Fourierov red

Neka je dana funkcija $f : [-\pi, \pi] \to \mathbb{R}$.

Definiramo Fourierove koeficijente:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\,dx \quad , \quad n \in \mathbb{N}_0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\,dx \quad , \quad n \in \mathbb{N}.$$

Fourierov red funkcije f u oznaci S:

$$S(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right)$$

Teorem (Dirichlet)

Neka je dana $f : [-\pi, \pi] \to \mathbb{R}$ takva da

ima konačan broj prekida prve vrste na $[-\pi, \pi]$;

i je po dijelovima monotona na $[-\pi, \pi]$.

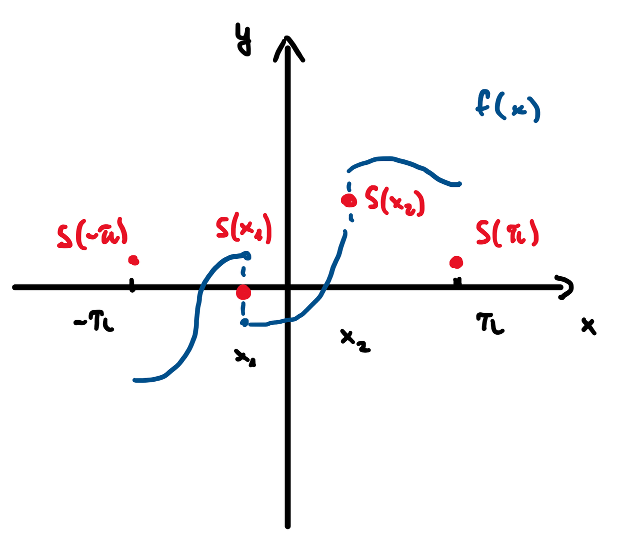
Tada Fourierov red $S(x)$ konvergira za svaki $x \in \mathbb{R}$.

Dodatno, ako je f neprekidna u točki $x \in \langle -\pi, \pi \rangle$ je $S(x) = f(x)$.

Ako ima prekid 1. vrste u točki x tada je

$$S(x) = \frac{f(x^-) + f(x^+)}{2}.$$

Općenito $$S(\pi) = S(-\pi) = \frac{f(-\pi^+) + f(\pi^-)}{2}$$



Razvijte funkciju u Fourierov red na $[-\pi, \pi]$:

a) $f(x) = x$

b) $f(x) = \begin{cases} 1 & , \ x < 0 \\ 0 & , \ x \geq 0 \end{cases}$

c) $f(x) = \sin x$

d) $f(x) = e^x$

Rj:

jer je $x \mapsto x\cos(nx)$ neparna f-ja na simetričnom intervalu $[-\pi, \pi]$.

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos(nx)\,dx = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx)\,dx = \begin{bmatrix} u = x & dv = \sin(nx)\,dx \\ du = dx & v = -\frac{1}{n}\cos(nx) \end{bmatrix}$$

$$= \frac{1}{\pi} \left(-\frac{x}{n} \cos(nx) \Big|_{-\pi}^{\pi} + \frac{1}{n} \int_{-\pi}^{\pi} \cos(nx)\,dx \right)$$

$$= \frac{1}{\pi} \left(-\frac{2\pi}{n} \cos(n\pi) + \underbrace{\frac{1}{n^2} \sin(nx) \Big|_{-\pi}^{\pi}}_{=0} \right)$$

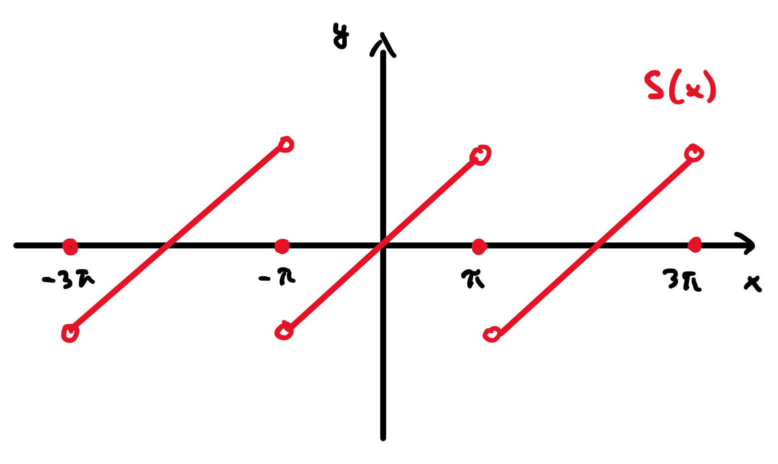
$$= -\frac{2}{n} \cos(n\pi) = -\frac{2}{n}(-1)^n$$

$$\Rightarrow \quad S(x) = \sum_{n=1}^{\infty} -\frac{2}{n}(-1)^n \sin(nx)$$

Koristeći Dirichletov tm. znamo da je

$S(x) = x$ za $\forall x \in \langle -\pi, \pi \rangle$.

$$S(-\pi) = S(\pi) = \frac{f(-\pi^+) + f(\pi^-)}{2} = \frac{-\pi + \pi}{2} = 0$$



b) $f(x) = \begin{cases} 1 & , \ x < 0 \\ 0 & , \ x \geq 0 \end{cases}$ za integrale najbolje f-je

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\,dx = \frac{1}{\pi} \int_{-\pi}^{0} \cos(nx)\,dx = \frac{1}{\pi n} \sin(nx) \Big|_{-\pi}^{0} = 0$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi} \int_{-\pi}^{0} dx = \frac{1}{\pi} \cdot (0 - (-\pi)) = 1$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\,dx = \frac{1}{\pi} \int_{-\pi}^{0} \sin(nx)\,dx = -\frac{1}{\pi n} \cos(nx) \Big|_{-\pi}^{0} = \frac{1}{\pi n}(-1 + \cos(n\pi))$$

$$= \begin{cases} 0 & , \ n \text{ paran} \\ -\frac{2}{\pi n} & , \ n \text{ neparan} \end{cases}$$

$$\Rightarrow \quad S(x) = \frac{1}{2} + \sum_{m=1}^{\infty} \frac{-2}{\pi(2m-1)} \sin((2m-1)x)$$

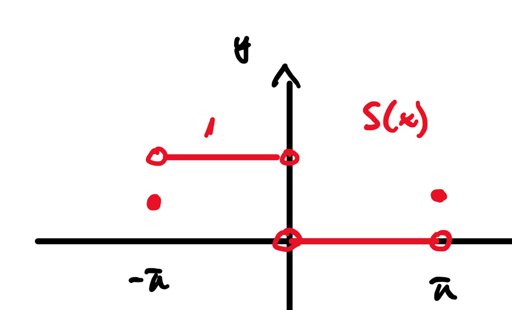
Koristeći Dirichletov tm. znamo da je

$S(x) = 1$ za $x \in \langle -1, 0 \rangle$

$S(x) = 0$ za $x \in \langle 0, 1 \rangle$

$$S(0) = \frac{f(0^-) + f(0^+)}{2} = \frac{1 + 0}{2} = \frac{1}{2}$$

$$S(\pi) = S(-\pi) = \frac{f(-\pi^+) + f(\pi^-)}{2} = \frac{1 + 0}{2} = \frac{1}{2}.$$



c) $f(x) = \sin x$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin x \cos(nx)\,dx = 0 \quad \text{za} \quad n \in \mathbb{N}_0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin x \sin(nx)\,dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos((n-1)x) - \cos((n+1)x)\,dx$$

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$$
$$\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$$
$$\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2 \sin\alpha \sin\beta$$

Jer je $\int_{-\pi}^{\pi} \cos((n-1)x) = \int_{-\pi}^{\pi} \cos(nx) = \int_{-\pi}^{\pi} \cos((n+1)x) = 0$ za $n > 1$

slijedi da je $b_n = 0$ za $n > 1$

za $n = 1$, $b_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} 1 - \cos(2x)\,dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} 1\,dx = 1$

$$\Rightarrow \quad S(x) = \sin x$$

općenito, ako je $f(x) = \frac{a_0}{2} + \sum_{n=1}^{m} (a_n \cos(nx) + b_n \sin(nx))$

tada je $S(x) = f(x) = \frac{a_0}{2} + \sum_{n \geq 1}^{m} (a_n \cos(nx) + b_n \sin(nx))$

d) $f(x) = e^x$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x\,dx = \frac{1}{\pi} e^x \Big|_{-\pi}^{\pi} = \frac{1}{\pi}(e^{\pi} - e^{-\pi}) = \frac{2\,\mathrm{sh}\,\pi}{\pi}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \cos(nx)\,dx = \begin{bmatrix} u = \cos(nx) & dv = e^x\,dx \\ du = -n\sin(nx)\,dx & v = e^x \end{bmatrix}$$

$$= \frac{1}{\pi} \left(\cos(nx) e^x \Big|_{-\pi}^{\pi} + n \int_{-\pi}^{\pi} \sin(nx) e^x\,dx \right) = \begin{bmatrix} u = \sin(nx) & dv = e^x\,dx \\ du = n\cos(nx)\,dx & v = e^x \end{bmatrix}$$

$$= \frac{1}{\pi} \left(\cos(n\pi)(e^{\pi} - e^{-\pi}) + \underbrace{\sin(nx) e^x \Big|_{-\pi}^{\pi}}_{=0} - n^2 \underbrace{\int_{-\pi}^{\pi} e^x \cos(nx)\,dx}_{a_n \cdot \pi} \right)$$

$$\Rightarrow \quad a_n = \frac{1}{\pi} \cos(n\pi)(e^{\pi} - e^{-\pi}) - n^2 a_n$$

$$\Rightarrow \quad (1 + n^2)\, a_n = \frac{1}{\pi} \cos(n\pi)(e^{\pi} - e^{-\pi})$$

$$a_n = \frac{2}{\pi} \frac{(-1)^n}{1 + n^2} \mathrm{sh}(\pi)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \sin(nx)\,dx = \begin{bmatrix} u = \sin(nx) & dv = e^x\,dx \\ du = n\cos(nx)\,dx & v = e^x \end{bmatrix}$$

$$= \frac{1}{\pi} \left(\underbrace{e^x \sin(nx) \Big|_{-\pi}^{\pi}}_{=0} - n \int_{-\pi}^{\pi} e^x \cos(nx)\,dx \right) = -n\, a_n$$

$$\Rightarrow \quad b_n = \frac{2}{\pi} \frac{n(-1)^{n+1}}{1 + n^2} \mathrm{sh}(\pi)$$

$$S(x) = \frac{2\,\mathrm{sh}(\pi)}{\pi} \left[\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{(-1)^n}{1 + n^2} \cos(nx) + \frac{(-1)^{n+1} n}{1 + n^2} \sin(nx) \right) \right]$$

Prema Dirichletu tm. vrijedi

$S(x) = e^x$ za $x \in \langle -\pi, \pi \rangle$

$$S(\pi) = S(-\pi) = \frac{f(-\pi^+) + f(\pi^-)}{2} = \frac{e^{-\pi} + e^{\pi}}{2} = \mathrm{ch}(\pi)$$