rho} = \frac{1}{2}$ je traženi radijus konvergencije.

Def. Neka je funkcija $f \in C^\infty(\langle c-r, c+r \rangle)$, $r > 0$.
Red potencija $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$

zovemo Taylorov red funkcije f oko točke c.
Taylorovi polinomi $\displaystyle T_n(x) = \sum_{m=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$ su

parcijalne sume Taylorovog reda.

Zad Odredite Taylorov red oko nule i njegov radijus konvergencije za funkciju.

a) $f(x) = e^x$, $c = 0$

$f'(x) = e^x \;\ldots\; f^{(n)}(x) = e^x$

$$f^{(n)}(0) = 1, \ \forall n \Rightarrow T(x) = \sum_{n=0}^{\infty} \underset{a_n}{\boxed{\frac{1}{n!}}} x^n$$

$$\rho = \lim_n \left| \frac{a_{n+1}}{a_n} \right| = \lim_n \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_n \frac{1}{n+1} = 0$$

$\Rightarrow r = \frac{1}{\rho} = +\infty$ je radijus konvergencije.

b) $f(x) = \sin x$

$$f'(x) = \cos x$$
$$f''(x) = -\sin x$$
$$f'''(x) = -\cos x$$
$$f^{(4)}(x) = \sin x$$
$$\vdots$$

$$\left.\begin{array}{ll} f(0) = 0 & n=0 \\ f'(0) = 1 & n=1 \\ f''(0) = 0 & n=2 \\ f'''(0) = -1 & n=3 \\ f^{(4)}(0) = 0 & n=4 \\ \vdots & \end{array}\right\} \Rightarrow a_n = \begin{cases} 0, & n \text{ paran} \\ \frac{1}{n!}, & n \text{ daje ostatak } 1 \text{ pri dijeljenju s } 4 \\ -\frac{1}{n!}, & n \text{ daje ostatak } 3 \text{ pri dijeljenju s } 4 \end{cases}$$

$$\Rightarrow T(x) = \sum_{m=0}^{\infty} (-1)^m \frac{1}{(2m+1)!} x^{2m+1}$$

Pokazali smo u prethodnom zadatku da je $\limsup_n \sqrt[n]{\frac{1}{n!}} = 0$

Uočimo da je $0 \overset{\text{def.}}{\leq} \limsup_n \sqrt[n]{|a_n|} \leq \limsup_n \sqrt[n]{\frac{1}{n!}} = 0$

jer je $|a_n| \leq \frac{1}{n!}$, čime je $\rho = \limsup_n \sqrt[n]{|a_n|} = 0$, te

je radijus konvergencije $r = \frac{1}{\rho} = +\infty$, opet isti.

c) $f(x) = \frac{1}{1-3x}$

prvi način: prepoznamo da je $\frac{1}{1-q} = \lim_n \frac{1-q^n}{1-q} = \lim_n (1+q+\dots+q^{n-1}) = \sum_{n=0}^{+\infty} q^n$

za $q = 3x$ slijedi da je $\frac{1}{1-(3x)} = \sum_{n=0}^{+\infty} \underbrace{3^n}_{a_n} x^n =: T(x)$

$\rho = \lim_n \sqrt[n]{3^n} = 3 \Rightarrow r = \frac{1}{\rho} = \frac{1}{3}$ je radijus konverga reda.

drugi način:

$g(y) = \frac{1}{1-y}, \quad f(x) = g(3x)$

$$g'(y) = -\frac{1}{(1-y)^2}(-1) = \frac{1}{(1-y)^2}$$

$$g''(y) = (-2)\frac{1}{(1-y)^3}(-1) = \frac{2}{(1-y)^3}$$

$\vdots$

$$g^{(n)}(y) = \frac{n!}{(1-y)^{n+1}}$$

$g(0) = 1$

$g'(0) = 1$

$g''(0) = 2$

$\vdots$

$g^{(n)}(0) = n!$

$y = 3x$ ✓

$$\Rightarrow \quad T_g(y) = \sum_{n=0}^{+\infty} \frac{g^{(n)}(0)}{n!} y^n = \sum_{n=0}^{+\infty} \frac{n!}{n!} y^n = \sum_{n=0}^{+\infty} y^n$$

$$\Rightarrow \quad T_f(x) = \sum_{n=0}^{+\infty} (3x)^n = \sum_{n=0}^{+\infty} 3^n x^n$$

d) $f(x) = x^4 - 2x^3 + 7x^2 - x - 6$

$f'(x) = 4x^3 - 6x^2 + 14x - 1$

$f''(x) = 12x^2 - 12x + 14$

$f'''(x) = 24x - 12$

$f^{(4)}(x) = 24$

$\vdots$

$f^{(n)}(x) = 0$, za $n \geq 5$

$f(0) = -6$

$f'(0) = -1$

$f''(0) = 14$

$f'''(0) = -12$

$f^{(4)}(0) = 24$

$f^{(n)}(0) = 0$ za $n \geq 5$

$$T(x) = -6 + \frac{-1}{1!}x + \frac{14}{2!}x^2 + \frac{-12}{3!}x^3 + \frac{24}{4!}x^4$$

$$= -6 - x + 7x^2 - 2x^3 + x^4 = f(x).$$

↗

prethodni korak je suvišan jer ako tražimo Taylorov razvoj oko nule f-je oblika $f(x) = \sum_{n=0}^{m} a_n x^n$, tada je $T(x) = f(x) = \sum_{n=0}^{m} a_n x^n$, tj.

f je već razvijen u Taylorov red.

Vidimo da je $\rho = \limsup_n \sqrt[n]{|a_n|} = \lim_n 0 = 0$ jer je $a_n = 0$ za $n \geq 5$.

čime je $r = \frac{1}{\rho} = +\infty$.

e) $f(x) = \operatorname{sh} x$

$f'(x) = \operatorname{ch} x$ $\Rightarrow$ $f(0) = 0$

$f''(x) = \operatorname{sh} x$ $\quad f'(0) = 1 \quad \Rightarrow \quad a_n = \begin{cases} 0 & , n \text{ paran} \\ \frac{1}{n!} & , n \text{ neparan} \end{cases}$

$\vdots$ $\quad f''(0) = 0$

$\Rightarrow T(x) = \sum_{m=0}^{+\infty} \frac{1}{(2m-1)!} x^{2m-1}$, radijus konverge $r = \infty$, (vidi argument od b))

f)
$$q = -\frac{2}{3}x$$
$$f(x) = \frac{1}{3+2x} = \frac{1}{3} \frac{1}{1-(-\frac{2}{3}x)} = \frac{1}{3} \frac{1}{1-q} = \frac{1}{3} \sum_{n=0}^{+\infty} q^n$$
$$= \frac{1}{3} \sum_{n=0}^{+\infty} \left(-\frac{2}{3}x\right)^n = \sum_{n=0}^{+\infty} \underbrace{\boxed{\frac{(-2)^n}{3^{n+1}}}}_{a_n} x^n$$
$$\rho = \lim_n \left(|a_n|\right)^{\frac{1}{n}} = \lim_n \left(\frac{2^n}{3^{n+1}}\right)^{\frac{1}{n}}$$
$$= \lim_n \frac{2}{3^{(1+\frac{1}{n})}} = \frac{2}{3}$$

$r = \frac{1}{\rho} = \frac{3}{2}$ je radijus konverge Taylora reda.

Zad Odredite Taylorov polinom i radijus konvergencije:

a) $f(x) = e^x$ oko točke $c = \ln 3$.

prvi način (preko poznatog reda): supstitucija $y = x - \ln 3$ koverg. za $\forall y$

$$f(x) = e^{x - \ln 3 + \ln 3} = e^{y + \ln 3} = 3e^y = 3 \sum_{n=0}^{+\infty} \frac{1}{n!} y^n$$
$$= \sum_{n=0}^{+\infty} \frac{3}{n!} (x - \ln 3)^n.$$ Radijus konverge $r = +\infty$

drugi način (direktni način):

$f'(x) = e^x$ $\qquad$ $f(\ln 3) = e^{\ln 3} = 3$

$f''(x) = e^x$ $\qquad$ $f'(\ln 3) = e^{\ln 3} = 3$

$\vdots$ $\qquad\qquad\qquad$ $\vdots$

$$\Rightarrow \quad T(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(\ln 3)}{n!} (x-\ln 3)^n = \sum_{n=0}^{+\infty} \frac{3}{n!} (x-\ln 3)^n$$

$$\varrho = \limsup_n \sqrt[n]{\frac{3}{n!}} = \lim_n \sqrt[n]{\frac{3}{n!}} = 0 \quad \left(\text{jer je } \lim_n \sqrt[n]{n!} = +\infty\right).$$

b) $f(x) = x^3 e^{x^4}$ oko točke $c = 0$

razvoj vrijedi $\forall y \in \mathbb{R}$.

$y = x^4$

$$f(x) = x^3 e^{x^4} = x^3 e^y = x^3 \sum_{n=0}^{+\infty} \frac{1}{n!} y^n$$

$$= x^3 \sum_{n=0}^{+\infty} \frac{1}{n!} (x^4)^n = \sum_{n=0}^{+\infty} \frac{1}{n!} x^{4n+3} = T(x)$$

razvoj vrijedi $\forall x \in \mathbb{R}$

radijus konv. je $r = +\infty$.

c) $f(x) = \ln(x+1)$ oko točke 0.

za $|x| < 1$.

$$f'(x) = \frac{1}{x+1} = \frac{1}{1-(-x)} = \sum_{n=0}^{+\infty} (-x)^n. \quad \Big/ \int dx$$

za $|x| < 1$

$$\Rightarrow \quad f(x) = C + \sum_{n=0}^{+\infty} (-1)^n \frac{x^{n+1}}{n+1} = C + \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{x^n}{n} = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} x^n$$

$$f(0) = \ln(0+1) = C \quad \Rightarrow \quad C = \ln 1 = 0$$

d) $f(x) = \operatorname{tg}^{-1} x$ oko točke $c = 0$

$$f'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{+\infty} (-x^2)^n$$

gdje red apsolutno konvergira akko
$|(-x^2)| < 1$
$\Leftrightarrow \quad |x| < 1$

$$= \sum_{n=0}^{+\infty} (-1)^n x^{2n} \quad \text{za } |x| < 1$$

$$f'(x) = \sum_{n=0}^{+\infty} (-1)^n x^{2n} \Big/ \int dx$$

$$\Rightarrow \quad f(x) = c + \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{za} \quad |x| < 1$$

$$f(0) = tg^{-1}(0) = c \quad \Rightarrow \quad c = 0$$

pa dobivamo da je $$f(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{za} \quad |x| < \textcircled{1}$$

↑ ovo je radijus konvergencije

e) $f(x) = x^3 + 2x^2 - 5$ oko točke $c = 1$.

$$\begin{aligned} f(x) &= ((x-1)+1)^3 + 2((x-1)+1)^2 - 5 \\ &= (x-1)^3 + 3(x-1)^2 + 3(x-1) + 1 + 2(x-1)^2 + 4(x-1) + 2 - 5 \\ &= (x-1)^3 + 5(x-1)^2 + 7(x-1) - 2, \end{aligned}$$

razvoj vrijedi $\forall x \in \mathbb{R}$ $\Leftrightarrow$ $r = +\infty$ je radijus konvergencije.

Teorem Neka je $n \in \mathbb{N}$, $I \subseteq \mathbb{R}$ otvoren interval, $0 \in I$, $f \in C^{n+1}(I)$. Tada $\forall x \in I$ postoji $c_x \in \langle -|x|, |x| \rangle$ t.d.

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k}_{T_n(x)} + \underbrace{\frac{f^{(n+1)}(c_x)}{(n+1)!} x^{n+1}}_{R_n(x)}$$

Zad Koristeći Taylorovu aproksimaciju izračunajte zadane vrijednosti funkcije f do na grešku 10^{-3}. Koristeći kalkulator izračunajte razliku između dobivene aproksimacije i stvarne vrijednosti:

a) $f(x) = \cos x$, $\cos(0.1)$

b) $f(x) = ch\, x$, $ch(0.3)$

c) $f(x) = \ln(1+x)$, $\ln(1.2)$

d) $f(x) = arctg\, x$, $arctg\ 0.1$

Rj: a)

$$\left.\begin{aligned} f(x) &= \cos x \\ f'(x) &= -\sin x \\ f''(x) &= -\cos x \\ f'''(x) &= \sin x \end{aligned}\right\} \Rightarrow \left.\begin{aligned} f(0) &= 1 \\ f'(0) &= 0 \\ f''(0) &= -1 \\ f'''(0) &= 0 \end{aligned}\right\} \Rightarrow T(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \dots$$

$$f^{(4)}(x) = f(x)$$

! Želimo da je apsolutna vrijednost razlike između Taylorovog polinoma i f-je f manja od $\varepsilon = 10^{-3}$ i to na segmentu $[-0.1, 0.1]$.

Tražim vrijednost $\cos(0,1)$

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + \frac{f^{(n+1)}(c_x)}{(n+1)!} x^{n+1}$$

Stavimo $n = 3$

$$f(x) = 1 - \frac{x^2}{2} + R_3(x)$$

$|\cos(c_x)| \leq 1$

$$\Rightarrow \left| f(x) - \left(1 - \frac{x^2}{2}\right) \right| = |R_3(x)| = \frac{|f^{(4)}(c_x)|}{4!} |x|^4$$

ako je $|x| < 0.1$

$$\leq \frac{1}{4!}|x|^4 \leq \frac{0.1^4}{4!} = 4.17 \cdot 10^{-4} < \varepsilon$$

ocjena greške

$$f(0.1) = \cos 0.1 \approx 0.995004$$

$$T_3(x) = T_2(x) = 1 - \frac{x^2}{2}$$

$$T_3(0.1) = 1 - \frac{0.1^2}{2} = 0.995$$

$$|f(0.1) - T_3(0.1)| \approx 4 \cdot 10^{-6} < 4.17 \cdot 10^{-3} < 10^{-3}$$

Istaknimo da je gornja ocjena uniformna i vrijedi za $x \in [-0,1, 0.1]$.
Općenito pokazali smo da je $|R_3(x)| \leq \frac{1}{24}|x|^4$

tj.

$$\left| f(x) - \left(1 - \frac{x^2}{2}\right) \right| \leq \frac{1}{24}|x|^4$$

Nap Korektno je koristiti i $\left| f(x) - \underbrace{\left(1 - \frac{x^2}{2}\right)}_{} \right| = |R_2(x)| \leq \frac{1}{6}|x|^3.$

$T_2(x)$

b) $f(x) = \operatorname{ch} x$, vrijednost $\operatorname{ch}(0.3)$ na točnost od barem 10^{-3}.

$f'(x) = \operatorname{sh} x$

$f''(x) = f(x) = \operatorname{ch} x$

$$\Rightarrow \quad T(x) = 1 + \frac{1}{2}x^2 + \frac{1}{4}x^4 + \dots$$

$$\Big|\operatorname{ch} x - \underbrace{\left(1 - \tfrac{1}{2}x^2\right)}_{= \, T_2(x)}\Big| = |R_2(x)| = \left|\frac{\overset{= \, \operatorname{sh}(c_x)}{f^{(3)}(c_x)}}{3!}x^3\right|$$

$\operatorname{sh}(c_x)$ gdje je $c_x \in \langle -0.3, 0.3\rangle$ za $|x| < 0.3$

$$\leq \frac{|\operatorname{sh}(c_x)|}{3!}|x|^3 \quad \text{za } |c_x| < 0.3$$

$$\leq \frac{0.3}{3!}|x|^3 \quad \Rightarrow \quad |R_2(x)| < \frac{0.3}{3!}|x|^3$$

$$|\operatorname{sh}(c_x)| < \operatorname{sh}(0.3) \approx 0.3$$

Vrijedi da je $|R_2(0.3)| < \frac{0.3}{3!}\,0.3^3 \approx 0.00135 \not< \varepsilon = 10^{-3}$

Potrebna je bolja ocjena!

$$\Big|\operatorname{ch} x - \underbrace{\left(1 - \tfrac{1}{2}x^2\right)}_{T_3(x) = T_2(x)}\Big| = |R_3(x)| = \left|\frac{\overset{= \, \operatorname{ch}(c_x)}{f^{(4)}(c_x)}}{4!}x^4\right| \leq \frac{5}{3 \cdot 4!}|x|^4$$

$$|\operatorname{ch}(c_x)| < \operatorname{ch}(0.3) \leq \operatorname{ch}(\ln 3) = \frac{e^{\ln 3} + e^{-\ln 3}}{2} = \frac{3 + \frac{1}{3}}{2} = \frac{10}{6} = \frac{5}{3}$$

jer je f-ja parna i rastuća na $\langle 0, +\infty\rangle$.

Tehnički $\operatorname{ch}(0.3)$ možete izračunati u kalkulatoru čime je zadatak odmah gotov, ali ako tražimo aproksimaciju od $\operatorname{ch}(0.3)$ nema smisla koristiti „točnu" vrijednost u ocjeni greške

$$|R_3(0.3)| < \frac{5}{3 \cdot 4!}\,0.3^4 \approx 0.0006 < \varepsilon$$

$$T_3(0.3) = 1 - \frac{1}{2}0.3^2 = 0.955$$

čime je vrijednost od $ch(0.3)$ u segmentu $[0.955 - 6\cdot 10^{-4}, 0.955 + 6\cdot 10^{-4}]$

Prava vrijednost je $\cos(0.3) \approx 0.9553364$

čime je stvarna greška $|\cos(0.3) - T_3(0.3)| \approx 0.0003 < |R_3(0.3)| < \varepsilon.$

c) $f(x) = \ln(1+x)$, $\ln(1.2) = ?$

$$f'(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{+\infty} (-x)^n \quad \Big/ \int dx \qquad \text{za } |x| < 1$$

$$\Rightarrow f(x) = c + \sum_{n=0}^{+\infty} (-1)^n \frac{x^{n+1}}{n+1} \quad \Big/ \quad \text{za } x = 0$$

$$f(0) = \ln(1) = 0 = c \quad \Rightarrow \quad f(x) = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{x^n}{n}$$

$$= \underbrace{x - \frac{x^2}{2} + \frac{x^3}{3}}_{T_3(x)} - \frac{x^4}{4} + \dots$$

$$f'(x) = \frac{1}{1+x}$$

$$f''(x) = \frac{-1}{(1+x)^2}$$

$$f'''(x) = \frac{2}{(1+x)^3}$$

$$\boxed{f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}}$$

Time je

$$|f(x) - T_n(x)| = |R_n(x)| = \left| \frac{\frac{\overset{1}{\cancel{n!}}}{(1+cx)^{n+1}}}{\underset{(n+1)}{\cancel{(n+1)!}}} \cdot x^{n+1} \right|$$

$$= \left| \frac{1}{(1+cx)^{n+1}(n+1)} \, x^{n+1} \right|$$

Preslikavanje $cx \longmapsto \frac{1}{(1+cx)^{n+1}}$ je strogo padajuće na $[-0.2, 0.2]$ i nenegativno čime je

$$\left| \frac{1}{(1+cx)^{n+1}} \right| \leq \frac{1}{0.8^{n+1}}$$

$$|R_3(x)| \leq \frac{1}{0.8^4 \cdot 4} \, 0.2^4 \approx 0.00098 < 10^{-3} = \varepsilon$$

čime je $T_3(0.2) = 0.2 - \frac{0.2^2}{2} + \frac{0.2^3}{3} \approx 0.18267$

dobra aproksimacija od $\ln(1.2)$.

Zaista $|\ln(1.2) - T_3(0.2)| \approx 0.0003 < 0.00098 < 10^{-3} = \varepsilon$.

d) $f(x) = \operatorname{arctg} x$, $\operatorname{arctg} 0.1$

$$f'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{+\infty} (-x^2)^n = \sum_{n=0}^{+\infty} (-1)^n x^{2n} \Big/ \int dx$$

$$\Rightarrow \quad f(x) = c + \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$\Rightarrow$ $c = 0$ jer je $f(0) = \operatorname{arctg} 0 = 0$.

$$f(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \frac{x}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

$$f''(x) = \frac{-2x}{(1+x^2)^2} \quad , \quad f'''(x) = \frac{-2(1+x^2)^2 + 8x^2(1+x^2)}{(1+x^2)^4} = -\frac{2}{(1+x^2)^4} + \frac{8x^2}{(1+x^2)^3}$$

$$|f'''(c_x)| \leq 2 + 8c_x^2 \leq 2 + 8 \cdot 0.1^2 = 2.08$$

$$\Rightarrow \quad |f(x) - T_2(x)| \leq |R_2(x)| = \left| \frac{f'''(c_x)}{3!} x^3 \right| \leq \frac{2.08}{3!} |x|^3$$

procjena greške

$$|\operatorname{arctg} 0.1 - 0.1| \leq \frac{0.1^3}{3!} < 3{,}47 \cdot 10^{-4} < \varepsilon$$

$\Rightarrow$ Zaista $\operatorname{arctg} 0.1 \approx 0.09967$ $\Rightarrow$ $|\operatorname{arctg} 0.1 - 0.1| \approx 3{,}3 \cdot 10^{-4}$

↑

prava greška,