Root separation for reducible integer polynomials of degrees four and five

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Abstract

For $n \geq 2$, let $e_{\mathrm{red}}(n)$ denote the root-separation exponent restricted to reducible integer polynomials of degree n. We prove $e_{\mathrm{red}}(4) \geq 5/2$, improving the previous lower bound $7/3$. Assuming Hall's conjecture, we obtain the exact value $e_{\mathrm{red}}(4) = 5/2$. We also prove $e_{\mathrm{red}}(5) \geq 22/7$, improving the previous lower bound 3. Both constructions use a bounded resultant between a factor of degree $n-1$ and a linear factor whose heights have different orders of growth. More generally, every exponent above $n-2$ is governed by rational approximation to a real algebraic number of degree $n-1$.

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1 Introduction

For a polynomial

$$P(x) = a_d x^d + \cdots + a_0 \in \mathbb{C}[x], \qquad a_d \neq 0,$$

put

$$H(P) = \max_{0 \leq j \leq d} |a_j|.$$

If $P \in \mathbb{Z}[x]$ is separable, its root separation is

$$\mathrm{sep}(P) = \min_{\substack{P(\alpha)=P(\beta)=0 \\ \alpha \neq \beta}} |\alpha - \beta|.$$

For $n \geq 2$, define

$$e_{\mathrm{red}}(n) = \limsup_{\substack{P \in \mathbb{Z}[x] \text{ reducible over } \mathbb{Q},\ \deg P = n \\ P \text{ separable},\ H(P) \to \infty}} \frac{-\log \mathrm{sep}(P)}{\log H(P)}.$$

If $P = cP_0$ with P_0 primitive, then $\mathrm{sep}(P) = \mathrm{sep}(P_0)$ and $H(P) = |c| H(P_0)$. When $-\log \mathrm{sep}(P_0) \geq 0$, passing to the primitive part can only increase the quotient defining the exponent; when it is negative, the quotient cannot contribute to a positive limsup. Since the exponents considered below are positive, it is therefore harmless to restrict to primitive polynomials.

Mahler's inequality [8] gives $e_{\mathrm{red}}(n) \leq n - 1$. Bugeaud and Dujella [3] proved the general lower bound

$$e_{\mathrm{red}}(n) \geq \frac{2n-1}{3} \qquad (n \geq 4).$$

This gives the lower bounds 7/3 in degree four and 3 in degree five. In the monic reducible quartic problem the exponent is exactly 2 [6]. For monic reducible quintics, the corresponding exponent lies between 7/3 and 3 [7]. Thus the nonmonic setting considered here is genuinely different.

Our first result is the following.

Theorem 1. *One has*

$$e_{\mathrm{red}}(4) \geq \frac{5}{2}.$$

If Hall's conjecture holds, then

$$e_{\mathrm{red}}(4) = \frac{5}{2}.$$

The connection between Hall's conjecture and root separation already appears in [4], where the assertion that the root-separation exponent for monic cubic polynomials equals 3/2 is shown to be equivalent to Hall's conjecture. In the present setting, Hall's conjecture enters through Badziahin's cubic–rational approximation estimate.

The quartic construction is based on the cubic–rational family in [1, Section 5]. See also [5] for recent related work on cubic–rational approximation. We verify directly the polynomial identity responsible for the approximation and use it to construct reducible quartics.

Our second result is unconditional.

Theorem 2. *One has*

$$e_{\mathrm{red}}(5) \geq \frac{22}{7}.$$

The proof uses explicit quartic polynomials F_k and primitive linear polynomials L_k with resultant equal to 1. To the best of our knowledge, neither this identity nor the resulting root-separation construction has appeared previously.

The two constructions follow the same principle. A factor of degree m has height of order T, a linear factor has height of order T^{ρ}, and their resultant remains bounded. When the two factors have exactly one common limiting root, simple in both, this produces the exponent

$$\frac{m\rho + 1}{\rho + 1}.$$

The quartic and quintic constructions correspond to $(m, \rho) = (3, 3)$ and $(m, \rho) = (4, 5/2)$, respectively.

We also prove the general structural estimate

$$\lambda_{n-1,1} \leq e_{\mathrm{red}}(n) \leq \max\{n - 2, \lambda_{n-1,1}\} \qquad (n \geq 3),$$

where $\lambda_{n-1,1}$ measures approximation between real algebraic numbers of degree $n - 1$ and rational numbers, with both heights allowed to vary. Thus an exponent larger than $n - 2$ can arise only from a real root of an irreducible factor of degree $n - 1$ approaching the rational root of a linear factor.

The paper is organized as follows. In Section 2 we prove the height, resultant, and separation estimates needed later and establish the structural reduction stated above. The degree-four and degree-five constructions are given in Sections 3 and 4, respectively.

2 Heights, resultants, and structural reductions

For a nonconstant polynomial $A(x) = a_r \prod_{i=1}^{r}(x - \alpha_i) \in \mathbb{C}[x]$, its Mahler measure is

$$M(A) = |a_r| \prod_{i=1}^{r} \max\{1, |\alpha_i|\}.$$

If

$$B(x) = b_s \prod_{j=1}^{s}(x - \beta_j) \in \mathbb{C}[x],$$

we use the standard formulas

$$\mathrm{Disc}(A) = a_r^{2r-2} \prod_{1 \le i < j \le r} (\alpha_i - \alpha_j)^2$$

and

$$\mathrm{Res}(A, B) = a_r^s b_s^r \prod_{i=1}^{r} \prod_{j=1}^{s} (\alpha_i - \beta_j) = a_r^s \prod_{i=1}^{r} B(\alpha_i).$$

In particular,

$$\mathrm{Res}(B, A) = (-1)^{rs} \mathrm{Res}(A, B).$$

For $A, B \in \mathbb{Z}[x]$, the discriminant and the resultant are integers. Moreover, $\mathrm{Disc}(A) = 0$ if and only if A has a multiple root, while $\mathrm{Res}(A, B) = 0$ if and only if A and B have a common root in $\mathbb{C}$.

All constants in this section depend only on the indicated degrees. For standard background on heights, Mahler measure, discriminants, and resultants, see [2, Appendix A].

Lemma 1 (Height of a product)**.** *If $A, B \in \mathbb{C}[x]$ have degrees r, s, then*

$$H(AB) \asymp_{r,s} H(A)H(B).$$

Proof. By Jensen's formula,

$$\log M(A) = \frac{1}{2\pi} \int_0^{2\pi} \log |A(e^{it})| \, dt.$$

Hence, by Jensen's inequality applied to the concave function log, followed by Parseval's identity,

$$\begin{aligned} M(A)^2 &= \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log |A(e^{it})|^2 \, dt \right) \\ &\le \frac{1}{2\pi} \int_0^{2\pi} |A(e^{it})|^2 \, dt = \sum_{j=0}^{r} |a_j|^2 \le (r+1) H(A)^2. \end{aligned}$$

Thus $M(A) \le \sqrt{r+1}\, H(A)$. On the other hand, every coefficient of A is an elementary symmetric function of its roots, and therefore $H(A) \le 2^r M(A)$. Since $M(AB) = M(A)M(B)$, applying these two estimates to A, B, and AB gives

$$H(AB) \le 2^{r+s} \sqrt{(r+1)(s+1)}\, H(A)H(B), \qquad H(A)H(B) \le 2^{r+s} \sqrt{r+s+1}\, H(AB).$$

This proves the assertion. □

Lemma 2 (Mahler's separation estimate)**.** *Let $A \in \mathbb{Z}[x]$ be separable of degree $r \ge 2$. Then*

$$\mathrm{sep}(A) \gg_r H(A)^{1-r}.$$

Proof. Write $A(x) = a_r \prod_{i=1}^{r}(x - \alpha_i)$ and choose α_1, α_2 with $|\alpha_1 - \alpha_2| = \operatorname{sep}(A)$. Since A is separable, its discriminant is a nonzero integer,

$$1 \le |\operatorname{Disc}(A)| = |a_r|^{2r-2} \prod_{i<j} |\alpha_i - \alpha_j|^2.$$

For every i, j, $|\alpha_i - \alpha_j| \le 2\max\{1, |\alpha_i|\}\max\{1, |\alpha_j|\}$. Bounding all factors except $|\alpha_1 - \alpha_2|$ in this way gives

$$1 \ll_r \operatorname{sep}(A)^2 M(A)^{2r-2}.$$

The result follows from the inequality $M(A) \le \sqrt{r+1}\, H(A)$ proved above. □

Lemma 3 (Roots of distinct factors)**.** *Let $A, B \in \mathbb{Z}[x]$ be primitive irreducible nonassociate polynomials of degrees r, s, respectively, and let α and β be roots of A and B. Then*

$$|\alpha - \beta| \gg_{r,s} H(A)^{-s} H(B)^{-r}.$$

Proof. Since A and B are nonassociate irreducible polynomials, their resultant is a nonzero integer. Writing all roots of A and B as α_i and β_j, and isolating the factor $|\alpha - \beta|$, we obtain

$$1 \le |\operatorname{Res}(A, B)| \ll_{r,s} |\alpha - \beta| M(A)^s M(B)^r.$$

The claimed estimate follows from $M(A) \ll_r H(A)$ and $M(B) \ll_s H(B)$. □

The product formula for the resultant has the following useful consequence for polynomial sequences whose normalized limits have a unique common simple root. The next proposition gives a precise form of this observation and the corresponding exponent calculation.

Proposition 1 (Roots approaching a common limit)**.** *Let $r, s \ge 1$, and let $a, b \ge 0$ with $a + b > 0$. Let $(T_k)_{k\ge1}$ be a sequence of positive real numbers tending to infinity, and, for every $k \ge 1$, let $A_k, B_k \in \mathbb{Z}[x]$ satisfy $\deg A_k = r$ and $\deg B_k = s$. Assume*

$$H(A_k) = T_k^{a+o(1)}, \qquad H(B_k) = T_k^{b+o(1)}.$$

Suppose that $A_k/H(A_k)$ and $B_k/H(B_k)$ converge coefficientwise to polynomials A_∞, B_∞ of degrees r, s. Assume that these limiting polynomials have exactly one common root γ, which is simple in both. Then, for all sufficiently large k, there are unique roots α_k of A_k and β_k of B_k in a fixed sufficiently small neighbourhood of γ, and

$$\alpha_k \longrightarrow \gamma, \qquad \beta_k \longrightarrow \gamma.$$

If, for all sufficiently large k, $\operatorname{Res}(A_k, B_k) \ne 0$ and

$$|\operatorname{Res}(A_k, B_k)| = T_k^{\sigma+o(1)}$$

for some $\sigma \in \mathbb{R}$, then

$$|\alpha_k - \beta_k| = T_k^{-sa-rb+\sigma+o(1)}.$$

Consequently,

$$e_{\mathrm{red}}(r+s) \ge \frac{sa + rb - \sigma}{a+b},$$

provided $A_k B_k$ is separable for all sufficiently large k.

Proof. Choose a closed disc D centred at γ which contains no other root of A_∞ or B_∞. Since γ is a simple root of each limiting polynomial, continuity of roots shows that, for all sufficiently large k, each of A_k and B_k has exactly one root in D. Denote these roots by α_k and β_k. Then $\alpha_k, \beta_k \to \gamma$.

Label the remaining roots so that

$$A_k(x) = a_{r,k} \prod_{i=1}^{r} (x - \alpha_{i,k}), \qquad B_k(x) = b_{s,k} \prod_{j=1}^{s} (x - \beta_{j,k}),$$

where $\alpha_{1,k} = \alpha_k$ and $\beta_{1,k} = \beta_k$, and so that each root converges, with multiplicity, to a root of the corresponding limiting polynomial. Since the limits have degrees r and s, their leading coefficients are nonzero, and hence

$$|a_{r,k}| = T_k^{a+o(1)}, \qquad |b_{s,k}| = T_k^{b+o(1)}.$$

The product formula for the resultant gives

$$|\operatorname{Res}(A_k, B_k)| = |a_{r,k}|^s |b_{s,k}|^r |\alpha_k - \beta_k| \prod_{\substack{1 \le i \le r,\ 1 \le j \le s \\ (i,j) \ne (1,1)}} |\alpha_{i,k} - \beta_{j,k}|.$$

The final product converges to a positive constant, because γ is the only common root of the limiting polynomials and is simple in each. Therefore

$$|\operatorname{Res}(A_k, B_k)| = T_k^{sa+rb+o(1)} |\alpha_k - \beta_k|.$$

Comparison with $|\operatorname{Res}(A_k, B_k)| = T_k^{\sigma+o(1)}$ gives the asserted estimate for $|\alpha_k - \beta_k|$.

By Lemma 1,

$$H(A_k B_k) = T_k^{a+b+o(1)} \longrightarrow \infty,$$

whereas $\operatorname{sep}(A_k B_k) \le |\alpha_k - \beta_k|$. Since $A_k B_k$ is a separable reducible integer polynomial of degree $r + s$ for all sufficiently large k, the claimed lower bound for $e_{\mathrm{red}}(r + s)$ follows. □

Corollary 1. *Let $m \ge 1$. For every $k \ge 1$, let $F_k \in \mathbb{Z}[x]$ be a polynomial of degree m, and let $L_k(x) = q_k x - p_k$ with $q_k \ne 0$. Suppose that there exist $\rho \ge 0$ and a sequence $(T_k)_{k \ge 1}$ of positive real numbers tending to infinity such that*

$$H(F_k) = T_k^{1+o(1)}, \qquad H(L_k) = T_k^{\rho+o(1)}.$$

Assume that $F_k/H(F_k)$ and $L_k/H(L_k)$ satisfy the limiting-polynomial hypotheses of Proposition 1, that

$$0 < |\operatorname{Res}(F_k, L_k)| \ll 1,$$

and that $F_k L_k$ is separable for all sufficiently large k. Then

$$e_{\mathrm{red}}(m + 1) \ge \Phi_m(\rho), \qquad \Phi_m(\rho) := \frac{m\rho + 1}{\rho + 1}.$$

Proof. The bounded nonzero resultants satisfy $|\operatorname{Res}(F_k, L_k)| = T_k^{o(1)}$. Apply Proposition 1 with

$$r = m, \qquad s = 1, \qquad a = 1, \qquad b = \rho, \qquad \sigma = 0.$$

The resulting exponent is

$$\frac{1 + m\rho}{1 + \rho} = \Phi_m(\rho). \qquad \square$$

For $m > 1$, the function Φ_m is increasing, since

$$\Phi_m(\rho) = m - \frac{m-1}{\rho+1}.$$

Thus the exponent increases when the height of the linear factor grows faster relative to the height of the degree-m factor. The constructions below realize

$$\Phi_3(3) = \frac{5}{2}, \qquad \Phi_4\Big(\frac{5}{2}\Big) = \frac{22}{7}.$$

For an algebraic number ξ, let $H(\xi)$ be the height of its primitive minimal polynomial in $\mathbb{Z}[x]$. For $r = p/q \in \mathbb{Q}$ in lowest terms, with $q > 0$, this gives $H(r) = \max\{|p|, q\}$. For $m \geq 2$, define

$$\lambda_{m,1} = \limsup_{\substack{\xi\in\mathbb{R},\ \deg\xi=m,\ r\in\mathbb{Q} \\ H(\xi)H(r)\to\infty}} \frac{-\log|\xi - r|}{\log(H(\xi)H(r))}.$$

Proposition 2 (Structural reduction)**.** *For every $n \geq 3$,*

$$\lambda_{n-1,1} \leq e_{\mathrm{red}}(n) \leq \max\{n-2, \lambda_{n-1,1}\}. \tag{2.1}$$

Consequently, if $\lambda_{n-1,1} \geq n-2$, then

$$e_{\mathrm{red}}(n) = \lambda_{n-1,1}.$$

Proof. For the lower bound, let A be the primitive minimal polynomial of a real algebraic number ξ of degree $n-1$, and let $r = p/q$ be in lowest terms, with $q > 0$. The polynomial $A(x)(qx-p)$ is primitive by Gauss's lemma and reducible. It is separable since A is separable and, being irreducible of degree $n-1 \geq 2$, has no rational root; hence A and $qx-p$ have no common root. Moreover, Lemma 1 gives

$$H\Big(A(x)(qx-p)\Big) \asymp_n H(\xi)H(r).$$

Its root separation is at most $|\xi - r|$. Moreover, Lemma 3 and the elementary bounds

$$|\xi| \leq H(\xi) + 1, \qquad |r| \leq H(r)$$

show that

$$|\log|\xi - r|| = O_n(\log(H(\xi)H(r))).$$

Together with the height estimate above, this gives

$$\frac{-\log|\xi-r|}{\log H\Big(A(x)(qx-p)\Big)} = \frac{-\log|\xi-r|}{\log(H(\xi)H(r))} + o(1)$$

as $H(\xi)H(r) \to \infty$. Taking the limsup gives $\lambda_{n-1,1} \leq e_{\mathrm{red}}(n)$.

For the upper bound, let (P_k) be an arbitrary sequence of primitive separable reducible polynomials of degree n with $H(P_k) \to \infty$, and put

$$\kappa_k := \frac{-\log \mathrm{sep}(P_k)}{\log H(P_k)}.$$

By Lemma 2, the sequence (κ_k) is bounded above. If $\limsup_{k\to\infty} \kappa_k = -\infty$, there is nothing to prove. Otherwise, after passing to a subsequence, we may assume that κ_k

converges to $\limsup_{j\to\infty}\kappa_j$. For each k, choose roots $\alpha_k \neq \beta_k$ with $|\alpha_k - \beta_k| = \mathrm{sep}(P_k)$, and write

$$P_k = \prod_{\ell=1}^{N_k} A_{\ell,k},$$

where the $A_{\ell,k}$ are pairwise nonassociate primitive irreducible polynomials. Since P_k is separable, these factors are distinct. For fixed n, repeated use of Lemma 1 gives

$$\log H(P_k) = \sum_{\ell=1}^{N_k} \log H(A_{\ell,k}) + O_n(1). \tag{2.2}$$

There are only finitely many possibilities for the factor degrees, for whether α_k and β_k lie in the same factor or in distinct factors, and for the degrees of the relevant factor or factors. Passing to a further subsequence, we may assume that these data are fixed. The sequence (κ_k) has the same limit along this further subsequence.

Suppose first that α_k and β_k belong to the same factor A_k, of degree d. Since A_k is a proper factor, $d \leq n-1$. By Lemma 2 and (2.2),

$$-\log|\alpha_k - \beta_k| \leq (d-1)\log H(A_k) + O_n(1) \leq (n-2)\log H(P_k) + O_n(1).$$

Suppose next that the two roots belong to distinct factors A_k and B_k, of degrees r and s. By Lemma 3 and (2.2),

$$\begin{aligned} -\log|\alpha_k - \beta_k| &\leq s\log H(A_k) + r\log H(B_k) + O_n(1) \\ &\leq \max\{r,s\}\log H(P_k) + O_n(1). \end{aligned}$$

Unless $\{r,s\} = \{n-1,1\}$, one has $\max\{r,s\} \leq n-2$, and hence $\kappa_k \leq n-2+o(1)$.

It remains to consider a factorization

$$P_k = A_k L_k,$$

where A_k is primitive irreducible of degree $n-1$, L_k is primitive linear, and the closest pair consists of a root α_k of A_k and the rational root r_k of L_k. If α_k is nonreal, then $\overline{\alpha_k}$ is another root of A_k and

$$\mathrm{sep}(A_k) \leq |\alpha_k - \overline{\alpha_k}| = 2|\operatorname{Im}\alpha_k| \leq 2|\alpha_k - r_k|.$$

Mahler's estimate and $H(A_k) \ll_n H(P_k)$ therefore give $\kappa_k \leq n-2+o(1)$ along the indices for which α_k is nonreal.

Finally, suppose that α_k is real. Since A_k and L_k are the primitive minimal polynomials of α_k and r_k, respectively,

$$H(P_k) \asymp_n H(\alpha_k)H(r_k).$$

The product on the right tends to infinity. As in the lower-bound argument,

$$|\log|\alpha_k - r_k|| = O_n(\log(H(\alpha_k)H(r_k))),$$

and therefore

$$\kappa_k = \frac{-\log|\alpha_k - r_k|}{\log H(P_k)} = \frac{-\log|\alpha_k - r_k|}{\log(H(\alpha_k)H(r_k))} + o(1).$$

The definition of $\lambda_{n-1,1}$ now gives an upper bound $\lambda_{n-1,1}$ for this case.

Since (κ_k) along the subsequence under consideration converges to the limsup of the original sequence, the preceding estimates prove the upper bound in (2.1). □

3 Reducible quartics

By Corollary 1, sequences of cubic and linear polynomials with heights $T_k^{1+o(1)}$ and $T_k^{3+o(1)}$, respectively, where $T_k \to \infty$, and with bounded nonzero resultants yield the lower bound $\Phi_3(3) = 5/2$. Badziahin's cubic–rational family [1, Section 5] has precisely these features. We now verify them directly.

Define integers u_k, v_k by

$$v_k + u_k\sqrt{2} = (1+\sqrt{2})^{k+1} \qquad (k \geq 0).$$

Equivalently,

$$u_0 = 1, \quad u_1 = 2, \quad u_{k+1} = 2u_k + u_{k-1}, \qquad v_0 = 1, \quad v_1 = 3, \quad v_{k+1} = 2v_k + v_{k-1}.$$

Taking norms gives

$$2u_k^2 - v_k^2 = (-1)^k.$$

This identity implies $\gcd(u_k, v_k) = 1$, and $v_k/u_k \to \sqrt{2}$.

Set

$$C_k(x) = u_k x^3 + (4u_k + v_k)x^2 - 2(u_k + 2v_k)x - 2v_k$$

and

$$p_k = 4u_k^3 + 16u_k^2 v_k + 14u_k v_k^2 + 4v_k^3,$$
$$q_k = 8u_k^3 + 14u_k^2 v_k + 8u_k v_k^2 + v_k^3.$$

The coefficient of x is the corrected version of [1, formula (18)], and the identity below includes the factor $(-1)^k$ omitted from [1, formula (20)].

Lemma 4. *Replace u_k, v_k by indeterminates u, v in the definitions of C_k, p_k, q_k, and denote the resulting quantities by C, p, q. Then*

$$q^3 C(p/q) = 2(2u^2 - v^2)^5.$$

Consequently,

$$q_k^3 C_k(p_k/q_k) = 2(-1)^k.$$

Proof. After clearing the denominator, the left-hand side is

$$up^3 + (4u + v)p^2 q - 2(u + 2v)pq^2 - 2vq^3.$$

Substitution of p and q, followed by collection of homogeneous terms, gives

$$2\Big(32u^{10} - 80u^8 v^2 + 80u^6 v^4 - 40u^4 v^6 + 10u^2 v^8 - v^{10}\Big),$$

which is $2(2u^2 - v^2)^5$. The second assertion follows from $2u_k^2 - v_k^2 = (-1)^k$. □

Lemma 5. *For every even k, the polynomial C_k is primitive and irreducible over $\mathbb{Q}$. Moreover,*

$$H(C_k) \asymp v_k, \qquad H(q_k x - p_k) \asymp v_k^3, \qquad \frac{p_k}{q_k} \longrightarrow \sqrt{2}.$$

Furthermore, coefficientwise,

$$\frac{C_k}{u_k} \longrightarrow g_3(x) := x^3 + (4 + \sqrt{2})x^2 - (2 + 4\sqrt{2})x - 2\sqrt{2},$$

and $\sqrt{2}$ is a simple root of g_3.

Proof. Since $\gcd(u_k, v_k) = 1$, one has $\gcd(u_k, 4u_k + v_k) = 1$, so C_k is primitive. Reducing the two recurrences modulo 3 shows that both sequences have period 8. At the even indices, the pairs (u_k, v_k) cycle through

$$(1,1),\ (2,1),\ (2,2),\ (1,2).$$

The corresponding reductions of C_k, up to nonzero scalar multiples, are

$$x^3 - x^2 + 1, \quad -x^3 + x + 1, \quad -x^3 + x^2 - 1, \quad x^3 - x - 1.$$

None has a root in $\mathbb{F}_3$, so each is irreducible. Hence C_k is irreducible over $\mathbb{Q}$.

Since $C_k/u_k \to g_3$ coefficientwise, $H(C_k)/u_k \to H(g_3) > 0$; in particular, $H(C_k) \asymp u_k \asymp v_k$. Since $v_k/u_k \to \sqrt{2}$, dividing the homogeneous formulas for p_k and q_k by u_k^3 gives $p_k/u_k^3 \to 32 + 24\sqrt{2}$ and $q_k/u_k^3 \to 24 + 16\sqrt{2}$. Consequently, $H(q_k x - p_k) \asymp u_k^3 \asymp v_k^3$ and $p_k/q_k \to \sqrt{2}$.

Finally,

$$g_3(x) = (x - \sqrt{2})\Big(x^2 + (4 + 2\sqrt{2})x + 2\Big),$$

and

$$g_3'(\sqrt{2}) = 8 + 4\sqrt{2} \neq 0. \qquad \square$$

Proof. (Proof of the unconditional part of Theorem 1) Restrict to even k and put

$$L_k(x) = q_k x - p_k.$$

By Lemma 4 and the symmetry of the resultant,

$$|\operatorname{Res}(C_k, L_k)| = 2.$$

Moreover,

$$H(C_k) = v_k^{1+o(1)}, \qquad H(L_k) = v_k^{3+o(1)}.$$

The convergence in Lemma 5 implies

$$\frac{C_k}{H(C_k)} \longrightarrow \frac{g_3}{H(g_3)}.$$

Since $p_k/q_k \to \sqrt{2}$ and $p_k, q_k > 0$, one also has

$$\frac{L_k}{H(L_k)} \longrightarrow \frac{x}{\sqrt{2}} - 1.$$

The two limiting polynomials have exactly one common root, namely $\sqrt{2}$, and it is simple in both. Thus Proposition 1, with $T_k = v_k$, $r = 3$, $s = 1$, $a = 1$, $b = 3$, and $\sigma = 0$, gives a root ξ_k of C_k such that

$$\xi_k \longrightarrow \sqrt{2}, \qquad \left|\xi_k - \frac{p_k}{q_k}\right| = v_k^{-10+o(1)}.$$

The root ξ_k is real for all sufficiently large k, since otherwise its conjugate would be a second root near the simple real root $\sqrt{2}$.

The identity in Lemma 4 also implies $\gcd(p_k, q_k) = 1$, since a common divisor would have its third power dividing 2. Hence the product $(q_k x - p_k)C_k(x)$ is primitive by Gauss's

lemma. It is separable, since C_k is irreducible over $\mathbb{Q}$ and hence separable, and its resultant with L_k is nonzero. By Lemma 1,

$$H\Big((q_k x - p_k)C_k(x)\Big) \asymp H(L_k)H(C_k) = v_k^{4+o(1)}.$$

Since the displayed cross-distance is one of the root distances of this product, Proposition 1 gives

$$e_{\mathrm{red}}(4) \geq \frac{10}{4} = \frac{5}{2}.$$

The same family gives

$$H(\xi_k) = H(C_k) = v_k^{1+o(1)}, \qquad H(p_k/q_k) = v_k^{3+o(1)},$$

and therefore $\lambda_{3,1} \geq 5/2$. Since this is larger than 2, Proposition 2 also gives the unconditional identity

$$e_{\mathrm{red}}(4) = \lambda_{3,1}. \qquad \square$$

We now record the conditional upper bound. We use the following weak form of Hall's conjecture, as in [1, Conjecture (Hall)]: for every $\epsilon > 0$, the inequality

$$0 < |X^3 - Y^2| < |X|^{1/2-\epsilon}$$

has only finitely many integer solutions (X, Y).

Proposition 3 (Badziahin's conditional estimate)**.** *Assume Hall's conjecture. Then*

$$\lambda_{3,1} \leq \frac{5}{2}.$$

Proof. Fix $\epsilon \in (0, 1/2)$. On p. 2 of [1], $D_{3,1}$ is defined as the set of pairs (μ, ν) for which there is a constant $c > 0$ such that

$$|\xi - r| \geq cH(\xi)^{-\mu}H(r)^{-\nu}$$

for every real cubic irrational ξ and every rational r. The paper also observes that requiring the inequality for all pairs or for all but finitely many pairs gives the same set $D_{3,1}$.

By [1, Theorem 3, formula (4)], for every θ with $1/2 + \epsilon \leq \theta \leq 1$, one has

$$\left(\theta + \frac{2(1-\theta)}{1/2 - \epsilon},\ 2 + \theta\right) \in D_{3,1}.$$

Taking $\theta = 1/2 + \epsilon$, both exponents are $5/2 + \epsilon$, since $\theta + 2(1-\theta)/(1/2-\epsilon) = 5/2 + \epsilon$ and $2 + \theta = 5/2 + \epsilon$. Thus there is a constant $c_\epsilon > 0$ such that

$$|\xi - r| \geq c_\epsilon H(\xi)^{-5/2-\epsilon}H(r)^{-5/2-\epsilon}$$

for every real cubic irrational ξ and every rational r. By the definition of $\lambda_{3,1}$, this implies $\lambda_{3,1} \leq 5/2 + \epsilon$. Letting $\epsilon \to 0$ proves the proposition. $\square$

Proof. (Completion of the proof of Theorem 1) Assume Hall's conjecture. Then Proposition 3 and the explicit construction give $\lambda_{3,1} = 5/2$. The last assertion of Proposition 2 then yields $e_{\mathrm{red}}(4) = 5/2$. $\square$

4 Reducible quintics

For $m = 4$, Corollary 1 gives the exponent

$$\Phi_4(\rho) = \frac{4\rho + 1}{\rho + 1}.$$

The value $22/7$ corresponds to $\rho = 5/2$. We therefore seek quartic polynomials of height T_k, linear polynomials of height $T_k^{5/2}$, and a bounded nonzero resultant, such that their normalized limiting polynomials have exactly one common real root, simple in both. The construction below has resultant equal to 1.

For $k \geq 0$, define positive integers c_k, d_k by

$$c_k + d_k\sqrt{2} = (1 + \sqrt{2})^{12k+1}.$$

Taking norms gives

$$c_k^2 - 2d_k^2 = -1. \tag{4.1}$$

We shall use the congruences

$$c_k \equiv d_k \equiv (-1)^k \pmod 3, \qquad c_k \equiv d_k \equiv (-1)^k \pmod{11}. \tag{4.2}$$

Indeed,

$$(1 + \sqrt{2})^{12} = 19601 + 13860\sqrt{2} \equiv -1 \pmod{33}$$

in $\mathbb{Z}[\sqrt{2}]$, and (4.2) follows after taking the kth power and multiplying by $1 + \sqrt{2}$.

Define

$$\begin{aligned} F_k(x) = {} & (15c_k^2 + 16c_kd_k + 18d_k^2)x^4 \\ & - 32(2c_k^2 + 3c_kd_k + 4d_k^2)x^3 \\ & + 12(-c_k^2 - 16c_kd_k + 2d_k^2)x^2 \\ & + 192c_kd_kx - 12(3c_k^2 + 16c_kd_k + 10d_k^2). \end{aligned}$$

Thus the coefficients of F_k are homogeneous quadratic forms in c_k, d_k. Define rational numbers a_k, b_k by

$$a_k + b_k\sqrt{2} = \frac{1}{3}\Big((7 + 5\sqrt{2})(c_k + d_k\sqrt{2})^5 - (2 - \sqrt{2})(c_k - d_k\sqrt{2})\Big). \tag{4.3}$$

They are in fact integers. By (4.2), the numerator in (4.3) is congruent modulo 3 in $\mathbb{Z}[\sqrt{2}]$ to

$$(-1)^k\Big((7 + 5\sqrt{2})(1 + \sqrt{2})^5 - (2 - \sqrt{2})(1 - \sqrt{2})\Big).$$

The expression in parentheses is

$$573 + 411\sqrt{2} = 3(191 + 137\sqrt{2}),$$

so $a_k, b_k \in \mathbb{Z}$. Moreover,

$$3b_k = 5c_k^5 + 35c_k^4d_k + 100c_k^3d_k^2 + 140c_k^2d_k^3 + 100c_kd_k^4 + 28d_k^5 + c_k + 2d_k > 0.$$

Put

$$L_k(x) = b_kx + a_k.$$

Theorem 3 (Quartic identity)**.** *For every $k \geq 0$,*

$$b_k^4 F_k\left(-\frac{a_k}{b_k}\right) = 1.$$

Consequently,

$$\operatorname{Res}(L_k, F_k) = 1,$$

$\gcd(a_k, b_k) = 1$*, and both L_k and F_k are primitive.*

Proof. Throughout this proof, write $s = \sqrt{2}$, and put

$$X = (1+s)^4(c_k + d_k s)^6, \qquad E = 81 b_k^4 F_k\left(-\frac{a_k}{b_k}\right).$$

Using $7 + 5s = (1+s)^3$, $(1+s)(2-s) = s$, and (4.1), the definition (4.3) gives

$$3(1+s)(c_k + d_k s)(a_k + b_k s) = (1+s)(7+5s)(c_k + d_k s)^6 - (1+s)(2-s)(c_k^2 - 2d_k^2) = X + s.$$

The norm of X is 1. Conjugating the last identity therefore gives

$$3(1-s)(c_k - d_k s)(a_k - b_k s) = X^{-1} - s.$$

We now evaluate the homogenized quartic at $(-a_k, b_k)$. Substitute

$$2a_k = (a_k + b_k s) + (a_k - b_k s), \qquad 2sb_k = (a_k + b_k s) - (a_k - b_k s)$$

and use the preceding two identities. Expanding and simplifying the homogenized value gives

$$\begin{aligned} E = {} & 2sX^{-1}(X+s)^4 + 2(X+s)^3(X^{-1} - s) \\ & - 3(X+s)^2(X^{-1} - s)^2 + 2(X+s)(X^{-1} - s)^3 - 2sX(X^{-1} - s)^4. \end{aligned}$$

Expanding and collecting the powers of X gives

$$E = (2 - s^2)(X^2 + X^{-2}) + 2s(s^2 - 2)(s^2 + 3)(X^{-1} - X) + 9s^4 + 24s^2 - 3.$$

Since $s^2 = 2$, the terms containing X vanish and the remaining constant is 81. Hence $E = 81$, which proves

$$b_k^4 F_k\left(-\frac{a_k}{b_k}\right) = 1.$$

If an integer ℓ divides both a_k and b_k, then ℓ^4 divides the left-hand side of the last identity; hence $|\ell| = 1$. Thus $\gcd(a_k, b_k) = 1$, and L_k is primitive. Every common divisor of the coefficients of F_k also divides the same value 1, so F_k is primitive. Finally, the formula for the resultant of a linear and a quartic polynomial gives

$$\operatorname{Res}(L_k, F_k) = b_k^4 F_k(-a_k/b_k) = 1. \qquad \square$$

The coefficients of F_k can be recovered systematically. Start with a general quartic whose coefficients are homogeneous quadratic forms in c_k, d_k, and put $t = c_k + d_k\sqrt{2}$. By (4.1), $c_k - d_k\sqrt{2} = -t^{-1}$. Expressing the left-hand side of the identity just proved in terms of t gives a Laurent polynomial whose coefficients depend linearly on the coefficients of the five quadratic forms. Requiring all nonconstant Laurent coefficients to vanish and the constant term to be 1 gives a linear system with a unique solution, namely the polynomial F_k defined above.

Lemma 6. *For every $k \geq 0$, the polynomial F_k is irreducible over $\mathbb{Q}$.*

Proof. By (4.2), $c_k \equiv d_k \equiv (-1)^k \pmod{11}$. Since every coefficient of F_k is a homogeneous quadratic form in c_k, d_k, the sign disappears and

$$F_k(x) \equiv 5h(x) \pmod{11}, \qquad h(x) = x^4 + 4x^3 + 8x^2 + x + 3.$$

If h were reducible over $\mathbb{F}_{11}$, it would have an irreducible factor of degree 1 or 2, and that factor would divide $x^{11^2} - x$. Binary exponentiation modulo $h(x)$ in $\mathbb{F}_{11}[x]$ gives

$$x^{121} - x \equiv x^3 + x^2 + 4x + 5 \pmod{h(x)}.$$

The Euclidean algorithm in $\mathbb{F}_{11}[x]$ then gives the Bézout identity

$$(5x^2 + 5x + 3)h(x) + (-5x^3 + 2x^2 - x + 5)(x^3 + x^2 + 4x + 5) \equiv 1 \pmod{11}.$$

Thus $\gcd(h, x^{121} - x) = 1$, so h is irreducible over $\mathbb{F}_{11}$. Since F_k is primitive by Theorem 3 and its reduction has degree 4, the standard reduction criterion proves that F_k is irreducible over $\mathbb{Q}$. □

Lemma 7 (Asymptotics of the quartic family)**.** *As $k \to \infty$,*

$$H(F_k) \asymp c_k^2 \asymp (1+\sqrt{2})^{24k}, \qquad a_k \asymp b_k \asymp c_k^5 \asymp (1+\sqrt{2})^{60k}, \qquad \frac{a_k}{b_k} \longrightarrow \sqrt{2}.$$

The normalized polynomials F_k/c_k^2 converge coefficientwise to

$$g_4(x) = (24 + 8\sqrt{2})x^4 - (128 + 48\sqrt{2})x^3 - 96\sqrt{2}\,x^2 + 96\sqrt{2}\,x - 96 - 96\sqrt{2}.$$

Moreover, $g_4(-\sqrt{2}) = 0$ and

$$g_4'(-\sqrt{2}) = -512 - 384\sqrt{2} \neq 0.$$

Proof. From the definition of c_k, d_k,

$$c_k = \frac{(1+\sqrt{2})^{12k+1} + (1-\sqrt{2})^{12k+1}}{2}, \qquad d_k = \frac{(1+\sqrt{2})^{12k+1} - (1-\sqrt{2})^{12k+1}}{2\sqrt{2}}.$$

Since $|1 - \sqrt{2}| < 1$, it follows that

$$c_k \sim \frac{1}{2}(1+\sqrt{2})^{12k+1}, \qquad d_k \sim \frac{1}{2\sqrt{2}}(1+\sqrt{2})^{12k+1},$$

so $d_k/c_k \to 1/\sqrt{2}$. Dividing the coefficients of F_k by c_k^2 gives the stated coefficientwise limit. Since $g_4 \neq 0$,

$$\frac{H(F_k)}{c_k^2} \longrightarrow H(g_4) > 0.$$

In (4.3), the first term has order $(1+\sqrt{2})^{60k}$ and the second has order $(1+\sqrt{2})^{-12k}$. Conjugating (4.3) shows that $a_k - b_k\sqrt{2}$ has order $(1+\sqrt{2})^{12k}$. Consequently,

$$a_k \sim \frac{1}{6}(7 + 5\sqrt{2})(c_k + d_k\sqrt{2})^5,$$

$$b_k \sim \frac{1}{6\sqrt{2}}(7 + 5\sqrt{2})(c_k + d_k\sqrt{2})^5.$$

This proves the remaining asymptotic estimates and $a_k/b_k \to \sqrt{2}$. The two assertions about g_4 follow by direct substitution. □

Proof. (Proof of Theorem 2) By Theorem 3,

$$\mathrm{Res}(L_k, F_k) = 1.$$

Put

$$T_k = c_k^2.$$

By Lemma 7,

$$H(F_k) = T_k^{1+o(1)}, \qquad H(L_k) = T_k^{5/2+o(1)}.$$

Moreover,

$$\frac{F_k}{H(F_k)} \longrightarrow \frac{g_4}{H(g_4)}.$$

Since $a_k/b_k \to \sqrt{2} > 1$, one has $H(L_k) = a_k$ for all sufficiently large k, and therefore

$$\frac{L_k}{H(L_k)} \longrightarrow \frac{x}{\sqrt{2}} + 1.$$

The limiting polynomials have exactly one common root, namely $-\sqrt{2}$, and this root is simple in both. Applying Proposition 1 gives a root α_k of F_k such that

$$\alpha_k \longrightarrow -\sqrt{2}, \qquad \left|\alpha_k + \frac{a_k}{b_k}\right| = T_k^{-11+o(1)}.$$

The root α_k is real for all sufficiently large k, since otherwise its conjugate would give a second root of F_k near the simple root $-\sqrt{2}$ of g_4.

The product $L_k F_k$ is primitive by Gauss's lemma. Lemma 6 shows that F_k is irreducible over $\mathbb{Q}$, hence separable. Since the resultant of the two factors is 1, the product is separable. By Lemma 1,

$$H(L_k F_k) \asymp H(L_k)H(F_k) = T_k^{7/2+o(1)}.$$

Hence

$$e_{\mathrm{red}}(5) \geq \frac{11}{7/2} = \frac{22}{7}.$$

Since F_k is the primitive minimal polynomial of α_k and L_k is the primitive minimal polynomial of $-a_k/b_k$, the same construction gives

$$H(\alpha_k) = H(F_k) = T_k^{1+o(1)}, \qquad H(-a_k/b_k) = H(L_k) = T_k^{5/2+o(1)},$$

so $\lambda_{4,1} \geq 22/7 > 3$. By Proposition 2,

$$e_{\mathrm{red}}(5) = \lambda_{4,1} \geq \frac{22}{7}.$$

No matching upper bound for $\lambda_{4,1}$ is obtained here. □

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