Red potencija

Neka je dan niz funkcija $(f_n)_n$ takva da
$f_n : I \subseteq \mathbb{C} \to \mathbb{C}$
Red $\sum_{n=0}^{\infty} f_n$ zovemo redom funkcija

Ako je $f_n(x) = a_n(x-c)^n$, $c \in I$ red

$$\sum_{n=0}^{\infty} a_n(x-c)^n$$

nazivamo redom potencija oko točke c.

Definira skup
$$K = \left\{ z \in \mathbb{C} \ : \ \text{red } \sum_{n=0}^{\infty} a_n(x-c)^n \text{ konvergira u točki } z \right\} \subseteq \mathbb{C}$$

Tada je radijus konvergencije reda $\sum_{n=0}^{\infty} a_n(x-c)^n$

$$r := \sup\{ |z-c| \ : \ z \in K \}$$

Ako je $|z-c| < r$ tada red konvergira apsolutno, dok
ako je $|z-c| > r$ red potencija divergira.

__Teorem__ Neka je $\sum_{n=0}^{\infty} a_n x^n$ red potencija, $\rho = \limsup_n \sqrt[n]{|a_n|}$.
Tada je radijus konvergencije $r = 1/\rho$ $\left(\text{uz } \frac{1}{0} = \infty, \ \frac{1}{\infty} = 0 \right)$

Dodatno, ako postoji $\rho_1 = \lim_n \sqrt[n]{|a_n|}$, tada je $r = \frac{1}{\rho_1}$

Dodatno, ako postoji $\rho_2 = \lim_n \left| \frac{a_{n+1}}{a_n} \right|$, tada je $r = \frac{1}{\rho_2}$.

__Zad__ Izračunajte radijus konvergencije redova:

a) $\sum_{n=0}^{\infty} x^n$

Rj: $a_n = 1$.

$\rho_1 := \sqrt[n]{|a_n|} = 1 \Rightarrow \rho = \limsup \sqrt[n]{|a_n|} = 1$

Rj: $a_n = 1$.

$$\lim_n \sqrt[n]{|a_n|} = 1 \Rightarrow \rho = \limsup_n \sqrt[n]{|a_n|} = 1$$

$r = \frac{1}{\rho} = 1$ je radijus konvergencije reda.

b) $\sum_{n=1}^{\infty} n^n x^n \Rightarrow a_n = n^n$

$$\rho = \lim_n \sqrt[n]{n^n} = \lim_n n = +\infty$$

$\Rightarrow r = \frac{1}{\rho} = 0$ je radijus konvergencije reda.

c) $\sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow a_n = \frac{1}{n!}$

$$\rho = \lim_n \left|\frac{a_{n+1}}{a_n}\right| = \lim_n \frac{\frac{1}{\cancel{(n+1)!}\,(n+1)}}{\frac{1}{\cancel{n!}}} = \lim_n \frac{1}{1+n} = 0$$

$\Rightarrow r = \frac{1}{\rho} = +\infty$ je radijus konvergencije reda.

d) $\sum_{n=1}^{\infty} \frac{\ln n}{2^n} x^n \Rightarrow a_n = \frac{\ln n}{2^n}$

$$\rho = \lim_n \sqrt[n]{|a_n|} = \lim_n \left(\frac{\ln n}{2^n}\right)^{\frac{1}{n}} = \lim_n \frac{\sqrt[n]{\ln n}}{2} = \frac{1}{2}$$

$\sqrt[n]{\ln n} \xrightarrow{n\to\infty} 1$

Znamo da je $\lim_n \sqrt[n]{\ln n} = 1$.

Zaista, $1 \leq \ln n \leq n$, $\forall n \geq 3$

te postoje limesi $\lim_n \sqrt[n]{1} = \lim_n \sqrt[n]{n} = 1$

i jednaki su, slijedi da je $\lim_n \sqrt[n]{\ln n} = 1$

tzv. teorem o sendviču.

$r = \frac{1}{\rho} = 2$ je radijus konvergencije reda.

Alternativno rješenje

gledam za $n>1$

$$\rho = \lim_n \left|\frac{a_{n+1}}{a_n}\right| = \lim_n \frac{\frac{\ln(n+1)}{\cancel{2^{n+1}}}\,2}{\frac{\ln n}{\cancel{2^n}}} = \lim_n \frac{\ln(n+1)}{2\ln n}$$

$$\overset{L'H}{=} \lim_n \frac{\frac{1}{n+1}}{2\,\frac{1}{n}} = \frac{1}{2}\lim_n \frac{n}{n+1} = \frac{1}{2}\lim_n \frac{1}{1+\frac{1}{n}} = \frac{1}{2}$$

$r = \frac{1}{\rho} = 2$ je traženi radijus konvergencije.

e) $\sum_{n=1}^{\infty} \frac{(3n)!}{n^{3n}} x^n \Rightarrow a_n = \frac{(3n)!}{n^{3n}}$

$$\rho = \lim_n \left|\frac{a_{n+1}}{a_n}\right| = \lim_n \frac{\frac{(3n+3)!}{(n+1)^{3n+3}}}{\frac{(3n)!}{n^{3n}}} = \lim_n \frac{\frac{(3n+3)(3n+2)(3n+1)\cancel{(3n)!}}{(n+1)^{3n}\,(n+1)^3}}{\frac{\cancel{(3n)!}}{n^{3n}}}$$

$$= \lim_n \underbrace{\frac{3n+3}{n+1}}_{\to 3}\cdot\underbrace{\frac{3n+2}{n+1}}_{\to 3}\cdot\underbrace{\frac{3n+1}{n+1}}_{\to 3}\cdot\left(\frac{1}{\underbrace{\left(1+\frac{1}{n}\right)^n}_{\to e}}\right)^3 = \frac{3^3}{e^3}$$

$r = \frac{1}{\rho} = \frac{e^3}{27}$ je radijus konvergencije reda //

f) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n \qquad a_n = \frac{(-1)^n}{n}$

$$\rho = \lim_n \sqrt[n]{|a_n|} = \lim_n \sqrt[n]{\frac{1}{n}} = \lim_n \frac{1}{\underbrace{\sqrt[n]{n}}_{\xrightarrow{n\to\infty} 1}} = 1$$

$$\rho = \lim_n \sqrt[n]{|a_n|} = \lim_n \sqrt[n]{\frac{1}{n}} = \lim_n \frac{1}{\sqrt[n]{n}} = 1$$

$\sqrt[n]{n} \xrightarrow{n \to \infty} 1$

$r = \frac{1}{\rho} = 1$ je traženi radijus konvergencije.

g) $\sum_{n=1}^{\infty} \frac{\cos(n^2)}{n^n} x^n \Rightarrow a_n = \frac{\cos(n^2)}{n^n}$

$$0 \leq \frac{|\cos(n^2)|}{n^n} \leq \frac{1}{n^n} \quad / \sqrt[n]{\cdot}$$

$$\Rightarrow 0 \leq \sqrt[n]{|a_n|} \leq \frac{1}{n} \xrightarrow{n \to \infty} 0$$

po tm. o sendviču je $\lim_n \sqrt[n]{|a_n|} = 0$

$r = \frac{1}{\rho} = +\infty$ je traženi radijus konvergencij.

h) $\sum_{n=1}^{\infty} (1 + (-1)^n) x^n \Rightarrow a_n = \begin{cases} 2, & n \text{ paran} \\ 0, & n \text{ neparan} \end{cases}$

$\rho = \limsup_n \sqrt[n]{|a_n|}$

$$0 \leq a_n \leq 2$$
$$\Rightarrow |a_n| \leq 2$$
$$\Rightarrow \sqrt[n]{|a_n|} \leq 2^{\frac{1}{n}}$$
$$\Rightarrow \limsup_n \sqrt[n]{|a_n|} \leq \liminf_n 2^{\frac{1}{n}} = \lim_n 2^{\frac{1}{n}} = 1$$

S druge strane $\lim_n \sqrt[n]{|a_{2n}|} = \lim_n \sqrt[n]{2} = 1$ pa je

$\limsup_n \sqrt[n]{|a_n|} = 1.$

$r = \frac{1}{\rho} = 1$ je traženi radijus konvergencije

Napomena: Gornji dokaz je za općenitu situaciju.

$\left.\lim_n \sqrt[n]{|a_{2n}|} = 1\right\} \Rightarrow \limsup \sqrt[n]{|a_n|} = \max\{0, 1\} = 1$

$$\left.\begin{array}{l}\lim_n \sqrt[n]{|a_{2n}|} = 1 \\ \lim_n \sqrt[n]{|a_{2n-1}|} = 0\end{array}\right\} \Rightarrow \limsup_n \sqrt[n]{|a_n|} = \max\{0,1\} = 1$$

i) $\sum_{n=1}^{\infty} (1+(-2)^n)\, x^n \qquad a_n = 1+(-2)^n$

$$\rho = \lim_n \left|\frac{a_{n+1}}{a_n}\right| = \lim_n \left|\frac{1+(-2)^{n+1}}{1+(-2)^n}\right| = \lim_n \left|\frac{\overbrace{\frac{1}{(-2)^{n+1}}}^{\to 0}+1}{\underbrace{\frac{1}{(-2)^{n+1}}}_{\downarrow\, 0}-\frac{1}{2}}\right|$$

$$= \left|\frac{1}{-\frac{1}{2}}\right| = 2$$

$r = \frac{1}{\rho} = \frac{1}{2}$ je traženi radijus konvergencije.

Def. Neka je funkcija $f \in C^{\infty}(\langle c-r, c+r\rangle)$, $r>0$.
Red potencija $\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$

zovemo Taylorov red funkcije f oko točke c.
Taylorovi polinomi $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$ su
parcijalne sume Taylorovog reda.

Zad Odredite Taylorov red oko nule i njegov radijus konvergencije za funkciju.

a) $f(x) = e^x$, $c = 0$

$f'(x) = e^x \quad \ldots \quad f^{(n)}(x) = e^x$

$f^{(n)}(0) = 1 \quad \forall n \quad \Rightarrow \quad T(x) = \sum \left(\frac{1}{\cdot}\right) x^n$

$$f^{(n)}(0)=1, \quad \forall n \Rightarrow T(x)=\sum_{n=0}^{\infty} \underbrace{\frac{1}{n!}}_{a_n} x^n$$

$$\rho=\lim_n \left|\frac{a_{n+1}}{a_n}\right| = \lim_n \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_n \frac{1}{n+1} = 0$$

$\Rightarrow r=\frac{1}{\rho}=+\infty$ je radijus konvergencije.

b) $f(x)=\sin x$

$f'(x)=\cos x$

$f''(x)=-\sin x$

$f'''(x)=-\cos x$

$f^{(4)}(x)=\sin x$

$\vdots$

$$\left.\begin{array}{ll} f(0)=0 & n=0 \\ f'(0)=1 & n=1 \\ f''(0)=0 & n=2 \\ f'''(0)=-1 & n=3 \\ f^{(4)}(0)=0 & n=4 \\ \vdots & \end{array}\right\} \Rightarrow a_n=\begin{cases} 0, & n \text{ paran} \\ \frac{1}{n!}, & n \text{ daje ostatak } 1 \text{ pri dijeljenju s } 4 \\ -\frac{1}{n!}, & n \text{ daje ostatak } 3 \text{ pri dijeljenju s } 4 \end{cases}$$

$$\Rightarrow T(x)=\sum_{m=0}^{\infty}(-1)^m \frac{1}{(2m+1)!} x^{2m+1}$$

Pokazali smo u prethodnom zadatku da je $\limsup_n \sqrt[n]{\frac{1}{n!}}=0$

Uočimo da je $0 \overset{\text{def.}}{\leq} \limsup_n \sqrt[n]{|a_n|} \leq \limsup_n \sqrt[n]{\frac{1}{n!}}=0$

jer je $|a_n| \leq \frac{1}{n!}$, čime je $\rho=\limsup_n \sqrt[n]{|a_n|}=0$, te

je radijus konvergencije $r=\frac{1}{\rho}=+\infty$, opet isti.

c) $f(x)=\frac{1}{1-3x}$

prvi način: prepoznamo da je $\frac{1}{1-q}=\lim_n \frac{1-q^n}{1-q}=\lim_n (1+q+\ldots+q^{n-1})=\sum_{n=0}^{+\infty} q^n$

za $y = 3x$ slijedi da je $\frac{1}{1-(3x)} = \sum_{n=0}^{+\infty} 3^n x^n =: T(x)$

a_n

$\varrho = \lim_n \sqrt[n]{3^n} = 3 \Rightarrow r = \frac{1}{\varrho} = \frac{1}{3}$ je radijus konvergencije reda.

drugi način:

$$g(y) = \frac{1}{1-y}, \qquad f(x) = g(3x)$$

$$g'(y) = -\frac{1}{(1-y)^2}(-1) = \frac{1}{(1-y)^2} \qquad g(0) = 1$$

$$g(0)=1, \quad g'(0) = 1$$

$$g''(y) = (-2)\frac{1}{(1-y)^3}(-1) = \frac{2}{(1-y)^3} \qquad g''(0) = 2$$

$$\vdots$$

$$g^{(n)}(y) = \frac{n!}{(1-y)^{n+1}} \qquad g^{(n)}(0) = n!$$

$y = 3x$

$$\Rightarrow T_g(y) = \sum_{n=0}^{+\infty} \frac{g^{(n)}(0)}{n!} y^n = \sum_{n=0}^{+\infty} \frac{n!}{n!} y^n = \sum_{n=0}^{+\infty} y^n$$

$$\Rightarrow T_f(x) = \sum_{n=0}^{+\infty} (3x)^n = \sum_{n=0}^{+\infty} 3^n x^n$$

d) $f(x) = x^4 - 2x^3 + 7x^2 - x - 6$

$f'(x) = 4x^3 - 6x^2 + 14x - 1$

$f''(x) = 12x^2 - 12x + 14$

$f'''(x) = 24x - 12$

$f^{(4)}(x) = 24$

$\vdots$

$f^{(n)}(x) = 0$, za $n \geq 5$

$f(0) = -6$

$f'(0) = -1$

$f''(0) = 14$

$f'''(0) = -12$

$f^{(4)}(0) = 24$

$f^{(n)}(0) = 0$ za $n \geq 5$

$$T(x) = -6 + \frac{-1}{1!}x + \frac{14}{2!}x^2 + \frac{-12}{3!}x^3 + \frac{24}{4!}x^4$$

$$= -6 - x + 7x^2 - 2x^3 + x^4 = f(x).$$

$$= -6 - x + 7x^2 - 2x^3 + x^4 = f(x).$$

prethodni korak je suvišan jer ako tražimo Taylorov razvoj oko nule f-je oblika

$f(x) = \sum_{n=0}^{m} a_n x^n$, tada je $T(x) = f(x) = \sum_{n=0}^{m} a_n x^n$, tj.

f je već razvijen u Taylorov red.

Vidimo da je $\rho = \limsup_n \sqrt[n]{|a_n|} = \lim_n 0 = 0$ jer je $a_n = 0$ za $n \geq 5$.

čime je $r = \frac{1}{\rho} = +\infty$.

e) $f(x) = \operatorname{sh} x$

$f'(x) = \operatorname{ch} x$ $\Rightarrow$ $f'(0) = 1$

$f''(x) = \operatorname{sh} x$ $\quad f''(0) = 0$

$\vdots$

$f(0) = 0$

$$\Rightarrow a_n = \begin{cases} 0, & n \text{ paran} \\ \frac{1}{n!}, & n \text{ neparan} \end{cases}$$

$$\Rightarrow T(x) = \sum_{m=0}^{+\infty} \frac{1}{(2m-1)!} x^{2m-1}$$, radijus konvergencije $r = \infty$, (vidi argument od b))

f) $f(x) = \frac{1}{3+2x} = \frac{1}{3} \frac{1}{1-(-\frac{2}{3}x)} = \frac{1}{3} \frac{1}{1-q} = \frac{1}{3} \sum_{n=0}^{+\infty} q^n$ $\quad q = -\frac{2}{3}x$

$$= \frac{1}{3} \sum_{n=0}^{+\infty} \left(-\frac{2}{3}x\right)^n = \sum_{n=0}^{+\infty} \underbrace{\frac{(-2)^n}{3^{n+1}}}_{a_n} x^n$$

$$\rho = \lim_n \left(|a_n|\right)^{\frac{1}{n}} = \lim_n \left(\frac{2^n}{3^{n+1}}\right)^{\frac{1}{n}}$$

$$= \lim_n \frac{2}{3^{(1+\frac{1}{n})}} = \frac{2}{3}$$

$r = \frac{1}{\varrho} = \frac{3}{2}$ je radijus konverge Taylora reda.

Zad Odredite Taylorov polinom i radijus konvergencije:

a) $f(x) = e^x$ oko točke $c = \ln 3$.

prvi način (preko poznatog reda): supstitucija $y = x - \ln 3$

konverg. za $\forall y$

$$f(x) = e^{x - \ln 3 + \ln 3} = e^{y + \ln 3} = 3e^{y} = 3\sum_{n=0}^{+\infty} \frac{1}{n!} y^n$$

$$= \sum_{n=0}^{+\infty} \frac{3}{n!} (x - \ln 3)^n.$$ Radijus konverge $r = +\infty$

drugi način (direktni način):

$f'(x) = e^x$ $\qquad f(\ln 3) = e^{\ln 3} = 3$

$f''(x) = e^x$ $\qquad f'(\ln 3) = e^{\ln 3} = 3$

$\vdots \qquad \vdots$

$$\Rightarrow \quad T(x) = \sum_{n=0}^{+\infty} \frac{f^{(n)}(\ln 3)}{n!} (x - \ln 3)^n = \sum_{n=0}^{+\infty} \frac{3}{n!} (x - \ln 3)^n$$

$$\varrho = \limsup_n \sqrt[n]{\frac{3}{n!}} = \lim_n \sqrt[n]{\frac{3}{n!}} = 0 \quad \left(\text{jer je } \lim_n \sqrt[n]{n!} = +\infty\right).$$

b) $f(x) = x^3 e^{x^4}$ oko točke $c = 0$

razvoj vrijedi $\forall y \in \mathbb{R}$.

$$f(x) = x^3 e^{x^4} \overset{y = x^4}{=} x^3 e^{y} = x^3 \sum_{n=0}^{+\infty} \frac{1}{n!} y^n$$

$$= x^3 \sum_{n=0}^{+\infty} \frac{1}{n!} (x^4)^n = \sum_{n=0}^{+\infty} \frac{1}{n!} x^{4n+3} = T(x)$$

razvoj vrijedi $\forall x \in \mathbb{R}$

radijus konv. je $r = +\infty$.

c) $f(x) = \ln(x+1)$ oko točke 0.

za $|x| < 1$.

$$f'(x) = \frac{1}{x+1} = \frac{1}{1-(-x)} = \sum_{n=0}^{+\infty} (-x)^n \quad \Big/ \int dx$$

za $|x| < 1$

$$\Rightarrow \quad f(x) = c + \sum_{n=0}^{+\infty} (-1)^n \frac{x^{n+1}}{n+1} = c + \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{x^n}{n} = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} x^n$$

$$f(0) = \ln(0+1) = c \quad \Rightarrow \quad c = \ln 1 = 0$$

d) $f(x) = tg^{-1} x$ oko točke $c = 0$

$$f'(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{+\infty} (-x^2)^n$$

gdje red apsolutno konvergira akko $|(-x^2)| < 1 \iff |x| < 1$

$$= \sum_{n=0}^{+\infty} (-1)^n x^{2n} \quad \text{za } |x| < 1$$

$$f'(x) = \sum_{n=0}^{+\infty} (-1)^n x^{2n} \quad \Big/ \int dx$$

$$\Rightarrow \quad f(x) = c + \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{za } |x| < 1$$

$$f(0) = tg^{-1}(0) = c \quad \Rightarrow \quad c = 0$$

pa dobivamo da je $\displaystyle f(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ za $|x| < 1$

ovo je radijus konvergencije

e) $f(x) = x^3 + 2x^2 - 5$ oko točke $c = 1$.

$$f(x) = ((x-1)+1)^3 + 2((x-1)+1)^2 - 5$$

$$= (x-1)^3 + 3(x-1)^2 + 3(x-1) + 1 + 2(x-1)^2 + 4(x-1) + 2 - 5$$

$$= (x-1)^3 + 5(x-1)^2 + 7(x-1) - 2,$$ razvoj vrijedi $\forall x \in \mathbb{R}$ $\iff$ $r = +\infty$ je

$\Leftrightarrow r = +\infty$ je radijus konvergencije.

<u>Teorem</u> Neka je $n \in \mathbb{N}$, $I \subseteq \mathbb{R}$ otvoren interval, $0 \in I$, $f \in C^{n+1}(I)$. Tada $\forall x \in I$ postoji $c_x \in \langle -|x|, |x| \rangle$ t.d.

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k}_{T_n(x)} + \underbrace{\frac{f^{(n+1)}(c_x)}{(n+1)!} x^{n+1}}_{R_n(x)}$$

<u>Zad</u> Koristeći Taylorovu aproksimaciju izračunajte zadane vrijednosti funkcije f do na grešku 10^{-3}. Koristeći kalkulator izračunajte razliku između dobivene aproksimacije i stvarne vrijednosti:

a) $f(x) = \cos x$, $\cos(0.1)$

b) $f(x) = \operatorname{ch} x$, $\operatorname{ch}(0.3)$

c) $f(x) = \ln(1+x)$, $\ln(1.2)$

d) $f(x) = \operatorname{arctg} x$, $\operatorname{arctg} 0.1$