# REAL LIMITS AND PERIOD LENGTHS OF SCHNEIDER'S $p$-ADIC CONTINUED FRACTIONS

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Abstract. For every quadratic $p$-adic number $\alpha$, we prove that the rational convergents of its Schneider $p$-adic continued fraction expansion converge in $\mathbb{R}$. The limit is a real image of $\alpha$ if and only if the expansion is eventually periodic. Starting from a periodic Schneider expansion, we construct a non-periodic continued fraction with partial quotients in a finite set of positive rational numbers. Its real and $p$-adic limits are roots of the same irreducible quadratic polynomial over $\mathbb{Q}$, disproving a conjecture of Romeo. For purely periodic quadratic Schneider expansions, we compare the Schneider period length with the ordinary continued fraction period length of the real limit. Explicit families show that either period length can remain fixed while the other tends to infinity. In a further family, both period lengths are bounded below by explicit linear functions of the parameter.

## 1. Introduction

Schneider's algorithm [18] associates to each $p$-adic unit a continued fraction with partial denominators in $\{1, \dots, p-1\}$ and partial numerators of the form $p^k$, where $k$ is a positive integer. We prove that its rational convergents converge in $\mathbb{R}$ for every quadratic input $\alpha$. The limit is a real image of $\alpha$ if and only if the expansion is eventually periodic. For periodic expansions, we compare the Schneider period length with the ordinary continued fraction period length of the real limit.

Periodicity of Schneider expansions of quadratic irrationals was studied by Bundschuh, de Weger, Becker, Tilborghs, and van der Poorten [7, 11, 4, 19, 20]. See also [16] for a survey of results. Conditions for real convergence were studied in [5, 14]. For quadratic inputs, we combine the convergence criterion of [14, Section 6] with integral recurrences for the quadratic polynomials satisfied by the complete quotients. The assertion of real convergence is a Schneider analogue of a result of Capuano, Veneziano and Zannier for Ruban continued fractions [9, Proposition 4.5].

The relation between real convergence and periodicity for several Browkin-type $p$-adic algorithms was studied in [17]. Other periodic continued-fraction representations giving simultaneous real and $p$-adic approximations were constructed in [1, 2].

For these Browkin-type algorithms and quadratic inputs whose fields admit a real embedding, Romeo proves that eventual periodicity implies convergence in $\mathbb{R}$ to a real image of the input [17, Proposition 1]. He also conjectured that a continued fraction with rational partial quotients which converges simultaneously in $\mathbb{R}$ and $\mathbb{Q}_p$ to a real quadratic irrational and a $p$-adic image of the same algebraic number must be periodic [17, Conjecture 16]. Example 2.6 disproves this conjecture, while Corollary 2.5 proves its analogue for Schneider continued fractions. To our knowledge, the corresponding conjecture for the specified Browkin-type algorithms [17, Conjecture 17] remains open.

Section 3 compares the ordinary continued fraction period length of a real quadratic irrational with the Schneider period length of its $p$-adic image, when the latter expansion is

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periodic. We obtain quadratic analogues of the three cases studied for finite expansions of rational numbers in [15]. We exhibit explicit purely periodic families that have arbitrarily large Schneider period lengths and ordinary period length two for odd $p$ and three for $p = 2$. In the opposite direction, for every prime $p$ and every $n \geq 2$, we give an expansion with Schneider period length two and ordinary period length exactly $3(n-1)$. Theorem 3.5 gives a further family with Schneider period length $2n + 1$ and ordinary period length greater than $c_p n - 5$, with an explicit constant $c_p > 0$ depending only on $p$.

Related constructions of prescribed period lengths for Browkin-type algorithms appear in [8, 13]. For Browkin's algorithm with odd $p$, Barbero, Murru and Urani [3] give infinite families of square roots of integers realizing each prescribed even period length.

Throughout, periodicity means eventual periodicity unless pure periodicity is explicitly stated.

## 2. Quadratic Schneider expansions

2.1. **Convergents.** Let $p$ be a prime. An infinite Schneider continued fraction has the form

$$[b_0, p^{a_1} : b_1, p^{a_2} : b_2, \ldots]_p := b_0 + \cfrac{p^{a_1}}{b_1 + \cfrac{p^{a_2}}{b_2 + \ddots}},$$

where the $b_n$ ($n \geq 0$) and $a_n$ ($n \geq 1$) are integers satisfying $1 \leq b_n \leq p-1$ and $a_n \geq 1$. Its rational convergents are

$$\frac{R_n}{S_n} = [b_0, p^{a_1} : b_1, p^{a_2} : b_2, \ldots, p^{a_n} : b_n]_p,$$

with $R_0/S_0 = b_0$. The numerators and denominators satisfy

$$\begin{aligned} &R_{-1} = 1, \quad R_0 = b_0, && S_{-1} = 0, \quad S_0 = 1, \\ &R_n = b_n R_{n-1} + p^{a_n} R_{n-2}, && S_n = b_n S_{n-1} + p^{a_n} S_{n-2} \quad (n \geq 1). \end{aligned} \tag{2.1}$$

Put $T_n = p^{a_1 + \cdots + a_n}$. The recurrences (2.1) give (see [14, (1.4)])

$$R_n S_{n-1} - R_{n-1} S_n = (-1)^{n-1} T_n. \tag{2.2}$$

The recurrence modulo $p$ shows that $p$ divides neither $R_n$ nor $S_n$. Consequently $R_n/S_n$ is reduced, since every common divisor of $R_n$ and $S_n$ divides $T_n$. For $n \geq 1$, put

$$\Delta_n := \left| \frac{R_n}{S_n} - \frac{R_{n-1}}{S_{n-1}} \right| = \frac{T_n}{S_n S_{n-1}}.$$

**Lemma 2.1** ([14, Section 6])**.** *The sequence $R_n/S_n$ converges in $\mathbb{R}$ if and only if $\Delta_n \to 0$. Moreover,*

$$\frac{\Delta_n}{\Delta_{n-1}} = \frac{p^{a_n} S_{n-2}}{S_n} = \frac{1}{1 + \dfrac{b_n S_{n-1}}{p^{a_n} S_{n-2}}} < 1 \tag{2.3}$$

*for $n \geq 2$.*

*Proof.* Equation (2.3) follows from the recurrence for $S_n$. Thus the absolute values of the consecutive differences decrease, while their signs alternate by (2.2). The sequence of convergents therefore converges if and only if its consecutive differences tend to zero, by the alternating-series criterion. □

We now recall how Schneider's algorithm expands a $p$-adic unit and, conversely, how an infinite continued fraction of the above form converges to a $p$-adic unit with that expansion.

Starting from a $p$-adic unit $\alpha_0$, choose at each step the unique $b_n \in \{1, \dots, p-1\}$ such that

$$\alpha_n \equiv b_n \pmod{p}.$$

If $\alpha_n = b_n$, the expansion terminates. Otherwise put

$$a_{n+1} = v_p(\alpha_n - b_n), \qquad \alpha_{n+1} = \frac{p^{a_{n+1}}}{\alpha_n - b_n},$$

where $v_p$ denotes the $p$-adic valuation normalized by $v_p(p) = 1$. The resulting finite or infinite Schneider expansion is unique. For every $n$ for which the complete quotient $\alpha_{n+1}$ is defined, we have [14, (1.10)]

$$\alpha_0 = \frac{R_n\alpha_{n+1} + p^{a_{n+1}}R_{n-1}}{S_n\alpha_{n+1} + p^{a_{n+1}}S_{n-1}}. \tag{2.4}$$

Since $S_n$ and $\alpha_{n+1}$ are $p$-adic units, (2.2) gives

$$\left|\alpha_0 - \frac{R_n}{S_n}\right|_p = T_{n+1}^{-1}.$$

Conversely, every infinite sequence of digits with $1 \le b_n \le p-1$ and $a_n \ge 1$ is the Schneider expansion of a $p$-adic unit. Indeed, its convergents form a Cauchy sequence in $\mathbb{Q}_p$, since

$$\left|\frac{R_n}{S_n} - \frac{R_{n-1}}{S_{n-1}}\right|_p = T_n^{-1} \longrightarrow 0.$$

The same holds for every tail. Let $\alpha_n$ be the $p$-adic value of the tail beginning with $b_n$. Its convergents are congruent to $b_n$ modulo $p$, so $\alpha_n \in \mathbb{Z}_p^\times$. Passing to the limit in $\mathbb{Q}_p$, we obtain

$$\alpha_n = b_n + \frac{p^{a_{n+1}}}{\alpha_{n+1}}. \tag{2.5}$$

Thus $\alpha_n \equiv b_n \pmod{p}$ and $v_p(\alpha_n - b_n) = a_{n+1}$, which recover exactly the prescribed Schneider digits and exponents.

2.2. **Periodicity.** We use the following sign criterion for eventual periodicity of Schneider expansions. For related results, see [8, Introduction] and, for Ruban expansions, [9, Theorem 1.3].

For a quadratic number $\alpha \in \mathbb{Q}_p$, the two embeddings of $\mathbb{Q}(\alpha)$ into $\mathbb{C}$ send $\alpha$ to the two roots of its minimal polynomial over $\mathbb{Q}$. We refer to $\mathbb{Q}(\alpha)$ as real quadratic when these roots are real and as imaginary quadratic otherwise.

**Lemma 2.2.** *Let $\alpha \in \mathbb{Q}_p$ be a $p$-adic unit quadratic over $\mathbb{Q}$, and let $\alpha_n$ be its complete quotients. The Schneider expansion of $\alpha$ is eventually periodic if and only if there is an embedding $\sigma : \mathbb{Q}(\alpha) \to \mathbb{R}$ such that $\sigma(\alpha_n) > 0$ for every $n \ge 0$. In this case, exactly one of the two real embeddings has this property.*

*Proof.* Assume first that the expansion is eventually periodic, and let $\alpha_N$ be a complete quotient at the beginning of a period. Composing (2.5) over one period gives

$$\alpha_N = \frac{r\alpha_N + s}{t\alpha_N + w}$$

for some nonnegative integers $r, s, t, w$ with $r, s, t > 0$, as follows by induction on the number of substitutions. Hence $\alpha_N$ satisfies

$$tX^2 + (w - r)X - s = 0.$$

The product of the two roots is $-s/t < 0$, so they are real and have opposite signs. Since $\alpha_N$ is quadratic, this polynomial is a nonzero scalar multiple of its minimal polynomial. Relation (2.5) gives $\mathbb{Q}(\alpha_{n+1}) = \mathbb{Q}(\alpha_n)$, so $\mathbb{Q}(\alpha_N) = \mathbb{Q}(\alpha)$. The two roots therefore correspond to the two embeddings of this field into $\mathbb{R}$. Choose the embedding $\sigma$ for which $\sigma(\alpha_N) > 0$. The other real embedding is negative on $\alpha_N$, so no other embedding can be positive on all complete quotients. The same relations propagate positivity backwards from $\sigma(\alpha_N) > 0$ through one period and then through the preperiod. Hence $\sigma(\alpha_n) > 0$ for every $n \geq 0$.

Conversely, suppose that $\sigma(\alpha_n) > 0$ for every $n$, and let $\tau$ be the other real embedding. Choose an irreducible polynomial $F_0(X) = A_0X^2 + B_0X + C_0 \in \mathbb{Z}[X]$ with root $\alpha_0$ and $A_0 > 0$. Define

$$F_{n+1}(X) = \frac{X^2}{p^{a_{n+1}}} F_n\left(b_n + \frac{p^{a_{n+1}}}{X}\right) = A_{n+1}X^2 + B_{n+1}X + C_{n+1}. \tag{2.6}$$

Then

$$A_{n+1} = \frac{F_n(b_n)}{p^{a_{n+1}}}, \qquad B_{n+1} = B_n + 2b_nA_n, \qquad C_{n+1} = p^{a_{n+1}}A_n.$$

Since $\alpha_n - b_n \in p^{a_{n+1}}\mathbb{Z}_p$ and $F_n(\alpha_n) = 0$, induction gives $F_n \in \mathbb{Z}[X]$. Each $F_n$ has the quadratic irrational $\alpha_n$ as a root, so $A_nC_n \neq 0$, and direct calculation gives $B_n^2 - 4A_nC_n = D$, where $D = B_0^2 - 4A_0C_0 > 0$.

Suppose that $\tau(\alpha_n) > 0$ for every $n$. Applying both embeddings to (2.5) shows that the two real roots $\sigma(\alpha_n)$ and $\tau(\alpha_n)$ of $F_n$ exceed $b_n$. Thus

$$\frac{F_n(b_n)}{A_n} = (b_n - \sigma(\alpha_n))(b_n - \tau(\alpha_n)) > 0.$$

The recurrence for $A_{n+1}$, together with $A_0 > 0$, now gives $A_n > 0$ by induction. Vieta's formulas then give $B_n < 0$ for all $n$, whereas

$$B_{n+1} = B_n + 2b_nA_n \geq B_n + 2,$$

which gives a contradiction for sufficiently large $n$. Hence $\tau(\alpha_m) < 0$ for some $m$, and (2.5) gives $\tau(\alpha_n) < 0$ for every $n \geq m$. For $n \geq m$, the roots have opposite signs, so

$$|B_n| < \sqrt{D}, \qquad 0 < |A_nC_n| \leq D/4.$$

Since $A_n$ and $C_n$ are nonzero integers, these bounds allow only finitely many triples $(A_n, B_n, C_n)$. Each corresponding polynomial has at most two roots in $\mathbb{Q}_p$, so only finitely many complete quotients can occur. A complete quotient must therefore repeat, and the uniqueness of Schneider's algorithm gives eventual periodicity. □

### 2.3. **Real convergence.**

**Theorem 2.3.** *Let $\alpha \in \mathbb{Q}_p$ be a p-adic unit quadratic over $\mathbb{Q}$. Then the rational convergents of its Schneider expansion converge in $\mathbb{R}$ to a limit $L$. Moreover,*

$$\left|L - \frac{R_n}{S_n}\right| = O_{\alpha,p}(n^{-2}).$$

*Proof.* Choose an irreducible quadratic polynomial $F_0 \in \mathbb{Z}[X]$ with $F_0(\alpha) = 0$, and define $F_n(X) = A_nX^2 + B_nX + C_n$ by the recurrence (2.6) in the proof of Lemma 2.2. As shown there, the coefficients are integers, $A_n \neq 0$, and $B_n^2 - 4A_nC_n = D$ for every $n \geq 0$, where $D = B_0^2 - 4A_0C_0$. These facts do not require a real embedding of $\mathbb{Q}(\alpha)$.

The recurrence (2.6) gives, for $n \geq 1$,

$$F_n(X) = \frac{(S_{n-1}X + p^{a_n}S_{n-2})^2}{T_n} F_0\left(\frac{R_{n-1}X + p^{a_n}R_{n-2}}{S_{n-1}X + p^{a_n}S_{n-2}}\right). \tag{2.7}$$

For $n = 1$ this is the definition of $F_1$. The induction step follows by substituting $b_n + p^{a_{n+1}}/X$ for $X$ and using the recurrences for $R_n$ and $S_n$.

Dividing (2.7) by $X^2$ and letting $X \to \infty$, we obtain

$$A_n = \frac{S_{n-1}^2}{T_n} F_0\left(\frac{R_{n-1}}{S_{n-1}}\right). \tag{2.8}$$

All partial numerators and denominators are positive, so

$$b_0 \leq \frac{R_n}{S_n} \leq b_0 + \frac{p^{a_1}}{b_1} \qquad (n \geq 0). \tag{2.9}$$

Thus $|F_0(R_n/S_n)| \leq M$ for some $M > 0$. Using $\Delta_n = T_n/(S_nS_{n-1})$, $T_n = p^{a_n}T_{n-1}$, and the recurrence for $S_n$, we obtain, for $n \geq 2$,

$$\begin{aligned} \frac{1}{\Delta_n} - \frac{1}{\Delta_{n-1}} &= \frac{S_{n-1}(S_n - p^{a_n}S_{n-2})}{T_n} = \frac{b_nS_{n-1}^2}{T_n} \\ &= \frac{b_n|A_n|}{|F_0(R_{n-1}/S_{n-1})|} \geq \frac{1}{M}. \end{aligned} \tag{2.10}$$

Formula (2.8) shows that $A_n$ and $F_0(R_{n-1}/S_{n-1})$ have the same sign, which justifies the absolute values. The last inequality uses $b_n \geq 1$ and $|A_n| \geq 1$. Hence $\Delta_n \to 0$, and Lemma 2.1 gives a real limit $L$. Since the consecutive differences alternate in sign and decrease in absolute value, we also have $|L - R_n/S_n| \leq \Delta_n$ for $n \geq 1$.

If $F_0(L) = 0$, the mean value theorem on the interval given by (2.9) gives

$$|F_0(R_{n-1}/S_{n-1})| \leq K|L - R_{n-1}/S_{n-1}| \leq K\Delta_{n-1}$$

for some $K > 0$. Identity (2.10) therefore gives

$$\Delta_n^{-1} \geq (1 + 1/K)\Delta_{n-1}^{-1}.$$

Thus the error decreases geometrically.

Suppose now that $F_0(L) \neq 0$. Replacing all $F_n$ by their negatives if necessary, we may assume that $F_0(L) > 0$. Formula (2.8) then gives $A_n > 0$ for all sufficiently large $n$. Since these coefficients are integers, $B_{n+1} - B_n = 2b_nA_n \geq 2$ for all sufficiently large $n$, and hence $B_n \to \infty$. We will show that $B_n \gg_{\alpha,p} n^2$.

For sufficiently large $n$, with $B_n \geq 1$ and $B_n^2 \geq 2|D|$, put

$$t = p^{a_n}, \qquad u = p^{a_{n+1}}, \qquad W_n = \max(A_{n-1}, A_n, A_{n+1}).$$

Using $C_n = p^{a_n}A_{n-1}$ in the discriminant identities gives

$$B_n^2 - D = 4tA_nA_{n-1}, \qquad B_{n+1}^2 - D = 4uA_nA_{n+1}.$$

Since the three coefficients in $W_n$ are positive and at most $W_n$, and $B_{n+1} \geq B_n$, these equalities imply $t, u \geq B_n^2/(8W_n^2)$. Subtracting the two discriminant identities, dividing by $4A_n$, and using $B_{n+1} = B_n + 2b_nA_n$, we obtain

$$uA_{n+1} - tA_{n-1} = b_nB_n + b_n^2A_n > 0.$$

Both $t$ and $u$ are powers of $p$, so $\min(t, u)$ divides both terms on the left and therefore divides the positive integer on the right. If $A_n \leq B_n$, it follows that

$$\frac{B_n^2}{8W_n^2} \leq \min(t, u) \leq b_nB_n + b_n^2A_n \leq p(p-1)B_n.$$

Hence $W_n \gg_p \sqrt{B_n}$. If $A_n > B_n$, the same conclusion follows from $W_n \geq A_n > B_n \geq \sqrt{B_n}$. Consequently,

$$B_{n+3} - B_n = 2\sum_{j=n}^{n+2} b_j A_j \geq 2W_{n+1} \gg_p \sqrt{B_n}.$$

Thus $B_{n+3} \geq B_n + c\sqrt{B_n}$ for some $c > 0$ and all sufficiently large $n$. Since $B_n \geq 1$,

$$\begin{aligned}\sqrt{B_{n+3}} - \sqrt{B_n} &\geq \sqrt{B_n + c\sqrt{B_n}} - \sqrt{B_n} \\ &= \frac{c}{\sqrt{1 + c/\sqrt{B_n}} + 1} \geq \frac{c}{\sqrt{1+c}+1}.\end{aligned}$$

Summing separately along each residue class modulo 3 gives $\sqrt{B_n} \gg_{\alpha,p} n$, and hence $B_n \gg_{\alpha,p} n^2$.

Finally, choose $N \geq 2$ so that $A_j > 0$ for every $j \geq N$. Summing (2.10) for $j = N, \ldots, n$ and using $|F_0(R_{j-1}/S_{j-1})| \leq M$, we obtain

$$\frac{1}{\Delta_n} \geq \frac{1}{M}\sum_{j=N}^{n} b_j A_j = \frac{B_{n+1} - B_N}{2M} \gg_{\alpha,p} n^2.$$

Therefore $|L - R_n/S_n| \leq \Delta_n = O_{\alpha,p}(n^{-2})$. □

**Corollary 2.4.** *Let $\alpha \in \mathbb{Q}_p$ be a quadratic $p$-adic unit such that $\mathbb{Q}(\alpha)$ is an imaginary quadratic field. Then the Schneider expansion of $\alpha$ is not eventually periodic, but its rational convergents converge in $\mathbb{R}$.*

*Proof.* The first assertion follows from Lemma 2.2, and the second from Theorem 2.3. □

For square roots of negative integers, the non-periodicity assertion was proved by de Weger [11, p. 114].

**Corollary 2.5.** *Let $\alpha$ be a quadratic $p$-adic unit, and let $L$ be the real limit of its Schneider convergents. The expansion is eventually periodic if and only if $L = \sigma(\alpha)$ for some real embedding $\sigma : \mathbb{Q}(\alpha) \to \mathbb{R}$. In this case, $\sigma$ is the unique real embedding that is positive on every complete quotient.*

*Proof.* If the expansion is eventually periodic, let $\sigma$ be the embedding supplied by Lemma 2.2. Since $\sigma(\alpha_{n+1}) > 0$, equation (2.4) places $\sigma(\alpha)$ between $R_{n-1}/S_{n-1}$ and $R_n/S_n$ for every $n \geq 1$. Both endpoints tend to $L$, so $L = \sigma(\alpha)$.

Conversely, suppose that $L = \sigma(\alpha)$ for a real embedding $\sigma$. Every complete quotient $\alpha_r$ is a quadratic $p$-adic unit. By Theorem 2.3, its Schneider convergents have a real limit $L_r \geq b_r \geq 1$. Passing to the limit in $\mathbb{R}$, we obtain

$$L_r = b_r + \frac{p^{a_{r+1}}}{L_{r+1}}.$$

Since $L_0 = L = \sigma(\alpha_0)$, this relation and (2.5) give by induction that $L_r = \sigma(\alpha_r) > 0$ for every $r \geq 0$. Lemma 2.2 proves eventual periodicity. □

When the Schneider expansion of $\alpha$ is not eventually periodic, we do not even know whether the real limit $L$ of the convergents can be rational.

For Ruban continued fractions, Capuano, Veneziano and Zannier proved that eventual periodicity implies convergence in $\mathbb{R}$ to a real image of the quadratic input [9, Theorem 6.5].

**Example 2.6.** Romeo conjectured that, for an odd prime $p$, a continued fraction with rational partial quotients which converges in $\mathbb{R}$ and $\mathbb{Q}_p$ to two images of the same quadratic irrational must be eventually periodic [17, Conjecture 16]. We retain powers of $p$ as partial numerators but allow both $p-1$ and $2p-1$, two representatives of the same residue class modulo $p$, as partial denominators. This relaxation of Schneider's rule, followed by an equivalence transformation, yields a counterexample.

Put $f(X) = X^2 - (2p-1)X - p$. Its discriminant $4p^2+1$ lies strictly between $(2p)^2$ and $(2p+1)^2$, so $f$ is irreducible over $\mathbb{Q}$. Let $\theta$ be a root of $f$, and denote its positive real image by $\theta^{(r)}$. Since $f(p-1) = -p^2$ and $f'(p-1) = -1$, Hensel's lemma gives a $p$-adic image $\theta^{(p)} \in \mathbb{Z}_p^\times$ with $\theta^{(p)} \equiv p-1 \pmod{p}$.

The identities

$$\theta = 2p - 1 + \frac{p}{\theta} = p - 1 + \frac{p^2}{p - 1 + p/\theta}$$

hold for both images. The second identity gives the two steps of the purely periodic Schneider expansion

$$\theta^{(p)} = [\overline{p-1, p^2 : p-1, p}]_p,$$

since $\theta^{(p)} - (p-1) = p^2/(p-1+p/\theta^{(p)})$ has valuation 2, and the next complete quotient $p-1+p/\theta^{(p)}$ differs from $p-1$ by a number of valuation 1. If $2p-1$ is also allowed, the first identity gives a return to $\theta$ in one step.

To keep the final tail unchanged in the equivalence transformation below, we prepend the step $\theta = 2p-1+p/\theta$ to each expression and use

$$\theta = [2p-1, p : 2p-1, p : \theta], \qquad \theta = [2p-1, p : p-1, p^2 : p-1, p : \theta]. \tag{2.11}$$

Here the final entry $\theta$ is the remaining tail. Starting with $\theta$, repeatedly replace its final occurrence using one of the expressions in (2.11). Use the second expression at stages $1, 4, 9, \ldots$ of this replacement process and the first at all other stages. This determines an infinite generalized continued fraction with partial denominators $p-1, 2p-1$ and partial numerators $p, p^2$. At every position the remaining tail is either $\theta$ or $p-1+p/\theta$, so its real image is positive and its $p$-adic image is a unit.

Let $R_n/S_n$ be its convergents, and define $\Delta_n$ and $T_n$ as before. All partial denominators are positive integers, so the recurrence for $S_n$ gives $S_{n-1} \geq S_{n-2}$. Since all exponents are at most 2, (2.3) gives

$$\Delta_n \leq \frac{1}{1+p^{-2}}\Delta_{n-1}$$

for $n \geq 2$. Thus $\Delta_n \to 0$. For $n \geq 1$, identity (2.4) remains valid and, since the real tails are positive, gives

$$\left|\theta^{(r)} - \frac{R_n}{S_n}\right| < \Delta_n.$$

Every partial denominator is congruent to $-1$ modulo $p$, so every $S_n$ is a $p$-adic unit. Since the $p$-adic tails are units, the same identity gives

$$\left|\theta^{(p)} - \frac{R_n}{S_n}\right|_p = T_{n+1}^{-1} \longrightarrow 0.$$

Hence the convergents tend to $\theta^{(r)}$ in $\mathbb{R}$ and to $\theta^{(p)}$ in $\mathbb{Q}_p$.

To put the continued fraction in the form required by [17, Conjecture 16], replace the partial numerators by 1 using the usual equivalence transformation. For the two blocks

in (2.11), this gives

$$[2p-1, p : 2p-1, p : t] = 2p - 1 + \cfrac{1}{\cfrac{2p-1}{p} + \cfrac{1}{t}},$$

$$[2p-1, p : p-1, p^2 : p-1, p : t] = 2p - 1 + \cfrac{1}{\cfrac{p-1}{p} + \cfrac{1}{\cfrac{p-1}{p} + \cfrac{1}{t}}},$$

where $t$ denotes the remaining tail. The tail is unchanged in both cases, so these transformations can be applied successively throughout the construction, preserving every convergent. The resulting partial quotients belong to the finite set $\{2p-1, (2p-1)/p, (p-1)/p\}$ of positive rational numbers. The entry $(p-1)/p$ occurs twice at each use of the second expression and nowhere else, at the same positions as $p-1$ before the transformation. Between the $k$-th and $(k+1)$-st such pairs there are $2k$ uses of the first expression. The partial denominator $p-1$ in the original expansion and the partial quotient $(p-1)/p$ in the transformed expansion each occur infinitely often with unbounded gaps between successive occurrences. Thus neither expansion is eventually periodic. The transformed continued fraction disproves the conjecture.

## 3. Schneider and ordinary period lengths

Let $\alpha$ be a quadratic $p$-adic unit with an eventually periodic Schneider expansion. By Corollary 2.5, its Schneider convergents tend in $\mathbb{R}$ to the positive real image $x$ of $\alpha$ obtained from the embedding that is positive on every complete quotient. The ordinary continued fraction of $x$ is therefore eventually periodic. We use *period* for a repeating block. Period lengths are always minimal, counting pairs $(b_n, a_{n+1})$ for Schneider expansions and partial quotients for ordinary continued fractions. We denote by $s_p(\alpha)$ the Schneider period length and by $r(\alpha)$ the period length of the ordinary continued fraction of $x$.

Ordinary real continued fractions are written in the standard form

$$[a_0, a_1, a_2, \ldots],$$

with an overline indicating the periodic part. We continue to write Schneider $p$-adic continued fractions as

$$[b_0, p^{a_1} : b_1, p^{a_2} : \ldots]_p.$$

For a purely periodic expansion $\alpha = \overline{[b_0, p^{a_1} : b_1, p^{a_2} : \ldots : b_{m-1}, p^{a_m}]}_p$, induction using (2.1) gives

$$\begin{pmatrix} b_0 & p^{a_1} \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} b_{m-1} & p^{a_m} \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} R_{m-1} & p^{a_m} R_{m-2} \\ S_{m-1} & p^{a_m} S_{m-2} \end{pmatrix}.$$

We call this product the period matrix of $\alpha$. Since $\alpha_m = \alpha$, (2.4) shows that $\alpha$ satisfies

$$S_{m-1} X^2 + \big(p^{a_m} S_{m-2} - R_{m-1}\big) X - p^{a_m} R_{m-2} = 0.$$

Since $S_{m-1} > 0$ and $R_{m-2} > 0$, this polynomial has two real roots of opposite signs. By Theorem 2.3 and Corollary 2.5, the Schneider convergents tend in $\mathbb{R}$ to $x = \sigma(\alpha)$, where $\sigma$ is positive on every complete quotient. Thus $x$ is the positive root, its conjugate $x'$ is negative, and (2.5) gives $x > b_0 \geq 1$.

Since $x' < 0$ and $\lfloor x \rfloor \geq 1$, the next complete quotient in the ordinary expansion satisfies

$$x_1 = \frac{1}{x - \lfloor x \rfloor} > 1, \qquad -1 < x_1' = \frac{1}{x' - \lfloor x \rfloor} < 0.$$

Thus $x_1$ is reduced. Galois's theorem [12, pp. 298–299] shows that its ordinary continued fraction is purely periodic. Hence the ordinary continued fraction of $x$ has a preperiod of length at most one. By uniqueness of Schneider's expansion, if the displayed Schneider block is not a repetition of a shorter block, then its Schneider period length is $m$.

3.1. **Bounded Schneider and ordinary period lengths.** Ordinary period length one occurs with different Schneider period lengths. For an odd prime $p$ and $n \geq 1$, the purely periodic expansion

$$\alpha_n = [\overline{2, p^{2n}}]_p$$

has positive real value

$$1 + \sqrt{p^{2n} + 1} = [p^n + 1, \overline{2p^n}],$$

and hence $(s_p(\alpha_n), r(\alpha_n)) = (1, 1)$. There are also examples with a larger Schneider period length. The purely periodic Schneider expansion

$$[\overline{p-2, p : p-1, p : p-2, p^2}]_p$$

has a period matrix whose fixed points satisfy

$$X^2 + 2X - p^2 = 0.$$

Its positive real value is

$$-1 + \sqrt{p^2 + 1} = [p - 1, \overline{2p}],$$

so in this case $(s_p, r) = (3, 1)$.

For a fixed odd prime $p$, it remains to be determined whether purely periodic quadratic Schneider expansions with ordinary period length one can have arbitrarily large Schneider period lengths.

3.2. **Unbounded Schneider and bounded ordinary period lengths.** The next result shows that ordinary period length two permits arbitrarily large Schneider period lengths for every odd prime.

**Theorem 3.1.** *Let $p$ be odd and let $m \geq 3$ be odd. The purely periodic Schneider expansion*

$$\alpha_m = [\overline{p-2, p : p-1, p : \ldots : p-1, p}]_p,$$

*where $p - 1$ occurs $m - 1$ times among the partial denominators, has $s_p(\alpha_m) = m$. Its positive real value is*

$$x_m = \left[p - 1, \overline{\frac{p(p^m + 1)}{p + 1}, p}\right],$$

*so $r(\alpha_m) = 2$.*

*Proof.* The displayed Schneider period is not a repetition of a shorter block, since $p - 2$ occurs exactly once among its partial denominators.

For any prime $p$, induction gives

$$\begin{pmatrix} p-1 & p \\ 1 & 0 \end{pmatrix}^n = \frac{1}{p+1} \begin{pmatrix} p^{n+1} + (-1)^n & p^{n+1} - p(-1)^n \\ p^n - (-1)^n & p^n + p(-1)^n \end{pmatrix} \qquad (n \geq 1). \tag{3.1}$$

By (3.1), the fixed-point equation for $\left(\begin{smallmatrix} p-2 & p \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} p-1 & p \\ 1 & 0 \end{smallmatrix}\right)^{m-1}$ simplifies to

$$\big(x_m - (p-1)\big)(x_m + 1) = \frac{p+1}{p^m + 1}.$$

Put

$$c_m = \frac{p(p^m + 1)}{p + 1}.$$

This is an integer because $m$ is odd. The positive number $[p-1, \overline{c_m, p}]$ satisfies

$$\big(x-(p-1)\big)(x+1) = \frac{p}{c_m} = \frac{p+1}{p^m+1},$$

and is therefore $x_m$. Since $c_m > p$, the displayed period has minimal length two. □

For $p = 2$, ordinary period lengths one and two cannot occur in the purely periodic Schneider case.

**Proposition 3.2.** *Let $\alpha$ be a quadratic irrational with a purely periodic Schneider expansion in $\mathbb{Q}_2$. Then $r(\alpha) \geq 3$.*

*Proof.* Write the period matrix as

$$\begin{pmatrix} A & B \\ C & E \end{pmatrix}.$$

Every factor $\left(\begin{smallmatrix} 1 & 2^a \\ 1 & 0 \end{smallmatrix}\right)$ is congruent modulo 2 to

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},$$

which is idempotent. Hence $A$ and $C$ are odd, while $B$ and $E$ are even. The fixed points satisfy

$$CX^2 + (E-A)X - B = 0.$$

Dividing by $\gcd(C, E-A, B)$, which is odd since $C$ is odd, gives a primitive polynomial $aX^2 + bX + c$. Since $E - A$ is odd and $B$ is even, its discriminant satisfies

$$b^2 - 4ac = \frac{(E-A)^2 + 4BC}{\gcd(C, E-A, B)^2} \equiv 1 \pmod 8.$$

The positive fixed point $x$ has a negative conjugate, so after at most one step of the ordinary continued fraction algorithm we obtain a reduced quadratic irrational, as explained at the beginning of this section. This step preserves the discriminant of the primitive polynomial. Indeed, putting $k = \lfloor x \rfloor$, we find that $1/(x-k)$ satisfies

$$(ak^2 + bk + c)Y^2 + (2ak + b)Y + a = 0.$$

The coefficients are coprime, since any common divisor divides $a$, then $b$, and then $c$. The discriminant is $(2ak+b)^2 - 4a(ak^2+bk+c) = b^2 - 4ac$.

A reduced real quadratic irrational of period length one has the form $[\overline{d}]$ and discriminant $d^2 + 4$, which is never 1 modulo 8. For period length two, write the reduced number as $[\overline{u, v}]$. A polynomial satisfied by it is

$$vX^2 - uvX - u.$$

If $g = \gcd(u, v)$, $u = gu_0$, and $v = gv_0$, then the discriminant of the primitive polynomial is

$$u_0 v_0 \big(g^2 u_0 v_0 + 4\big).$$

This is either even or congruent to 5 modulo 8. Thus neither period length one nor period length two is possible. □

The lower bound in Proposition 3.2 is attained with arbitrarily large Schneider period lengths.

**Theorem 3.3.** *Let $m \geq 3$ be odd. The purely periodic Schneider expansion*

$$\alpha_m = [\overline{1, 4 : 1, 2 : \ldots : 1, 2}]_2,$$

*where the initial partial numerator is* 4 *and the remaining* $m-1$ *partial numerators are* 2*, has* $s_2(\alpha_m) = m$*. Its positive real value is*

$$x_m = \left[2, 1, \overline{\frac{2^{m+1}-1}{3}, 3}\right],$$

*so* $r(\alpha_m) = 3$*.*

*Proof.* The displayed Schneider period is not a repetition of a shorter block because the partial numerator 4 occurs exactly once. Put

$$c_m = \frac{2^{m+1}-1}{3},$$

which is an integer because $m$ is odd. Using (3.1) with $p = 2$, we find that the fixed points of $\left(\begin{smallmatrix}1 & 4\\ 1 & 0\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & 2\\ 1 & 0\end{smallmatrix}\right)^{m-1}$ satisfy

$$(c_m+1)X^2 - 2c_mX - 3c_m - 5 = 0.$$

The positive number $[2, \overline{1, c_m, 3}]$ satisfies this equation and is therefore $x_m$. The ordinary block $(1, c_m, 3)$ is not a repetition of a shorter block, so $r(\alpha_m) = 3$. □

Thus $s_p(\alpha)/r(\alpha)$ is unbounded even among purely periodic Schneider expansions. For odd $p$ this already holds with $r(\alpha) = 2$, while for $p = 2$ the least possible ordinary period length is 3.

3.3. **Bounded Schneider and unbounded ordinary period lengths.** For every prime $p$, the ordinary period length is unbounded even when the Schneider period length is one. Indeed, the positive real value of

$$\alpha_n = [\overline{1, p^{2n+1}}]_p, \qquad n \geq 1,$$

is

$$x_n = \frac{1+\sqrt{1+4p^{2n+1}}}{2}.$$

The number $\alpha_n$ is irrational by Bundschuh's criterion [7], since all partial numerators in its Schneider expansion are larger than $p$. Its minimal polynomial over $\mathbb{Q}$ is therefore $X^2 - X - p^{2n+1}$, so the positive real root $x_n$ of this polynomial is irrational as well. Apply Bugeaud and Luca's result [6, Corollary 2.2] to the power sum $f(n) = 4p(p^2)^n + 1$. The required condition holds because neither $4p$ nor $4p^3$ is a square. Hence the ordinary period length of $\sqrt{f(n)}$ tends to infinity. Cohen's bound for multiplication by 2 [10, Corollary 4.4(b)] shows that the ordinary period length of $2x_n$ is at most $5r(\alpha_n)$. Since $2x_n = 1 + \sqrt{f(n)}$ has the same period length as $\sqrt{f(n)}$, it follows that $r(\alpha_n) \to \infty$. The following elementary construction gives Schneider period length two and ordinary period length exactly $3(n-1)$, for every prime $p$ and every $n \geq 2$.

**Theorem 3.4.** *Let* $p$ *be any prime and* $n \geq 2$*. The purely periodic Schneider expansion*

$$\alpha_{p,n} = [\overline{1, p : 1, p^n}]_p$$

*has* $s_p(\alpha_{p,n}) = 2$ *and* $r(\alpha_{p,n}) = 3(n-1)$*. More precisely, if* $x_{p,n}$ *is its positive real value, then*

$$x_{p,n} = [1, \overline{p^{n-1}-1,\ (p^j-1, 1, p^{n-1-j}-1)_{j=1}^{n-2},\ p^{n-1}-1,\ p^n-p+1}],$$

*where the triples are written in increasing order of* $j$ *and are absent when* $n = 2$*.*

*Proof.* The displayed Schneider block is not a repetition of a shorter block. Its period matrix is $\left(\begin{smallmatrix}1&p\\1&0\end{smallmatrix}\right)\left(\begin{smallmatrix}1&p^n\\1&0\end{smallmatrix}\right)=\left(\begin{smallmatrix}p+1&p^n\\1&p^n\end{smallmatrix}\right)$, so $x=x_{p,n}$ satisfies $x^2+(p^n-p-1)x-p^n=0$. The polynomial on the left is negative at 1 and positive at 2, with respective values $-p$ and $p^n-2p+2>0$. Hence $1<x<2$. Put $y=1/(x-1)$. Substituting $x=1+1/y$ gives

$$py^2-(p^n-p+1)y-1=0. \tag{3.2}$$

The polynomial on the left equals $-1$ at $p^{n-1}-1+1/p$ and equals $(p-1)p^{n-1}-1>0$ at $p^{n-1}$, so $p^{n-1}-1+1/p<y<p^{n-1}$.

Equation (3.2) gives

$$(py-1)(y+1)=p^ny, \qquad (py-1)(y+1-p^{n-1})=p^{n-1}.$$

The second identity yields

$$\frac{1}{y-(p^{n-1}-1)}=\frac{py-1}{p^{n-1}}.$$

Since $y+1>p^{n-1}$, the first identity gives, for $1\le j\le n-1$,

$$\frac{py-1}{p^{n-j}}=p^j-\frac{p^j}{y+1},$$

and hence

$$p^j-1<\frac{py-1}{p^{n-j}}<p^j.$$

The bounds for $y$ also give, for $1\le j\le n-2$,

$$p^{n-1-j}-1<\frac{y+1-p^j}{p^j}<p^{n-1-j}.$$

For $1\le j\le n-2$, we also have $y+1>p^{n-1}\ge 2p^j$, and hence

$$1<1+\frac{p^j}{y+1-p^j}<2.$$

Using the two product identities above, we obtain, for $1\le j\le n-2$,

$$\frac{1}{\dfrac{py-1}{p^{n-j}}-(p^j-1)}=1+\frac{p^j}{y+1-p^j}, \qquad \frac{1}{\dfrac{y+1-p^j}{p^j}-(p^{n-1-j}-1)}=\frac{py-1}{p^{n-1-j}}.$$

For each $1\le j\le n-2$, these formulas give three successive partial quotients $p^j-1$, 1 and $p^{n-1-j}-1$. The next complete quotient has the same form with $j$ replaced by $j+1$.

Finally, dividing (3.2) by $y$ gives

$$py=p^n-p+1+\frac{1}{y}.$$

Since $y>1$, the integer part of $py$ is $p^n-p+1$. The last two steps are therefore

$$\frac{1}{\dfrac{py-1}{p}-(p^{n-1}-1)}=py, \qquad \frac{1}{py-(p^n-p+1)}=y.$$

If $n=2$, the successive complete quotients follow the cycle

$$y\longmapsto\frac{py-1}{p}\longmapsto py\longmapsto y.$$

If $n \geq 3$, they follow the cycle

$$y \longmapsto \frac{py-1}{p^{n-1}} \longmapsto 1 + \frac{p}{y+1-p} \longmapsto \frac{y+1-p}{p}$$
$$\longmapsto \frac{py-1}{p^{n-2}} \longmapsto \cdots \longmapsto \frac{py-1}{p} \longmapsto py \longmapsto y.$$

In both cases, the preceding inequalities identify the integer parts and give the displayed ordinary continued fraction expansion. The displayed block has length $1+3(n-2)+2 = 3(n-1)$. The last entry $p^n - p + 1$ is larger than every preceding entry and occurs only once, so the displayed block cannot be a repetition of a shorter block. □

The ordinary period length $3(n-1)$ in the preceding theorem also follows from Williams's formula [21, Section 3, Case C(7)].

Theorem 3.4 shows that $r(\alpha)/s_p(\alpha)$ is unbounded even among purely periodic Schneider expansions.

3.4. **Families in which both period lengths grow.** The following family is a quadratic analogue of the rational families with simultaneous growth of both lengths in [15, Theorems 3.2 and 3.3].

**Theorem 3.5.** *Let $p$ be any prime and $n \geq 1$. The purely periodic Schneider expansion*

$$\alpha_n = [\overline{1, p^2 : 1, p^2 : \ldots : 1, p^2 : 1, p^3}]_p,$$

*where the partial numerator $p^2$ occurs $2n$ times and the final partial numerator is $p^3$, represents a quadratic irrational and satisfies*

$$s_p(\alpha_n) = 2n+1, \qquad r(\alpha_n) > c_p n - 5,$$

*where*

$$\lambda_p = \frac{1+\sqrt{1+4p^2}}{2}, \qquad c_p = \frac{\log(\lambda_p/(\lambda_p-1))}{\log(2p)} > 0.$$

*Proof.* By Bundschuh's criterion [7], a nonterminating Schneider expansion of a rational number eventually has the tail $[p-1, p : p-1, p : p-1, \ldots]_p$. Thus $\alpha_n$ is irrational, and periodicity shows that it is quadratic. Since $p^3$ occurs exactly once in the displayed period, this period is not a repetition of a shorter block. Hence $s_p(\alpha_n) = 2n+1$.

Let $x_n$ be the real limit of the Schneider convergents of $\alpha_n$. We first estimate its distance from $\lambda_p$, the positive fixed point of the map $t \mapsto 1 + p^2/t$ used at each of the first $2n$ steps of the period. The number $\lambda_p$ satisfies $\lambda_p^2 - \lambda_p = p^2$ and $p < \lambda_p < p+1$. Direct calculation gives

$$\lambda_p = [p, \overline{1, 1, 2p-1}].$$

We will use this expansion to obtain a lower bound for the ordinary period length of $x_n$.

By Lemma 2.2 and Corollary 2.5, there is a real embedding sending $\alpha_n$ to $x_n$ and all its Schneider complete quotients to positive numbers. Let $y_0 = x_n, y_1, \ldots, y_{2n}$ be the images of the successive complete quotients in one period under this embedding. Applying (2.5) gives

$$y_j = 1 + \frac{p^2}{y_{j+1}} \quad (0 \leq j < 2n), \qquad y_{2n} = 1 + \frac{p^3}{y_0}.$$

In particular, every $y_j$ exceeds 1. For $t > 1$, substituting $p^2 = \lambda_p(\lambda_p - 1)$ gives

$$\frac{1 + p^2/t - \lambda_p}{p^2/t + \lambda_p} = -\frac{\lambda_p - 1}{\lambda_p} \frac{t - \lambda_p}{t + \lambda_p - 1}.$$

Applying this identity with $t = y_1, \ldots, y_{2n}$, we obtain

$$\left|\frac{x_n - \lambda_p}{x_n + \lambda_p - 1}\right| = \left(\frac{\lambda_p - 1}{\lambda_p}\right)^{2n} \left|\frac{y_{2n} - \lambda_p}{y_{2n} + \lambda_p - 1}\right| < \left(\frac{\lambda_p - 1}{\lambda_p}\right)^{2n},$$

where the last inequality follows from $y_{2n} > 1$ and $\lambda_p > 1$. Also, $x_n = 1 + p^2/y_1 < 1 + p^2$, so $x_n + \lambda_p - 1 < \lambda_p^2$. If $x_n = \lambda_p$, the first $2n$ relations for the $y_j$ would give $y_1 = \cdots = y_{2n} = \lambda_p$, whereas the last would give $\lambda_p = 1 + p^3/\lambda_p$, contradicting $\lambda_p^2 - \lambda_p = p^2$. Therefore

$$0 < |x_n - \lambda_p| < \lambda_p^2 \left(\frac{\lambda_p - 1}{\lambda_p}\right)^{2n}.$$

We now relate this distance to $r = r(\alpha_n)$. As shown at the beginning of Section 3, the ordinary continued fraction of $x_n$ is periodic after its initial partial quotient, with period length $r$. Its first $r+4$ partial quotients cannot all agree with those of $\lambda_p$, counting the initial partial quotient. Otherwise, the entries in positions 3 and $r+3$ after the initial partial quotient would be equal by periodicity. In the expansion of $\lambda_p$, the value $2p-1$ occurs exactly in positions divisible by 3, so this equality would imply $3 \mid r$. Both expansions would then repeat their common first $r$ entries after the initial partial quotient, giving $x_n = \lambda_p$, a contradiction.

Write $P_j/Q_j$ for the ordinary convergents of $\lambda_p$, indexed from $P_0/Q_0 = p$. The irrational numbers sharing its first $r+4$ partial quotients lie between

$$\frac{P_{r+3}}{Q_{r+3}} \quad \text{and} \quad \frac{P_{r+3} + P_{r+2}}{Q_{r+3} + Q_{r+2}},$$

and every irrational number between these endpoints has those partial quotients. Thus $\lambda_p$ lies inside this interval and $x_n$ outside it. We estimate the distance from $\lambda_p$ to each endpoint.

All partial quotients of $\lambda_p$ are at most $2p-1$. The recurrence for $Q_j$ therefore gives $Q_j \le 2pQ_{j-1}$ and hence $Q_j \le (2p)^j$. Each endpoint $a/b$ above belongs to $[p, p+1]$ and has $b \le 2(2p)^{r+3}$. Using $\lambda_p^2 - \lambda_p = p^2$, we have

$$a^2 - ab - p^2b^2 = b^2\left(\frac{a}{b} - \lambda_p\right)\left(\frac{a}{b} + \lambda_p - 1\right).$$

Since $\lambda_p$ is irrational, the left-hand side is a nonzero integer, so its absolute value is at least 1. Together with $a/b + \lambda_p - 1 < 2p + 1$, this gives

$$\left|\lambda_p - \frac{a}{b}\right| \ge \frac{1}{(2p+1)b^2} \ge \frac{1}{4(2p+1)(2p)^{2(r+3)}}.$$

Since $x_n$ lies outside the interval, its distance from $\lambda_p$ is at least the smaller of the distances to the two endpoints. Combining this with the upper bound already proved gives

$$\frac{1}{4(2p+1)(2p)^{2(r+3)}} \le |x_n - \lambda_p| < \lambda_p^2 \left(\frac{\lambda_p - 1}{\lambda_p}\right)^{2n}.$$

Taking logarithms and using the definition of $c_p$, we conclude that

$$r > c_p n - 3 - \frac{\log\bigl(4(2p+1)\lambda_p^2\bigr)}{2\log(2p)} > c_p n - 5.$$

The last inequality follows from $4(2p+1)\lambda_p^2 < 4(2p+1)(p+1)^2 < (2p)^4$ for $p \ge 2$. □

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