Real and Schneider's p-adic continued fractions of rational numbers

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Abstract

Fix a prime p. We consider positive rational p-adic units with finite Schneider's p-adic continued fraction expansion and compare this expansion with the ordinary real continued fraction expansion. We construct examples showing that the two lengths may behave in essentially different ways: one of them may remain bounded while the other grows, and there are also families in which both grow logarithmically with the height. We also study common ordinary and Schneider's convergents. For rational numbers of height at most H, we prove a general upper bound for the maximal possible number of common convergents and a lower bound of order $\log\log H$. For $p = 2$, we also obtain a sharper construction.

1 Introduction

Schneider introduced in [9] one of the classical continued fraction algorithms in the field of p-adic numbers [8]. For rational inputs, the behaviour of the algorithm was studied by Bundschuh [2] and, more recently, in [1, 4, 5, 6, 7]. In this paper, we consider positive rational p-adic units, that is, elements of $\mathbb{Q}_{>0} \cap \mathbb{Z}_p^\times$.

Unlike real continued fractions, Schneider's p-adic continued fraction expansions of rational numbers need not terminate. Infinite expansions occur frequently in this setting, see [2, 5, 6]. Here we restrict attention to rational numbers whose Schneider's expansion is finite.

For such a rational number α, let $r(\alpha)$ denote the length of its real continued fraction expansion, written in the usual normalized form with last partial quotient at least 2, and let $s_p(\alpha)$ denote the length of its finite Schneider's p-adic continued fraction expansion. We consider two comparison problems.

The first concerns the possible simultaneous behaviour of $r(\alpha)$ and $s_p(\alpha)$ under a height bound on α. We construct explicit families of rational numbers showing that the pair $(r(\alpha), s_p(\alpha))$ may behave in rather different ways: the Schneider's length may be large while the ordinary length stays bounded, both lengths may grow logarithmically, and the Schneider's length may stay bounded while the ordinary length tends to infinity.

The second problem concerns common convergents. Write $C_p(\alpha)$ for the number of rational numbers which occur both as ordinary convergents and as Schneider's convergents of α. We prove a general upper bound for $C_p(\alpha)$ in terms of the height, and we also give constructions showing that arbitrarily many common convergents can occur.

The next section collects several auxiliary facts on Schneider's p-adic continued fractions and ordinary real continued fractions that will be used throughout the paper. In Section 3, we compare the lengths of these two expansions for rational numbers and exhibit explicit examples showing several different types of behaviour. In Section 4, we study common ordinary and Schneider's convergents, proving a general upper bound and giving explicit constructions with many such common convergents.

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2 Preliminaries

We collect the notation and a few auxiliary facts used later. Most of them are standard, and we include only some short proofs such as the elementary height bound for finite Schneider's expansions. Throughout the paper, p denotes a fixed prime.

For

$$\alpha = \frac{a}{b} \in \mathbb{Q}_{>0}, \qquad \gcd(a,b) = 1,$$

we write

$$H(\alpha) = \max\{|a|, |b|\}$$

for the naive height.

Every positive rational number α has a unique normalized finite ordinary continued fraction expansion

$$\alpha = [c_0; c_1, \dots, c_n], \qquad c_0 \in \mathbb{Z}_{\geq 0},\ c_j \in \mathbb{N}\ (j \geq 1),\ c_n \geq 2 \text{ if } n \geq 1.$$

We denote the length of such an expansion by

$$r(\alpha) = n + 1.$$

We now recall Schneider's algorithm on $\mathbb{Q}_{>0} \cap \mathbb{Z}_p^\times$, for more details see e.g. [5].

Let $\alpha \in \mathbb{Q}_{>0} \cap \mathbb{Z}_p^\times$. Starting from $\alpha_0 = \alpha$, one defines inductively digits $b_j \in \{1, \dots, p-1\}$ and exponents $a_{j+1} \geq 1$ by

$$\alpha_j \equiv b_j \pmod{p},$$

and, if $\alpha_j \neq b_j$,

$$a_{j+1} = v_p(\alpha_j - b_j), \qquad \alpha_{j+1} = \frac{p^{a_{j+1}}}{\alpha_j - b_j},$$

where v_p is the p-adic valuation normalized so that $v_p(p) = 1$.

If the algorithm terminates after the digit b_{m-1}, i.e. $\alpha_{m-1} = b_{m-1}$, we write

$$\alpha = [b_0, p^{a_1} : b_1, p^{a_2} : \dots, p^{a_{m-1}} : b_{m-1}]_p := b_0 + \cfrac{p^{a_1}}{b_1 + \cfrac{p^{a_2}}{\ddots + \frac{p^{a_{m-1}}}{b_{m-1}}}} \tag{1}$$

and call this the finite Schneider's p-adic continued fraction expansion of α with length

$$s_p(\alpha) = m.$$

For rational inputs, the alternative [2] is that the expansion becomes eventually periodic with tail

$$p-1,\ p : p-1,\ p : p-1, \dots,$$

which occurs if a complete quotient α_j becomes negative, equivalently when -1 appears as a complete quotient, see [2, 5, 6, 7, 10].

For finite Schneider's expansion (1) we use the standard convergents P_n/Q_n defined by

$$P_{-2} = 0, \quad P_{-1} = 1, \qquad Q_{-2} = 1, \quad Q_{-1} = 0,$$

and, with the convention $a_0 = 0$,

$$P_n = b_n P_{n-1} + p^{a_n} P_{n-2}, \qquad Q_n = b_n Q_{n-1} + p^{a_n} Q_{n-2} \qquad (0 \leq n \leq m-1).$$

Then

$$[b_0, p^{a_1} : b_1, \dots, p^{a_n} : b_n]_p = \frac{P_n}{Q_n} \qquad \text{for } 0 \leq n \leq m-1,$$

and all such P_n, Q_n are positive. Since $b_0 \in \{1, \dots, p-1\}$, one has $p \nmid P_0$, and the congruence

$$P_n \equiv b_n P_{n-1} \pmod{p}$$

shows inductively that $p \nmid P_n$ for all $n \geq 0$. Similarly, $p \nmid Q_n$ for $n \geq 0$. Also, as in [5], we obtain

$$P_n Q_{n-1} - P_{n-1} Q_n = (-1)^{n+1} p^{a_1 + \cdots + a_n} \qquad (n \geq 0), \tag{2}$$

and therefore

$$\gcd(P_n, Q_n) = 1 \qquad (n \geq 0).$$

Hence if

$$\alpha = [b_0, p^{a_1} : b_1, \dots, p^{a_{m-1}} : b_{m-1}]_p = \frac{P_{m-1}}{Q_{m-1}},$$

then

$$H(\alpha) = \max\{P_{m-1}, Q_{m-1}\} = P_{m-1}.$$

We use Vinogradov's symbols $\ll$, $\gg$, and $\asymp$ with implicit constants depending at most on p, unless stated otherwise.

The following simple proposition gives the maximal possible finite Schneider's length under a height bound.

Proposition 2.1. *Define a sequence $(U_m)_{m \geq 0}$ by*

$$U_0 = U_1 = 1, \qquad U_m = U_{m-1} + pU_{m-2} \quad (m \geq 2).$$

Let $\alpha \in \mathbb{Q}_{>0} \cap \mathbb{Z}_p^\times$ have finite Schneider's expansion, and suppose $s_p(\alpha) = m$. Then

$$H(\alpha) \geq U_m.$$

Equality is attained only for

$$\rho_m := [1, p : 1, p : 1, \dots, p : 1]_p = \frac{U_m}{U_{m-1}}$$

(m symbols 1). Consequently,

$$\max\{s_p(\alpha) : \alpha \in \mathbb{Q}_{>0} \cap \mathbb{Z}_p^\times,\ H(\alpha) < H,\ s_p(\alpha) < \infty\} = \max\{m \geq 1 : U_m < H\}.$$

Proof. Write

$$\alpha = [b_0, p^{a_1} : b_1, \dots, p^{a_{m-1}} : b_{m-1}]_p = \frac{P_{m-1}}{Q_{m-1}}.$$

We prove by induction that

$$P_j \geq U_{j+1}, \qquad Q_j \geq U_j \qquad (0 \leq j \leq m-1).$$

For $j = 0$ and $j = 1$ this is immediate. If the claim holds for $j-1$ and $j-2$, then, since $b_j \geq 1$ and $a_j \geq 1$,

$$P_j = b_j P_{j-1} + p^{a_j} P_{j-2} \geq P_{j-1} + pP_{j-2} \geq U_j + pU_{j-1} = U_{j+1},$$

and similarly $Q_j \geq U_j$. Hence

$$H(\alpha) = \max\{P_{m-1}, Q_{m-1}\} \geq P_{m-1} \geq U_m.$$

Equality $P_{m-1} = U_m$ is attained only if $b_j = 1$ and $a_j = 1$ for all j, and then the same recurrence gives

$$P_{m-1} = U_m, \qquad Q_{m-1} = U_{m-1}.$$

□

Remark 2.2. Let
$$\lambda_p = \frac{1+\sqrt{4p+1}}{2}, \qquad \mu_p = \frac{1-\sqrt{4p+1}}{2}.$$
Then
$$U_m = \frac{\lambda_p^{m+1} - \mu_p^{m+1}}{\lambda_p - \mu_p},$$
and thus $U_m \asymp_p \lambda_p^m$.

The next proposition is the standard Fibonacci lower bound for denominators of ordinary convergents, and is essentially Lamé's classical estimate for the Euclidean algorithm.

Proposition 2.3. *There exists an absolute constant $C > 0$ such that for every rational number α,*
$$r(\alpha) \le C \log H(\alpha) + C.$$
More precisely, if $r(\alpha) = n+1$, then
$$F_{n+1} \le H(\alpha),$$
where (F_k) denotes the Fibonacci sequence.

Proof. Denote by q_k the denominators of the convergents in the ordinary continued fraction expansion of $\alpha = [c_0; c_1, \dots, c_n]$. Write $\alpha = a/b$ in lowest terms. In its normalized ordinary continued fraction expansion, the final denominator is $q_n = b \le H(\alpha)$. Since the recurrence
$$q_k = c_k q_{k-1} + q_{k-2} \qquad (k \ge 1)$$
and the inequalities $c_k \ge 1$ imply inductively that $q_k \ge F_{k+1}$, we obtain
$$F_{r(\alpha)} = F_{n+1} \le q_n \le H(\alpha),$$
which yields the stated logarithmic bound. □

For the reader's convenience, we record the following two standard facts from the theory of ordinary continued fractions.

Lemma 2.4. *Let $\xi = [a_0; a_1, a_2, \dots]$ be an irrational real number whose partial quotients satisfy*
$$a_n \le B \qquad (n \ge 1).$$
Let p_n/q_n be the convergents of ξ. Then there exists a constant $c(B) > 0$ such that, whenever a real number x satisfies
$$|x - \xi| < \frac{c(B)}{q_n^2},$$
the numbers x and ξ have the same first $n+1$ partial quotients. One may take
$$c(B) = \frac{1}{2(B+1)(B+2)}.$$
Moreover,
$$q_n \le (B+1)^n \qquad (n \ge 0).$$
Consequently, if a sequence (x_m) of rational numbers satisfies
$$|x_m - \xi| \le A\eta^m,$$
for some $A > 0$ and $0 < \eta < 1$, then there exist positive constants c_1, c_2 depending only on ξ, A, η, such that
$$r(x_m) \ge c_1 m - c_2.$$

Proof. Fix $n \geq 0$. Let $\xi_{n+1} = [a_{n+1}; a_{n+2}, \dots]$ be the $(n+1)$st complete quotient of ξ. Then

$$\xi = \frac{p_n \xi_{n+1} + p_{n-1}}{q_n \xi_{n+1} + q_{n-1}}.$$

The map

$$y \longmapsto \frac{p_n y + p_{n-1}}{q_n y + q_{n-1}}$$

is monotone on $(0, \infty)$, and the set of real numbers whose continued fraction expansion begins with $(a_0, \dots, a_n)$ is the interval between

$$\frac{p_n}{q_n} \qquad \text{and} \qquad [a_0; \dots, a_n, 1] = \frac{p_n + p_{n-1}}{q_n + q_{n-1}}.$$

A direct computation gives

$$\left| \xi - \frac{p_n}{q_n} \right| = \frac{1}{q_n (q_n \xi_{n+1} + q_{n-1})}$$

and

$$\left| \xi - \frac{p_n + p_{n-1}}{q_n + q_{n-1}} \right| = \frac{\xi_{n+1} - 1}{(q_n \xi_{n+1} + q_{n-1})(q_n + q_{n-1})}.$$

Since every partial quotient of ξ is at most B, one has $\xi_{n+1} < B + 1$ and

$$\xi_{n+1} - 1 \geq \cfrac{1}{a_{n+2} + \cfrac{1}{a_{n+3} + \cdots}} > \frac{1}{B+1}.$$

Also $q_{n-1} \leq q_n$, so

$$q_n \xi_{n+1} + q_{n-1} \leq (B+2) q_n$$

and

$$q_n + q_{n-1} \leq 2 q_n.$$

Therefore,

$$\left| \xi - \frac{p_n}{q_n} \right| \geq \frac{1}{(B+2) q_n^2}$$

and

$$\left| \xi - \frac{p_n + p_{n-1}}{q_n + q_{n-1}} \right| \geq \frac{1}{2(B+1)(B+2) q_n^2}.$$

Hence every real number x with

$$|x - \xi| < \frac{1}{2(B+1)(B+2) q_n^2}$$

remains in the same cylinder as ξ, so it has the same first $n+1$ partial quotients.

The recurrence $q_{n+1} = a_{n+1} q_n + q_{n-1}$ and the bound $a_{n+1} \leq B$ imply inductively that $q_n \leq (B+1)^n$. If $|x_m - \xi| \leq A \eta^m$, then the first part applies as soon as

$$A \eta^m < \frac{c(B)}{(B+1)^{2n}},$$

which holds for all

$$n < \frac{m \log(\eta^{-1}) - \log(A / c(B))}{2 \log(B+1)}.$$

Thus $r(x_m) \geq n + 1$, which yields the claimed linear lower bound. $\square$

Lemma 2.5. *Let $\frac{a}{b}$ be a positive rational number in lowest terms. Then every real number x satisfying*

$$0 < \left| x - \frac{a}{b} \right| < \frac{1}{2b^2}$$

has every ordinary convergent of $\frac{a}{b}$ among its ordinary convergents.

Proof. By Legendre's theorem on continued fractions, see e.g. [3, Theorem 8.26], we obtain that $\frac{a}{b}$ is an ordinary convergent of x. Every convergent in the finite real continued fraction expansion of $\frac{a}{b}$ is also an ordinary convergent of x. □

3 Lengths of ordinary and Schneider's expansions

In this section we compare the possible sizes of the ordinary and Schneider's lengths. We treat three regimes: long Schneider's and short ordinary expansions, logarithmic growth in both theories, and bounded Schneider's length with unbounded ordinary length.

3.1 Long Schneider's expansion and short ordinary expansion

We first exhibit a simple family of rational numbers for which the Schneider's length tends to infinity while the ordinary length stays uniformly bounded.

Define

$$W_0 = 1, \qquad W_1 = p-1, \qquad W_m = (p-1)W_{m-1} + pW_{m-2} \quad (m \geq 2),$$

and

$$\tau_m := [p-1, p : p-1, p : p-1, \ldots, p : p-1]_p$$

with m digits $p-1$.

Theorem 3.1. *For every $m \geq 1$ one has*

$$\tau_m = \frac{W_m}{W_{m-1}}$$

and

$$s_p(\tau_m) = m, \qquad r(\tau_m) \leq 3, \qquad H(\tau_m) = W_m \asymp p^m.$$

More explicitly,

$$\tau_m = \begin{cases} [p-1], & m = 1, \\ [3], & (p,m) = (2,2), \\ [p; W_{m-1}], & m \geq 2 \text{ even and } (p,m) \neq (2,2), \\ [p-1; 1, W_{m-1}-1], & m \geq 3 \text{ odd.} \end{cases}$$

Proof. The identity $\tau_m = W_m/W_{m-1}$ follows from the recurrence for Schneider's convergents. Set

$$D_m := W_m - pW_{m-1}.$$

Using the recurrence for W_m, we obtain

$$D_m = (p-1)W_{m-1} + pW_{m-2} - pW_{m-1} = -(W_{m-1} - pW_{m-2}) = -D_{m-1}.$$

Since $D_1 = (p-1) - p = -1$, it follows that

$$D_m = (-1)^m, \qquad \text{i.e.} \qquad W_m = pW_{m-1} + (-1)^m.$$

Therefore

$$\tau_m = p + \frac{(-1)^m}{W_{m-1}}.$$

If m is even, this is the ordinary continued fraction $[p; W_{m-1}]$, except in the special case $(p, m) = (2, 2)$, when $\tau_2 = 3$. If $m \geq 3$ is odd, then

$$p - \frac{1}{W_{m-1}} = (p-1) + \frac{W_{m-1} - 1}{W_{m-1}} = [p-1; 1, W_{m-1} - 1].$$

Hence in all cases $r(\tau_m) \leq 3$.

Finally, the characteristic polynomial of (W_m) is $(x-p)(x+1)$, so $W_m \asymp p^m$. □

Note that there are also infinite families for which both lengths remain bounded. For instance,

$$[1, p^n : 1]_p = p^n + 1 \qquad (n \geq 1)$$

has Schneider's length 2 and ordinary length 1.

3.2 Long Schneider's expansion and long ordinary expansion

We next give families of rational numbers for which both the ordinary and the Schneider's lengths grow logarithmically with the height. The odd prime case and the case $p = 2$ are treated separately.

Theorem 3.2. *Assume $p > 2$, and let*

$$\rho_m = [1, p : 1, \ldots, p : 1]_p = \frac{U_m}{U_{m-1}}$$

with m digits 1. There exist constants $c_p, C_p, c'_p, C'_p > 0$ such that for all $m \geq 1$,

$$s_p(\rho_m) = m, \qquad c_p m - C_p \leq r(\rho_m) \leq c'_p m + C'_p, \qquad H(\rho_m) \asymp \lambda_p^m,$$

where $\lambda_p = (1 + \sqrt{4p+1})/2$. In particular,

$$s_p(\rho_m) \asymp \log H(\rho_m), \qquad r(\rho_m) \asymp \log H(\rho_m).$$

Proof. The equalities $s_p(\rho_m) = m$ and $\rho_m = U_m/U_{m-1}$ follow from Proposition 2.1; the height estimate follows from Remark 2.2. Put

$$\lambda = \frac{1 + \sqrt{4p+1}}{2}, \qquad \mu = \frac{1 - \sqrt{4p+1}}{2}.$$

By the Binet formula,

$$\rho_m = \frac{\lambda^{m+1} - \mu^{m+1}}{\lambda^m - \mu^m} = \lambda \, \frac{1 - (\mu/\lambda)^{m+1}}{1 - (\mu/\lambda)^m},$$

so

$$|\rho_m - \lambda| \ll \left(\frac{|\mu|}{\lambda} \right)^m.$$

Since λ is a quadratic irrational, its ordinary continued fraction is eventually periodic and in particular has bounded partial quotients. Lemma 2.4 yields

$$r(\rho_m) \geq c_p m - C_p.$$

The upper bound follows from Proposition 2.3 and the estimate $H(\rho_m) \asymp \lambda^m$. □

For $p = 2$, the sequence (ρ_m) from Theorem 3.2 coincides with the sequence (τ_m) from Theorem 3.1. Consequently, this sequence has bounded ordinary length. To obtain logarithmic growth in both settings, we instead use partial numerators equal to 4.

Define

$$V_0 = V_1 = 1, \qquad V_m = V_{m-1} + 4V_{m-2} \quad (m \geq 2),$$

and

$$\sigma_m := [1, 4 : 1, 4 : 1, \ldots, 4 : 1]_2 = \frac{V_m}{V_{m-1}}.$$

Theorem 3.3. *There exist constants $c_2, C_2, c_2', C_2' > 0$ such that for all $m \geq 1$,*

$$s_2(\sigma_m) = m, \qquad c_2 m - C_2 \leq r(\sigma_m) \leq c_2' m + C_2', \qquad H(\sigma_m) \asymp \left(\frac{1+\sqrt{17}}{2}\right)^m.$$

In particular,

$$s_2(\sigma_m) \asymp \log H(\sigma_m), \qquad r(\sigma_m) \asymp \log H(\sigma_m).$$

Proof. The recurrence for V_m has characteristic polynomial $x^2 - x - 4$, whose roots are $(1 \pm \sqrt{17})/2$. Hence

$$V_m = \frac{\left(\frac{1+\sqrt{17}}{2}\right)^{m+1} - \left(\frac{1-\sqrt{17}}{2}\right)^{m+1}}{\sqrt{17}},$$

so

$$H(\sigma_m) = V_m \asymp \left(\frac{1+\sqrt{17}}{2}\right)^m$$

and

$$\left|\sigma_m - \frac{1+\sqrt{17}}{2}\right| \ll \left(\frac{\sqrt{17}-1}{\sqrt{17}+1}\right)^m.$$

The number $(1+\sqrt{17})/2 = [2; \overline{1, 1, 3}]$ has bounded partial quotients, and Lemma 2.4 gives the linear lower bound for $r(\sigma_m)$. The upper bound follows from Proposition 2.3. □

3.3 Short Schneider's expansion and long ordinary expansion

We now turn to the opposite phenomenon: the Schneider's length stays bounded, while the ordinary length tends to infinity. Again we first treat odd primes and then the case $p = 2$.

The first nontrivial Schneider's length here is 3. If $s_p(\alpha) = 1$, then $\alpha = b_0$ is an integer, whereas if $s_p(\alpha) = 2$, then

$$\alpha = [b_0, p^a : b_1]_p = \frac{b_0 b_1 + p^a}{b_1}, \qquad 1 \leq b_0, b_1 \leq p - 1,$$

so the denominator of α is at most $p - 1$, and therefore $r(\alpha)$ is bounded in terms of p alone. In particular, for $p = 2$ one gets $\alpha = [1, 2^a : 1]_2 = 2^a + 1$, so $r(\alpha) = 1$.

Fix an odd prime p and put

$$\beta_p = 2 \log_p 2, \qquad \varphi = \frac{1+\sqrt{5}}{2}.$$

Since p is odd, the number β_p is irrational. Let A_n/B_n be the convergents of the ordinary continued fraction of β_p. There are infinitely many indices n such that

$$\delta_n := \beta_p B_n - A_n > 0.$$

For such n, and after discarding finitely many initial indices, define

$$m_n := \left\lfloor \frac{1-\log_p \varphi}{\delta_n} \right\rfloor \geq 1, \qquad t_n := m_n B_n, \qquad N_n := m_n A_n + 1,$$

and then

$$x_n := [1, p^{2t_n N_n} : 2, p^{N_n} : 1]_p = 1 + \frac{p^{2t_n N_n}}{p^{N_n}+2}.$$

Theorem 3.4. *For every odd prime p, the sequence (x_n) is well defined for all sufficiently large chosen indices n, and satisfies*

$$s_p(x_n) = 3.$$

Moreover there exist constants $c_p, C_p > 0$ such that

$$r(x_n) \geq c_p \log\log H(x_n) - C_p.$$

In particular, one obtains bounded Schneider's length and unbounded ordinary length.

Proof. Since p is odd, the displayed Schneider's expansion of x_n is admissible. Hence $s_p(x_n) = 3$.

From the definition of m_n we have

$$0 \leq N_n - \beta_p t_n - \log_p \varphi = 1 - \log_p \varphi - m_n \delta_n < \delta_n.$$

Also,

$$p^{N_n} = p^{\beta_p t_n + \log_p \varphi}\, p^{N_n - \beta_p t_n - \log_p \varphi} = 2^{2t_n} \varphi\, p^{N_n - \beta_p t_n - \log_p \varphi}.$$

Set

$$\alpha_n := \frac{p^{N_n}+2}{2^{2t_n}}.$$

Then

$$\alpha_n = \varphi\, p^{N_n - \beta_p t_n - \log_p \varphi} + 2^{1-2t_n}.$$

For all sufficiently large chosen indices n we have $\delta_n < 1$, and then

$$0 \leq p^{N_n - \beta_p t_n - \log_p \varphi} - 1 \leq p(\log p)\, \delta_n.$$

We also have $1 - \log_p \varphi > 0$ since $p \geq 3$. As $\delta_n \to 0$, the definition of m_n gives

$$m_n \geq \frac{1-\log_p \varphi}{2\delta_n}$$

for all sufficiently large chosen indices n. Since $t_n = m_n B_n$ and $B_n \geq 1$, it follows that

$$t_n \geq \frac{1-\log_p \varphi}{2\delta_n},$$

and hence

$$2^{1-2t_n} = 2e^{-2t_n \log 2} \leq 2e^{-(1-\log_p \varphi)\log 2/\delta_n} \leq \frac{2}{(1-\log_p \varphi)\log 2}\, \delta_n.$$

Thus

$$|\alpha_n - \varphi| \ll_p \delta_n.$$

Let

$$\beta'_{n+1} = [a_{n+1}; a_{n+2}, \dots]$$

be the $(n+1)$-st complete quotient of β_p. Since

$$\delta_n = \frac{1}{B_n \beta'_{n+1} + B_{n-1}},$$

while

$$a_{n+1} < \beta'_{n+1} < a_{n+1} + 1 \qquad \text{and} \qquad B_{n+1} = a_{n+1} B_n + B_{n-1},$$

we obtain

$$B_{n+1} < B_n \beta'_{n+1} + B_{n-1} < B_{n+1} + B_n \leq 2B_{n+1},$$

and therefore

$$\frac{1}{2B_{n+1}} < \delta_n < \frac{1}{B_{n+1}}.$$

Consequently,

$$|\alpha_n - \varphi| \ll_p B_{n+1}^{-1}.$$

The denominator of the j-th ordinary convergent of $\varphi = [1; \overline{1}]$ is $F_j \asymp \varphi^j$. Choose j maximal such that

$$F_j^2 \leq \frac{B_{n+1}}{12C},$$

where $C > 0$ depending only on p is such that

$$|\alpha_n - \varphi| < \frac{C}{B_{n+1}}$$

for all sufficiently large chosen indices n. Then

$$|\alpha_n - \varphi| < \frac{1}{12F_j^2},$$

and Lemma 2.4 shows that the first $j+1$ ordinary convergents of φ are also ordinary convergents of α_n. Hence

$$r(\alpha_n) \geq j + 1 \geq c_p \log B_{n+1} - C_p$$

with suitable positive constants depending only on p.

Next,

$$N_n \geq \beta_p t_n + \log_p \varphi,$$

so

$$p^{N_n} \geq 2^{2t_n} \varphi > 2^{2t_n}.$$

Thus

$$0 < 2^{2t_n} < p^{N_n} + 2.$$

Moreover,

$$p^{2t_n N_n} \equiv (p^{N_n})^{2t_n} \equiv (-2)^{2t_n} \equiv 2^{2t_n} \pmod{p^{N_n} + 2}.$$

Hence

$$x_n = \left\lfloor 1 + \frac{p^{2t_n N_n}}{p^{N_n} + 2} \right\rfloor + \frac{1}{\alpha_n}.$$

Since $\alpha_n \to \varphi > 1$, we have $\alpha_n > 1$ for all sufficiently large chosen indices n, and therefore

$$r(x_n) = 1 + r(\alpha_n).$$

It follows that

$$r(x_n) \geq c_p \log B_{n+1} - C_p$$

with suitable positive constants depending only on p.

Finally,

$$x_n = \frac{p^{2t_nN_n} + p^{N_n} + 2}{p^{N_n} + 2},$$

and this fraction is already in lowest terms, since any common divisor would divide both $p^{2t_nN_n}$ and $p^{N_n} + 2$. Therefore,

$$\log H(x_n) \asymp_p t_nN_n.$$

Also, from

$$\frac{1}{2B_{n+1}} < \delta_n < \frac{1}{B_{n+1}}$$

and the definition of m_n, we obtain

$$m_n \asymp_p B_{n+1}.$$

Thus

$$t_n = m_nB_n \asymp_p B_nB_{n+1} \leq B_{n+1}^2.$$

Since $A_n/B_n \to \beta_p > 0$, we also have $A_n \asymp_p B_n$, and therefore

$$N_n = m_nA_n + 1 \asymp_p m_nB_n = t_n.$$

Consequently,

$$\log H(x_n) \ll_p t_nN_n \ll_p B_{n+1}^4.$$

Hence

$$\log B_{n+1} \geq \frac{1}{4} \log\log H(x_n) - C_p$$

for some constant $C_p > 0$, and the stated lower bound for $r(x_n)$ follows. □

Proposition 3.5. *If α has Schneider's expansion of length 3 for $p = 2$, then $r(\alpha) \leq 4$. Thus the regime "bounded 2-adic length and unbounded ordinary length" cannot occur with $s_2(\alpha) = 3$.*

Proof. Every such number has the form

$$\alpha = [1, 2^{a_1} : 1, 2^{a_2} : 1]_2 = 1 + \frac{2^{a_1}}{2^{a_2} + 1}.$$

Write

$$a_1 = ka_2 + r, \qquad 0 \leq r < a_2,$$

and set

$$q := 2^{a_2} + 1.$$

Since

$$2^{a_2} \equiv -1 \pmod{q},$$

we have

$$2^{a_1} \equiv (-1)^k 2^r \pmod{q}.$$

Hence there exists an integer $M \geq 0$ such that

$$\alpha = \begin{cases} 1 + M + \dfrac{2^r}{q}, & \text{if } k \text{ is even,} \\ 1 + M + \dfrac{q - 2^r}{q}, & \text{if } k \text{ is odd.} \end{cases}$$

Assume first that k is even. Then

$$\alpha = 1 + M + \frac{1}{q/2^r}.$$

If $r = 0$, then $q/2^r = q$ is an integer, so $r(\alpha) = 2$. If $r > 0$, then

$$\frac{q}{2^r} = \frac{2^{a_2} + 1}{2^r} = 2^{a_2 - r} + \frac{1}{2^r} = [2^{a_2-r}; 2^r],$$

and thus $r(\alpha) = 3$.

Assume now that k is odd. Then

$$\alpha = 1 + M + \frac{q - 2^r}{q} = 1 + M + \frac{1}{q/(q - 2^r)}.$$

If $r = 0$, then

$$\frac{q}{q-1} = 1 + \frac{1}{2^{a_2}} = [1; 2^{a_2}],$$

so $r(\alpha) = 3$. If $r > 0$, then

$$\frac{q - 2^r}{2^r} = \frac{2^{a_2} + 1 - 2^r}{2^r} = 2^{a_2 - r} - 1 + \frac{1}{2^r} = [2^{a_2-r} - 1; 2^r],$$

and hence

$$\frac{q}{q - 2^r} = [1; 2^{a_2-r} - 1, 2^r].$$

Therefore, $r(\alpha) = 4$. □

For $p = 2$ we therefore pass to Schneider's length 4. Put

$$\beta_2 = \log_2 9, \qquad \varphi = \frac{1 + \sqrt{5}}{2}.$$

The number β_2 is irrational. Let A_n/B_n be the convergents of the ordinary continued fraction of β_2. Again there are infinitely many indices n such that

$$\delta_n := \beta_2 B_n - A_n > 0.$$

For such n, and after discarding finitely many initial indices, define

$$m_n := \left\lfloor \frac{3 - \log_2(3\varphi)}{\delta_n} \right\rfloor \geq 1, \qquad t_n := m_n B_n, \qquad N_n := m_n A_n + 3,$$

and then

$$y_n := [1, 2^{2t_n N_n} : 1, 2^{N_n} : 1, 2 : 1]_2 = 1 + \frac{3 \cdot 2^{2t_n N_n}}{2^{N_n} + 3}.$$

Theorem 3.6. *The sequence (y_n) is well defined for all sufficiently large chosen indices n, and satisfies*

$$s_2(y_n) = 4.$$

Moreover there exist constants $c_2, C_2 > 0$ such that

$$r(y_n) \geq c_2 \log\log H(y_n) - C_2.$$

Thus for $p = 2$ one also obtains bounded Schneider's length and unbounded ordinary length.

We omit the proof of this theorem since it is completely analogous to the proof of Theorem 3.4.

4 Common ordinary and Schneider's convergents

We now study the size of the intersection between the ordinary convergents and the Schneider's convergents of a rational number with finite Schneider's expansion. We first prove a general upper bound in terms of the height and then construct explicit families of rational numbers with many common convergents.

Let

$$\alpha = [c_0; c_1, \ldots, c_n]$$

be the normalized ordinary continued fraction expansion of a positive rational number, and let

$$\frac{u_j}{v_j} = [c_0; \ldots, c_j] \qquad (0 \le j \le n)$$

be its ordinary convergents. Suppose also that

$$\alpha = [b_0, p^{a_1} : b_1, \ldots, p^{a_{m-1}} : b_{m-1}]_p \tag{3}$$

is the finite Schneider's p-adic continued fraction expansion of α, and let

$$\frac{P_k}{Q_k} \qquad (0 \le k \le m-1)$$

be its Schneider's convergents. We write

$$C_p(\alpha) := \#\left\{\frac{u_j}{v_j} : 0 \le j \le n,\ \frac{u_j}{v_j} = \frac{P_k}{Q_k} \text{ for some } 0 \le k \le m-1\right\}.$$

We also consider the extremal function

$$M_p(H) := \max\{C_p(\alpha) : \alpha \in \mathbb{Q}_{>0} \cap \mathbb{Z}_p^\times,\ s_p(\alpha) < \infty,\ H(\alpha) \le H\}.$$

At the opposite extreme from the constructions below, the intersection may reduce to the number α itself. Indeed, for even m, the family τ_m from Theorem 3.1 satisfies $C_p(\tau_m) = 1$. If $(p, m) = (2, 2)$, then $\tau_2 = 3$ is an integer, so this is immediate. Otherwise

$$\tau_m = [p; W_{m-1}],$$

so its only proper ordinary convergent is p. On the Schneider side, the convergents are $\tau_1, \ldots, \tau_m$, with $\tau_1 = p - 1$ and $\tau_j \notin \mathbb{Z}$ for $j \ge 2$. Thus none of the Schneider's convergents equals p, and therefore τ_m is the only common convergent.

4.1 A general upper bound

We begin with a simple divisibility property of Schneider's convergents and derive from it a general upper bound for the number of common convergents.

Lemma 4.1. *Let*

$$\frac{P_0}{Q_0}, \frac{P_1}{Q_1}, \ldots, \frac{P_{m-1}}{Q_{m-1}}$$

be the Schneider's convergents of a finite Schneider's expansion (3)*. Then, for every* $0 \le k \le \ell \le m-1$*, one has*

$$p^{k+1} \mid (P_\ell Q_k - P_k Q_\ell).$$

Proof. We argue by induction on $\ell - k$.

If $\ell = k$, the claim is obvious. If $\ell = k+1$, then the formula (2) from Section 2 gives

$$P_{k+1}Q_k - P_k Q_{k+1} = (-1)^{k+2} p^{a_1+\cdots+a_{k+1}},$$

and since each $a_r \geq 1$, this is divisible by p^{k+1}.

Now assume $\ell \geq k+2$. Using the recurrence relations for Schneider's convergents, we obtain

$$\begin{aligned} P_\ell Q_k - P_k Q_\ell &= (b_\ell P_{\ell-1} + p^{a_\ell} P_{\ell-2}) Q_k - P_k (b_\ell Q_{\ell-1} + p^{a_\ell} Q_{\ell-2}) \\ &= b_\ell (P_{\ell-1} Q_k - P_k Q_{\ell-1}) + p^{a_\ell} (P_{\ell-2} Q_k - P_k Q_{\ell-2}). \end{aligned}$$

By the induction hypothesis, both terms on the right-hand side are divisible by p^{k+1}, and hence so is $P_\ell Q_k - P_k Q_\ell$. □

Theorem 4.2. *For every $H \geq 1$ one has*

$$M_p(H) \leq \left\lfloor \frac{1 + \sqrt{1 + 8 \log_p H}}{2} \right\rfloor .$$

Proof. Let $\alpha = a/b > 0$ be a rational number counted by $M_p(H)$, and list the common convergents in increasing ordinary order:

$$\frac{u_{i_1}}{v_{i_1}}, \frac{u_{i_2}}{v_{i_2}}, \ldots, \frac{u_{i_t}}{v_{i_t}} = \alpha.$$

Then $t = C_p(\alpha)$.

For each $1 \leq s \leq t$, write

$$\frac{u_{i_s}}{v_{i_s}} = \frac{P_{r_s}}{Q_{r_s}}$$

for the corresponding Schneider index. Since $\alpha > 0$, the numerators of its ordinary convergents form a strictly increasing sequence. Likewise, the numerators of the Schneider's convergents form a strictly increasing sequence by the recurrence relations. Therefore the common convergents occur in the same order in both sequences, and hence

$$r_1 < r_2 < \cdots < r_t.$$

We first note that two consecutive ordinary convergents can never both be Schneider's convergents. Indeed, if u_i/v_i and u_{i+1}/v_{i+1} were both Schneider's convergents, then

$$u_{i+1} v_i - u_i v_{i+1} = (-1)^i,$$

whereas Lemma 4.1 would imply that this determinant is divisible by p, a contradiction.

Now fix $s \in \{1, \ldots, t-1\}$. By the previous paragraph, we have

$$i_{s+1} \geq i_s + 2.$$

Write

$$[c_{i_s+1}; \ldots, c_{i_{s+1}}] = \frac{A_s}{B_s}, \qquad A_s, B_s \in \mathbb{N},\ \gcd(A_s, B_s) = 1.$$

Then

$$\frac{u_{i_{s+1}}}{v_{i_{s+1}}} = [c_0; \ldots, c_{i_s}, A_s/B_s] = \frac{A_s u_{i_s} + B_s u_{i_s-1}}{A_s v_{i_s} + B_s v_{i_s-1}}.$$

This fraction is reduced. Indeed, if

$$d := \gcd(A_s u_{i_s} + B_s u_{i_s-1},\ A_s v_{i_s} + B_s v_{i_s-1}),$$

then

$$d \mid v_{i_s}(A_s u_{i_s} + B_s u_{i_s-1}) - u_{i_s}(A_s v_{i_s} + B_s v_{i_s-1}) = B_s(u_{i_s-1}v_{i_s} - u_{i_s}v_{i_s-1}),$$

and

$$d \mid v_{i_s-1}(A_s u_{i_s} + B_s u_{i_s-1}) - u_{i_s-1}(A_s v_{i_s} + B_s v_{i_s-1}) = A_s(u_{i_s}v_{i_s-1} - u_{i_s-1}v_{i_s}).$$

Since

$$u_{i_s-1}v_{i_s} - u_{i_s}v_{i_s-1} = (-1)^{i_s},$$

it follows that $d \mid A_s$ and $d \mid B_s$. As $\gcd(A_s, B_s) = 1$, we get $d = 1$. Therefore,

$$u_{i_{s+1}} = A_s u_{i_s} + B_s u_{i_s-1}, \qquad v_{i_{s+1}} = A_s v_{i_s} + B_s v_{i_s-1}.$$

Moreover,

$$u_{i_{s+1}}v_{i_s} - u_{i_s}v_{i_{s+1}} = B_s(u_{i_s-1}v_{i_s} - u_{i_s}v_{i_s-1}) = (-1)^{i_s}B_s.$$

Since

$$\frac{u_{i_s}}{v_{i_s}} = \frac{P_{r_s}}{Q_{r_s}} \qquad \text{and} \qquad \frac{u_{i_{s+1}}}{v_{i_{s+1}}} = \frac{P_{r_{s+1}}}{Q_{r_{s+1}}},$$

Lemma 4.1 yields

$$p^{r_s+1} \mid (P_{r_{s+1}}Q_{r_s} - P_{r_s}Q_{r_{s+1}}) = (u_{i_{s+1}}v_{i_s} - u_{i_s}v_{i_{s+1}}),$$

so

$$p^{r_s+1} \mid B_s.$$

Hence

$$B_s \geq p^{r_s+1}.$$

Since $A_s \geq B_s$, we have

$$v_{i_{s+1}} = A_s v_{i_s} + B_s v_{i_s-1} > B_s v_{i_s} \geq p^{r_s+1}v_{i_s}.$$

Because $r_s \geq s-1$, it follows that

$$v_{i_{s+1}} \geq p^s v_{i_s}.$$

Iterating these inequalities for $s = 1, \dots, t-1$, we obtain

$$v_{i_t} \geq p^{1+2+\cdots+(t-1)}v_{i_1} \geq p^{t(t-1)/2},$$

since $v_{i_1} \geq 1$. But $v_{i_t} = b \leq H(\alpha) \leq H$, hence

$$p^{t(t-1)/2} \leq H,$$
$$t^2 - t - 2\log_p H \leq 0.$$

Therefore,

$$t \leq \left\lfloor \frac{1 + \sqrt{1 + 8\log_p H}}{2} \right\rfloor.$$

Taking the maximum over all admissible α proves the theorem. ◻

4.2 Explicit constructions

We next construct families with arbitrarily many common convergents. For this we use a stability result for ordinary convergents together with a simple bound for the linear fractional map obtained by replacing the final digit 1 in a finite Schneider's expansion.

For $n \geq 1$ put

$$T_{p,n} := [1, p : 1, p^n : 1]_p = 1 + \frac{p}{1 + p^n} = \frac{p^n + p + 1}{p^n + 1}.$$

Let θ be a positive rational number with finite Schneider's expansion ending in the digit 1. If $\theta = 1$, we set

$$F_\theta(x) := x.$$

Otherwise we write

$$\theta = [b_0, p^{a_1} : b_1, \dots, p^{a_m} : 1]_p$$

and define

$$F_\theta(x) := [b_0, p^{a_1} : b_1, \dots, p^{a_m} : x]_p.$$

Writing

$$F_\theta(x) = \frac{Ax + B}{Cx + D},$$

one has

$$A = P_{m-1}, \qquad B = p^{a_m} P_{m-2}, \qquad C = Q_{m-1}, \qquad D = p^{a_m} Q_{m-2},$$

where P_k/Q_k are the Schneider's convergents of θ. Therefore, if $\theta = a/b$ in lowest terms, then

$$A + B = a, \qquad C + D = b. \tag{4}$$

Lemma 4.3. *Let $\theta = a/b > 0$ be a positive rational number with finite Schneider's expansion ending in the digit 1, and let*

$$F_\theta(x) = \frac{Ax + B}{Cx + D}$$

be defined as above. Then

$$A + B, C + D \leq H(\theta),$$

and, for every $x \geq 1$,

$$|F_\theta(x) - \theta| \leq H(\theta)^2 |x - 1|.$$

Proof. If $\theta = 1$, then $F_\theta(x) = x$, so the claim is trivial. Assume therefore that $\theta \neq 1$.

The inequalities $A + B = a \leq H(\theta)$ and $C + D = b \leq H(\theta)$ follow from (4). Next,

$$F_\theta(x) - F_\theta(1) = \frac{(AD - BC)(x - 1)}{(Cx + D)(C + D)}.$$

Since $x \geq 1$ and integers $C > 0$, $D \geq 0$, the denominator is at least 1. On the other hand,

$$|AD - BC| < (A + B)(C + D) = ab \leq H(\theta)^2.$$

Therefore,

$$|F_\theta(x) - \theta| = |F_\theta(x) - F_\theta(1)| \leq H(\theta)^2 |x - 1|.$$

□

Theorem 4.4. *Set $\theta_1 = 1$. Inductively, once θ_m is constructed, write*

$$H_m := H(\theta_m), \qquad n_m := 2 + \left\lceil 4 \log_p H_m \right\rceil,$$

and define

$$\theta_{m+1} := F_{\theta_m}(T_{p,n_m}).$$

Then, for every $m \geq 1$, the following hold:

(i) θ_m *has a finite Schneider's p-adic continued fraction ending in the digit* 1*;*

(ii) $\theta_1, \dots, \theta_m$ *are common ordinary and Schneider's convergents of* θ_m*;*

(iii) $H_{m+1} \le 3p^3 H_m^5$.

Consequently,

$$C_p(\theta_m) \ge m, \qquad H_m \le (3p^3)^{(5^{m-1}-1)/4} \qquad (m \ge 1),$$

and, in particular,

$$n_m < 5^m \qquad (m \ge 1).$$

Proof. We argue by induction on m. For $m = 1$, assertions (i) and (ii) are clear, since $\theta_1 = 1$ has Schneider's expansion $[1]_p$ and is trivially a common ordinary and Schneider's convergent of itself. Also $H_1 = 1$.

Assume that θ_m has already been constructed and satisfies (i) and (ii). Since its Schneider's expansion ends in 1, the number

$$\theta_{m+1} = F_{\theta_m}(T_{p,n_m})$$

is obtained by replacing the final digit 1 in the Schneider's expansion of θ_m by the block

$$[1, p : 1, p^{n_m} : 1]_p.$$

Thus θ_{m+1} again has a finite Schneider's expansion ending in 1. This proves (i) for $m + 1$.

Every Schneider's convergent of θ_m remains a Schneider's convergent of θ_{m+1}, because the new expansion is obtained by replacing the final digit 1 by a longer block. On the ordinary side, Lemma 4.3 gives

$$|\theta_{m+1} - \theta_m| \le H_m^2 |T_{p,n_m} - 1| = H_m^2 \frac{p}{1 + p^{n_m}} < H_m^2 p^{1-n_m}.$$

Since

$$n_m = 2 + \lceil 4 \log_p H_m \rceil,$$

we have

$$p^{1-n_m} \le p^{-1} H_m^{-4} \le \frac{1}{2H_m^4},$$

because $p \ge 2$. Hence

$$|\theta_{m+1} - \theta_m| < \frac{1}{2H_m^2}.$$

Now $\theta_m = a/b$ in lowest terms, with $b \le H_m$, so Lemma 2.5 implies that every ordinary convergent of θ_m remains an ordinary convergent of θ_{m+1}. By the induction hypothesis, $\theta_1, \dots, \theta_m$ are common convergents of θ_m, hence also of θ_{m+1}.

Finally, θ_{m+1} itself is a common convergent of θ_{m+1}. Since the ordinary convergents of a rational number are pairwise distinct, θ_{m+1} is different from $\theta_1, \dots, \theta_m$. Therefore,

$$\theta_1, \dots, \theta_{m+1}$$

are common ordinary and Schneider's convergents of θ_{m+1}. This proves (ii) for $m + 1$.

The bound

$$C_p(\theta_m) \ge m$$

follows immediately from (ii).

It remains to prove (iii). Write

$$T_{p,n_m} = \frac{u_m}{v_m} = \frac{p^{n_m} + p + 1}{p^{n_m} + 1}.$$

Then

$$\max\{u_m, v_m\} \le 3p^{n_m}.$$

Since

$$\theta_{m+1} = F_{\theta_m}(T_{p,n_m}) = \frac{Au_m + Bv_m}{Cu_m + Dv_m},$$

and Lemma 4.3 gives $A+B, C+D \le H_m$, we obtain

$$H_{m+1} \le \max\{(A+B)\max\{u_m, v_m\}, (C+D)\max\{u_m, v_m\}\} \le 3H_m p^{n_m}.$$

Using again the definition of n_m, we get

$$p^{n_m} \le p^2\, p^{\lceil 4\log_p H_m \rceil} \le p^3 H_m^4,$$

so

$$H_{m+1} \le 3p^3 H_m^5.$$

This proves (iii). From the previous inequality, we obtain by iteration that

$$H_m \le (3p^3)^{1+5+\cdots+5^{m-2}} = (3p^3)^{(5^{m-1}-1)/4}.$$

Hence

$$n_m = 2 + \left\lceil 4\log_p H_m \right\rceil \le 2 + \left\lceil (5^{m-1}-1)\log_p(3p^3) \right\rceil \le 2 + \left\lceil 5(5^{m-1}-1) \right\rceil < 5^m,$$

since $\log_p(3p^3) = 3 + \log_p 3 < 5$. □

Corollary 4.5. *For every $H \ge 1$ one has*

$$M_p(H) \ge 1 + \left\lfloor \log_5\left(1 + \frac{4\log H}{\log(3p^3)}\right)\right\rfloor$$

and as a consequence

$$M_p(H) \gg_p \log\log H.$$

Proof. Let

$$m(H) := 1 + \left\lfloor \log_5\left(1 + \frac{4\log H}{\log(3p^3)}\right)\right\rfloor.$$

Then

$$5^{m(H)-1} \le 1 + \frac{4\log H}{\log(3p^3)}.$$

Hence

$$\frac{5^{m(H)-1}-1}{4}\log(3p^3) \le \log H.$$

Therefore Theorem 4.4 yields

$$H(\theta_{m(H)}) \le (3p^3)^{(5^{m(H)-1}-1)/4} \le H.$$

Since $C_p(\theta_{m(H)}) \ge m(H)$, we conclude that

$$M_p(H) \ge m(H).$$

The final estimate follows immediately. □

Proposition 4.6. *The sequence $(n_m)_m$ in Theorem 4.4 is not optimal. For $p = 2$, set*

$$\xi_1 = 1, \qquad n_1 := 1,$$

and define inductively

$$\xi_{m+1} := F_{\xi_m}(T_{2,n_m}) \qquad (m \geq 1),$$

where

$$n_m := 5 \cdot 2^{m-2} - 1 \qquad (m \geq 2).$$

Then, for every $m \geq 1$, the number ξ_m has a finite Schneider's 2-adic continued fraction ending in the digit 1, and

$$\xi_1, \dots, \xi_m$$

are common ordinary and Schneider's convergents of ξ_m. In particular,

$$C_2(\xi_m) \geq m.$$

Proof. The claim is clear for $m = 1$. Also,

$$\xi_2 = T_{2,1} = [1, 2 : 1, 2 : 1]_2 = \frac{5}{3},$$

so ξ_2 has a finite Schneider's 2-adic continued fraction ending in the digit 1, and ξ_1, ξ_2 are common ordinary and Schneider's convergents of ξ_2.

Now let $m \geq 2$, and write

$$F_{\xi_m}(x) = \frac{A_m x + B_m}{C_m x + D_m}.$$

Let

$$\Delta_m := A_m D_m - B_m C_m$$

denote the determinant of this linear fractional transformation. Since

$$[1, 2 : 1, 2^n : x]_2 = \frac{3x + 2^n}{x + 2^n},$$

we have

$$F_{\xi_{m+1}}(x) = F_{\xi_m}\left(\frac{3x + 2^{n_m}}{x + 2^{n_m}}\right),$$

whence a direct computation gives

$$\Delta_{m+1} = 2^{n_m+1} \Delta_m.$$

Because $\Delta_2 = 4$, an induction yields

$$\Delta_m = 2^{5 \cdot 2^{m-2} - 3} \qquad (m \geq 2),$$

and thus

$$2^{n_m} = 4\Delta_m \qquad (m \geq 2).$$

Write $\xi_m = \tilde{a}_m / \tilde{b}_m$ in lowest terms. Since $\tilde{b}_m = C_m + D_m$ and

$$F_{\xi_m}(x) - F_{\xi_m}(1) = \frac{\Delta_m (x - 1)}{(C_m x + D_m)(C_m + D_m)},$$

we obtain, by substituting

$$T_{2,n_m} = \frac{2^{n_m} + 3}{2^{n_m} + 1},$$

that

$$|\xi_{m+1}-\xi_m| = |F_{\xi_m}(T_{2,n_m}) - F_{\xi_m}(1)| = \frac{2\Delta_m}{(2^{n_m}+1)(C_mT_{2,n_m}+D_m)\tilde{b}_m}.$$

Since $T_{2,n_m} > 1$ and $C_m, D_m > 0$, we have

$$C_mT_{2,n_m} + D_m > C_m + D_m = \tilde{b}_m.$$

Therefore,

$$|\xi_{m+1}-\xi_m| < \frac{2\Delta_m}{(2^{n_m}+1)\tilde{b}_m^2} < \frac{2\Delta_m}{2^{n_m}\tilde{b}_m^2} = \frac{1}{2\tilde{b}_m^2}.$$

By Lemma 2.5, every ordinary convergent of ξ_m remains an ordinary convergent of ξ_{m+1}. Moreover, ξ_{m+1} is obtained by replacing the final digit 1 in the Schneider's expansion of ξ_m by the block $[1, 2 : 1, 2^{n_m} : 1]_2$, so ξ_{m+1} again has a finite Schneider's 2-adic continued fraction ending in the digit 1, and every Schneider convergent of ξ_m remains a Schneider convergent of ξ_{m+1}. Hence every common convergent of ξ_m remains a common convergent of ξ_{m+1}.

Since ξ_1, ξ_2 are common ordinary and Schneider's convergents of ξ_2, it now follows by induction that $\xi_1, \ldots, \xi_m$ are common ordinary and Schneider's convergents of ξ_m for every $m \geq 1$. □

For example,

$$\xi_5 = [1, 2 : 1, 2 : 1, 2 : 1, 2^4 : 1, 2 : 1, 2^9 : 1, 2 : 1, 2^{19} : 1]_2 = \frac{24486917971}{14260136637},$$

and its common ordinary and Schneider's convergents are

$$1,\ \frac{5}{3},\ \frac{91}{53},\ \frac{46705}{27199},\ \xi_5.$$

Combining Theorem 4.2 with Corollary 4.5, we obtain

$$\log_5\left(1 + \frac{4\log H}{\log(3p^3)}\right) \leq M_p(H) \leq \frac{1+\sqrt{1+8\log_p H}}{2}.$$

Thus the maximal number of common ordinary and Schneider's p-adic convergents of a positive rational number of height at most H grows at least on the order of $\log\log H$ and at most on the order of $\sqrt{\log H}$.

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