On rational numbers with nonterminating p-adic continued fraction expansion

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Abstract

We determine all pairs (p, n), where p is a prime and n a positive integer, such that there exists a reduced fraction $u/v > 1$ with $u+v = n$ and u/v has a nonterminating Schneider's p-adic continued fraction expansion. We also prove a bound on the length of the preperiod in the p-adic continued fraction of u/v when $\max\{|u|, |v|\} < p$.

1 Introduction

For a prime number p, Schneider's p-adic continued fraction is an expression of the form

$$b_0 + \cfrac{p^{a_1}}{b_1 + \cfrac{p^{a_2}}{b_2 + \cfrac{p^{a_3}}{\ddots}}} \tag{1}$$

written more succinctly as $[b_0, p^{a_1} : b_1, p^{a_2} : b_2, p^{a_3} : \ldots]$, where all a_i are positive integers and $b_i \in \{1, 2, \ldots, p-1\}$.

The numbers b_i are called partial denominators and by splitting the expression (1) at any point, we obtain the convergents $[b_0, p^{a_1} : b_1, \ldots, p^{a_k} : b_k]$ and the complete quotients $[b_k, p^{a_{k+1}} : b_{k+1}, \ldots]$.

A finite Schneider's p-adic continued fraction clearly represents a positive rational number. The value attached to an infinite p-adic continued fraction is the limit of the sequence of its convergents in the p-adic field. Conversely, every p-adic unit can be expressed as the value of a unique Schneider's p-adic continued fraction. For more details on Schneider's continued fractions see

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[6, 9, 10], while a more general overview of different p-adic continued fraction algorithms can be found in the survey [8].

Unlike the usual simple continued fraction algorithm in the reals, an expansion of a rational number into Schneider's p-adic continued fraction can be infinite. Here and throughout, we can restrict ourselves to rationals which are p-adic units simply by factoring out at the start any power of p from the numerator or the denominator. As Bundschuh [2] showed, a rational number has a nonterminating p-adic continued fraction if and only if a negative complete quotient is encountered at some point in its p-adic continued fraction expansion. This clearly happens if we start with a negative rational number, but can also happen (more often than not, see [7]) in the expansion of a positive rational number. In that case, eventually -1 has to appear as a complete quotient [2]. Since $-1 = [p-1, p : p-1, p : p-1, \ldots]$, this is equivalent to saying that the continued fraction expansion of a rational number does not terminate if and only if from some point onwards in the expansion, the block $p-1, p$ repeats indefinitely.

Following [3] and [7], for a positive integer n, denote by $S_p(n)$ the set of all positive rational numbers u/v, where $u+v=n$ and $\gcd(u,v) = \gcd(uv,p) = 1$. Also, let $T_p(n)$ be the set of elements in $S_p(n)$ with terminating p-adic continued fraction expansion.

In the first part of this paper, we completely determine all pairs (p, n) such that there is a nontrivial fraction $\frac{u}{v} \in S_p(n) \setminus T_p(n)$. In other words, we determine for which (p, n) there is a fraction $\frac{u}{v} \in S_p(n)$ greater than 1 with nonterminating p-adic continued fraction expansion. A previous partial result [7] only considered the case of p an odd prime and n an odd positive integer such that $n-1$ is not divisible by p. Let us explain why we call $\frac{u}{v} > 1$ a nontrivial example. Since $\frac{u}{v} \in S_p(n)$ if and only if $\frac{v}{u} \in S_p(n)$, when $S_p(n)$ is nonempty, it certainly contains a number smaller than 1, whose p-adic continued fraction expansion is obviously nonterminating (we immediately obtain a negative complete quotient). However, such an example tells us only that $S_p(n)$ is nonempty and has little to do with p-adic continued fractions for particular p and n. Thus, we look for nontrivial examples of elements of $S_p(n)$ with infinite expansion, i.e. we require that such examples be greater than 1.

In the second part of the paper, we prove two results on p-adic continued fraction expansion of rational numbers with both numerator and denominator less than p.

2 Sets containing rationals with nonterminating expansion

As is easily seen [3, Proposition 6], an integer $n \not\equiv 0 \pmod{p}$ has a finite p-adic continued fraction if and only if $n \in \{1, 2, \ldots, p-1\}$ or $n = b + p^a$, where $b \in \{1, 2, \ldots, p-1\}$ and a is a positive integer. We will also frequently use the following simple lemma.

Lemma 1. *Let $\frac{u}{v} > 1$ be a reduced fraction such that the prime p does not divide uv.*

If for every $k \in \{1, \ldots, \lfloor \frac{u}{v} \rfloor\}$, the integer $u - kv$ is not divisible by p, then the p-adic continued fraction expansion of $\frac{u}{v}$ is nonterminating.

If $p > u > v > 1$, then $\frac{u}{v}$ has nonterminating p-adic continued fraction expansion.

If $p > u > v = 1$, then $\frac{u}{v} = u$ has terminating expansion.

Proof. The condition that $u - kv$ is not divisible by p for $k \in \{1, \ldots, \lfloor \frac{u}{v} \rfloor\}$ implies that the first partial denominator (b_0 in (1)) is greater than $\lfloor \frac{u}{v} \rfloor$. Therefore, already the next complete quotient in the p-adic continued fraction expanion of $\frac{u}{v}$ is negative, so the expansion does not terminate. The other two statements are then obvious. □

Our main theorem is the following.

Theorem 2. *For every prime p and every positive integer $n \notin \{1, 2, 3, 4, 6\}$ such that $(p, n) \notin E$, there is a rational number $\frac{u}{v} > 1$ such that $\frac{u}{v} \in S_p(n)$ and the p-adic continued fraction expansion of $\frac{u}{v}$ is nonterminating. Here E is the set containing all pairs $(2, 2k+1)$, $k \geq 2$ and the following elements*

$$\begin{aligned} &(2, 10), \\ &(3, 5),\ (3, 7),\ (3, 10),\ (3, 11),\ (3, 13),\ (3, 16),\ (3, 28), \\ &(5, 8),\ (5, 9), \\ &(7, 10),\ (7, 12),\ (7, 15). \end{aligned}$$

For $(p, n) \in E$, such a rational number does not exist.

We excluded some small positive integers as values of n since the results are trivial in those cases. Namely,

$$\begin{aligned} &S_p(1) = \emptyset, \quad S_p(2) \cap (1, +\infty) = \emptyset, \\ &S_p(n) \cap (1, +\infty) \subseteq \left\{\tfrac{n-1}{1}\right\} \text{ for } n \in \{3, 4, 6\}. \end{aligned}$$

Proof. We prove the theorem by considering the following six cases:

1. $p = 2$,
2. $p > 3$, n odd,
3. $p > 3$, n even,
4. $p > 3$, $n \in \{2p-2, 2p-1, 2p+1\}$,
5. $p = 3$, n odd,
6. $p = 3$, n even.

Case 1. Let $p = 2$.

For odd n, the set $S_2(n)$ is empty, so we take $n > 6$ to be an even integer. Write $n = 2 + 2^t r$, where $t \geq 1$ and r is an odd positive integer.

If $r > 1$, then

$$\frac{n-1}{1} = 1 + \frac{2^t}{\frac{1}{r}},$$

where $\frac{1}{r} \in (0,1)$, so $\frac{n-1}{1} \in S_2(n) \setminus T_2(n)$.

If $r = 1$, we have $n = 2 + 2^t$ and

$$\frac{\frac{n}{2}+2}{\frac{n}{2}-2} = 1 + \frac{4}{2^{t-1}-1}.$$

For $t \geq 4$,

$$2^{t-1} - 1 = 1 + \frac{2}{\frac{1}{2^{t-2}-1}},$$

and $2^{t-2} - 1 \geq 2^2 - 1 > 1$ implying $(\frac{n}{2}+2)/(\frac{n}{2}-2) \in S_2(n) \setminus T_2(n)$. For $t \leq 2$, we get $n \leq 6$, so the only remaining possibility is $t = 3$, $n = 10$ in which case

$$S_2(10) \cap (1, +\infty) = \left\{\tfrac{9}{1}, \tfrac{7}{3}\right\} = T_2(10).$$

Case 2. Let $p > 3$ and $n > 3$ odd.

If p does not divide $(n-1)(n+1)$, then the fraction

$$\frac{\frac{n+1}{2}}{\frac{n-1}{2}} = 1 + \frac{1}{\frac{n-1}{2}}$$

is in $S_p(n)$ and lies in the interval $(1,2)$, so Lemma 1 immediately implies it is not in $T_p(n)$.

Suppose $n \equiv -1 \pmod{p}$, so $n = 2kp - 1$ for a positive integer k. The fraction $\frac{n-1}{1} = p-2+(2k-1)p$ is in $S_p(n)$ and it is in $T_p(n)$ only if $2k-1 = p^t$ for a nonnegative integer t. If $t = 0$, then $k = 1$ and $n = 2p - 1$ which is a case that we deal with later. Thus, take t to be a positive integer and $n = p^{t+1} + p - 1$. Now, the fraction

$$\frac{\frac{n+p}{2}}{\frac{n-p}{2}} = 1 + \frac{p}{\frac{p-1}{2} + \frac{\frac{p}{2}}{p^t-1}}$$

is in $S_p(n)$, but it is not in $T_p(n)$ since $\frac{2}{p^t-1} < 1$.

Next, assume $n \equiv 1 \pmod{p}$, so $n = 2kp + 1$ for a positive integer k. The fraction $\frac{n-2}{2} = \frac{p-1}{2} + \frac{2k-1}{2}p$ is in $S_p(n)$ and it belongs to $T_p(n)$ only if $2k - 1 = p^t$ for a nonnegative integer t. If $t = 0$, then $k = 1$ and $n = 2p + 1$, which we consider later, so take $t \geq 1$ and $n = p^{t+1} + p + 1$. Then the fraction

$$\frac{\frac{n+p}{2}}{\frac{n-p}{2}} = 1 + \frac{p}{\frac{p+1}{2} + \frac{\frac{p}{2}}{p^t-1}}$$

is again in $S_p(n) \setminus T_p(n)$.

Case 3. Let $p > 3$ and $n > 6$ even.

Consider the fractions

$$\frac{\frac{n}{2} + k}{\frac{n}{2} - k} = 1 + \frac{2k}{\frac{n}{2} - k} \tag{2}$$

for $k \in \{1, 2, 3, 4\}$.

If $n > 24$, we have $6k < n$, so $2k < \frac{n}{2} - k$ and all of the fractions in (2) belong to the interval $(1, 2)$. Since p is neither 2 nor 3, we also see that p does not divide $2k$. Thus, if any of these fractions is in $S_p(n)$, Lemma 1 immediately implies it is not in $T_p(n)$ and we are finished.

We have that $\gcd(\frac{n}{2} - k, \frac{n}{2} + k) = \gcd(\frac{n}{2} - k, 2k)$ divides 8 unless $k = 3$ and n is divisible by 3.

If $\frac{n}{2}$ is odd, all the numbers in the set $\{\frac{n}{2} - 4, \frac{n}{2} - 2, \frac{n}{2} + 2, \frac{n}{2} + 4\}$ are odd and at most one of them is divisible by p, so (2) is an element of $S_p(n)$ for at least one $k \in \{2, 4\}$.

If $\frac{n}{2}$ is even, all the numbers in the set $\{\frac{n}{2} - 3, \frac{n}{2} - 1, \frac{n}{2} + 1, \frac{n}{2} + 3\}$ are odd and at most one of them is divisible by p, so a fraction (2) is in $S_p(n)$ for at least one $k \in \{1, 3\}$ unless p divides $\frac{n}{2} - 1$ or $\frac{n}{2} + 1$ and n is divisible by 3.

If $\frac{n}{2} - 1$ is divisible by p, then $n = 2rp^t + 2$ for some positive integers r and t such that p does not divide r. Then

$$\frac{n-1}{1} = 1 + 2rp^t = 1 + \frac{p^t}{\frac{1}{2r}} \in S_p(n) \setminus T_p(n).$$

Assume now that $\frac{n}{2}$ is even, $\frac{n}{2} + 1$ is divisible by p and n is divisible by 3, so that $n = 2kp - 2$ for some odd positive integer k. The fraction

$\frac{n-1}{1} = p - 3 + (2k-1)p$ is in $S_p(n) \setminus T_p(n)$ unless $2k - 1 = p^t$ for some nonnegative integer t. If $t = 0$, then $k = 1$ and $n = 2p - 2$ which we consider later, so take $t \geq 1$ and $n = p^{t+1} + p - 2$. For the fraction

$$\begin{aligned}\frac{\frac{n}{2}+p}{\frac{n}{2}-p} &= 1 + \frac{2p}{\frac{n}{2}-p} = 1 + \frac{p}{\frac{p^{t+1}-p-2}{4}} \\ &= 1 + \frac{p}{\frac{p-1}{2} + p \cdot \frac{p^t-3}{4}},\end{aligned} \tag{3}$$

we have $\gcd(\frac{n}{2}+p, \frac{n}{2}-p) = \gcd(\frac{n}{2}+p, 2p) = 1$, so this fraction is in $S_p(n)$. Since $\frac{p^t-3}{4}$ is not divisible by p and it is strictly greater than 1 for $p^t > 7$, we conclude that in this case (3) is not in $T_p(n)$. Checking $p^t = 5$, we get that $n = 25+5-2 = 28$ is not divisible by 3, while for $p^t = 7$, $n = 49+7-2 = 54$ and we take $\frac{43}{11} = [2, 7 : 6, 7 : 4, 7 : -1] \in S_7(54) \setminus T_7(54)$.

With the help of Lemma 1, we inspect even numbers n, $8 \leq n \leq 24$, and, excluding the possibility $n = 2p-2$, which we will soon study, the only exception we obtain is $S_7(10) \cap (1, +\infty) = \{\frac{9}{1}\} = T_7(10)$.

Case 4. Let $p > 3$ and $n \in \{2p-2, 2p-1, 2p+1\}$.

A result by Nagura [4] states that for any real number $x \geq 25$, there is a prime q in the interval $(x, 6x/5)$. Taking $x = \frac{n+2}{2}$, we see that for $n \geq 48$, we have $x \geq 25$ and

$$\begin{aligned}p = \frac{2p-2+2}{2} \leq \frac{n+2}{2} &< q \\ &< \frac{6}{5} \cdot \frac{n+2}{2} \leq \frac{3}{5}(2p+1+2) \leq 2p-2 \leq n.\end{aligned}$$

Thus, $\gcd(q, n-q) = \gcd(q, n) = 1$, and $gcd(q, p) = 1$, but also $\gcd(n-q, p) = 1$ since

$$1 \leq n - q < n - \frac{n+2}{2} = \frac{n-2}{2} \leq \frac{2p+1-2}{2} < p.$$

Therefore, $\frac{q}{n-q}$ is in $S_p(n)$. Also, from

$$\frac{n}{2} < \frac{n+2}{2} < q < \frac{3}{5}(n+2) < \frac{2}{3}n,$$

we get $\frac{q}{n-q} \in (1, 2)$. However,

$$\frac{q}{n-q} = 1 + \frac{2q-n}{n-q},$$

and $2 < 2q - n < p$, so p does not divide $2q - n$ and we conclude that $\frac{q}{n-q}$ is not in $T_p(n)$.

We inspect n of the given form for $7 \leq n \leq 47$ and obtain the following exceptions

$$\begin{aligned} S_5(8) \cap (1, +\infty) &= \{\tfrac{7}{1}\} = T_5(8), \\ S_5(9) \cap (1, +\infty) &= \{\tfrac{8}{1}, \tfrac{7}{2}\} = T_5(9), \\ S_7(12) \cap (1, +\infty) &= \{\tfrac{11}{1}\} = T_7(12), \\ S_7(15) \cap (1, +\infty) &= \{\tfrac{13}{2}, \tfrac{11}{4}\} = T_7(15). \end{aligned}$$

Case 5. Let $p = 3$ and $n > 3$ odd.

If $n \equiv 0 \pmod 3$, then

$$\frac{\frac{n+1}{2}}{\frac{n-1}{2}} = 1 + \frac{1}{\frac{n-1}{2}}$$

is in $(1, 2)$ and we immediately see that it lies in $(S_3(n) \setminus T_3(n)) \cap (1, +\infty)$.

Assume now that $n \not\equiv 0 \pmod 3$ and $n > 13$. Then

$$\frac{\frac{n+3}{2}}{\frac{n-3}{2}} = 1 + \frac{3}{\frac{n-3}{2}}$$

is an element of $S_3(n) \cap (1, 2)$ which has a finite 3-adic continued fraction expansion if and only if $\frac{n-3}{2} = 1 + 3^t$ or $\frac{n-3}{2} = 2 + 3^t$, i.e. $n = 5 + 2 \cdot 3^t$ or $n = 7 + 2 \cdot 3^t$ for some positive integer t. The condition $n > 13$ implies $t \geq 2$.

If $n = 5 + 2 \cdot 3^t$, then $\frac{n-1}{1} = 1 + 3(1 + 2 \cdot 3^{t-1})$ is in $S_3(n) \setminus T_3(n)$ since $1 + 2 \cdot 3^{t-1}$ is strictly greater than 1 and not divisible by 3.

If $n = 7 + 2 \cdot 3^t$, then

$$\frac{n-2}{2} = 1 + 3 \cdot \frac{1 + 2 \cdot 3^{t-1}}{2}$$

is a fraction in $S_3(n) \setminus T_3(n)$ since $1 + 2 \cdot 3^{t-1}$ is strictly greater than 2 and not divisible by 3.

The remaining cases to be checked are $3 < n \leq 13$ and n not divisible by 3. We get $S_3(n) \cap (1, +\infty) = T_3(n)$ for $n \in \{5, 7, 11, 13\}$.

Case 6. Let $p = 3$ and $n > 6$ even.

Suppose first that $n \not\equiv 1 \pmod 3$. Then $\frac{n-1}{1} \in S_3(n)$ and, since $n > 6$ is even, this fraction is in $T_3(n)$ only if $n = 3 + 3^t$ for an integer $t \geq 2$.

If t is even, then $3^t - 2 \equiv \pm 1 - 2 \not\equiv 0 \pmod 5$, so

$$\frac{n-5}{5} = \frac{3^t - 2}{5} = 2 + 3 \cdot \frac{3^{t-1} - 4}{5}$$

is in $S_3(n)$, but

$$\frac{3^{t-1}-4}{5} < 0 \text{ for } t = 2 \text{ and } \frac{3^{t-1}-4}{5} > 1 \text{ for } t \geq 4,$$

so $\frac{n-5}{5}$ is not in $T_3(n)$.

If t is odd, then $3^t + 3 \equiv 2 \pmod 4$, so $\frac{n}{2}$ is odd. The fraction

$$\frac{\frac{n}{2}+2}{\frac{n}{2}-2} = 1 + \frac{8}{3^t-1} = 2 + \frac{9-3^t}{3^t-1}$$

is in $S_3(n)$, but because of $9 - 3^t \leq 9 - 27 < 0$, it is not in $T_3(n)$.

Finally, we assume that $n \equiv 1 \pmod 3$. We have

$$\frac{\frac{n}{2}+3k}{\frac{n}{2}-3k} = 1 + 3 \cdot \frac{2k}{\frac{n}{2}-3k} \quad \text{and} \quad \frac{2k}{\frac{n}{2}-3k} > 1$$

as soon as $\frac{n}{10} < k < \frac{n}{6}$. From

$$\begin{aligned} &\gcd\left(\tfrac{n}{2}+3k, 3\right) = \gcd\left(\tfrac{n}{2}-3k, 3\right) = 1 \quad \text{and} \\ &\gcd\left(\tfrac{n}{2}+3k, \tfrac{n}{2}-3k\right) = \gcd\left(\tfrac{n}{2}+3k, 6k\right) \\ &\quad = \gcd\left(\tfrac{n}{2}+3k, 2k\right) = \gcd\left(\tfrac{n}{2}+k, 2k\right), \end{aligned}$$

we see it is sufficient to demand that $\frac{n}{2}$ and k be coprime numbers of different parities, for the fraction $\left(\frac{n}{2}+3k\right)/\left(\frac{n}{2}-3k\right)$ to be in $S_3(n)$. If, moreover, $\frac{n}{10} < k < \frac{n}{6}$ and k is not divisible by 3, then this fraction is not in $T_3(n)$.

Let $q_1 \in \left(\frac{n}{20}, \frac{n}{12}\right)$ and $q_2 \in \left(\frac{n}{10}, \frac{n}{6}\right)$ be prime numbers, different from 3, that do not divide n. If $\frac{n}{2}$ is odd, we take $k = 2q_1$, and if $\frac{n}{2}$ is even, we set $k = q_2$. We only need to confirm that such primes q_1 and q_2 exist.

For $n \geq 500$, we have $\frac{n}{20} \geq 25$, so by the already mentioned result of Nagura [4], there are prime numbers

$$\begin{aligned} q' &\in \left(\tfrac{n}{20}, \tfrac{6}{5}\cdot\tfrac{n}{20}\right) = \left(\tfrac{n}{20}, \tfrac{3n}{50}\right) \subset \left(\tfrac{n}{20}, \tfrac{n}{12}\right) \text{ and} \\ q'' &\in \left(\tfrac{3n}{50}, \tfrac{6}{5}\cdot\tfrac{3n}{50}\right) = \left(\tfrac{3n}{50}, \tfrac{9n}{125}\right) \subset \left(\tfrac{n}{20}, \tfrac{n}{12}\right). \end{aligned}$$

If both of these prime numbers divided n, we would have $n = r'q' = r''q''$, where $r', r'' \in \{13, 14, 15, 16, 17, 18, 19\}$, which would imply that some prime from the set $\{2, 3, 5, 7, 13, 17, 19\}$ divides the prime $q'' > 25$ which is impossible. Therefore, we take q_1 to be one of the numbers q', q'' which does not divide n. Completely analogously, we show the existence of the required prime q_2.

For $6 < n < 500$, $n \equiv 4 \pmod 6$, we easily check by computer that $\big(S_3(n) \setminus T_3(n)\big) \cap (1, +\infty)$ is nonempty except in the cases $n \in \{10, 16, 28\}$ where the following holds

$$S_3(10) = \emptyset, \quad S_3(16) \cap (1, +\infty) = \{\tfrac{11}{5}\} = T_3(16),$$
$$S_3(28) \cap (1, +\infty) = \{\tfrac{23}{5}, \tfrac{17}{11}\} = T_3(28).$$

This completes the proof of the theorem. □

3 On rationals with numerator and denominator bounded by p

Theorem 3. *Let $\frac{u}{v} \notin \{-1, 1, 2, \ldots, p-1\}$ be a rational number such that*

$$\gcd(u, v) = \gcd(uv, p) = 1 \quad \textit{and} \quad |u| < p, \ |v| < p.$$

Then the p-adic continued fraction expansion of $\frac{u}{v}$ is infinite

$$\frac{u}{v} = [b_0, p : b_1, p : b_2, \ldots, p : b_k, p : -1],$$

where we noted the first appearance of -1 as a complete quotient.

Moreover, for $0 \le i \le k$, we have $k \le i + b_i$ and thus $k \le p - 1$.

Proof. Let $(x_0, x_1) = (u, v)$. Expanding x_0/x_1 into a continued fraction (1), denote one complete quotient $x_i/x_{i+1} = [b_i, p^{a_{i+1}} : b_{i+1}, \ldots]$, where x_i, x_{i+1} are coprime integers. Then the next complete quotient is

$$\frac{p^{a_{i+1}}}{\frac{x_i}{x_{i+1}} - b_i} = \frac{x_{i+1}}{\frac{x_i - b_i x_{i+1}}{p^{a_{i+1}}}} = \frac{x_{i+1}}{x_{i+2}}, \tag{4}$$

where $x_{i+2} = (x_i - b_i x_{i+1})/p^{a_{i+1}}$ is an integer and $\gcd(x_{i+1}, x_{i+2}) = 1$. This is how we obtain the sequence $(x_i)_{i \ge 0}$ of integers which terminates with $x_i/x_{i+1} \in \{1, 2, \ldots, p-1\}$ for some i if and only if the continued fraction expansion of x_0/x_1 is finite. Otherwise, we reach $x_i/x_{i+1} = -1$ for the first time for some positive integer i, which marks the start of the periodic part of the nonterminating continued fraction expansion of x_0/x_1.

From $|x_0|, |x_1| < p$, we obtain

$$|x_0 - b_0 x_1| \le (p-1) + (p-1)^2 = p(p-1) < p^2,$$

which implies that $a_1 = 1$. Also, (4) gives

$$|x_2| = \frac{|x_0 - b_0 x_1|}{p} \le \frac{p(p-1)}{p} = p - 1.$$

Analogously, we conclude that $a_i = 1$ and $|x_i| \le p-1$ for all i.

If $x_1 x_2 > 0$, then $|x_0| = |b_0 x_1 + p x_2| = b_0|x_1| + p|x_2| > p$, so we deduce that $x_1 x_2 < 0$ and, analogously, $x_i x_{i+1} < 0$ for $i \ge 1$. We see that the p-adic continued fraction of $\frac{x_0}{x_1}$ is infinite. Without loss of generality, we can take $x_1 > 0$, so $(-1)^i x_i < 0$ for all $i \ge 1$.

The equality $|x_1| = |x_2|$, together with $\gcd(x_1, x_2) = 1$ and $x_1 x_2 < 0$, would imply $x_1 = 1$, $x_2 = -1$, indicating that we have reached the periodic part of the expansion.

For $|x_2| > |x_1|$, we would have $x_2 \le -x_1 - 1 < 0$ and

$$x_0 = b_0 x_1 + p x_2 \le b_0 x_1 - p x_1 - p = (b_0 - p) x_1 - p < -p$$

which is impossible. Thus, we conclude that for $x_1/x_2 \ne -1$, the inequality $|x_2| < |x_1|$, i.e. $0 < -x_2 < x_1$ has to hold. In the same way, we see that if $x_i/x_{i+1} \ne -1$, then $|x_{i+1}| < |x_i|$.

Let $k \ge 0$ be the largest integer such that $x_k/x_{k+1} \ne -1$, i.e. $k+1$ is the length of the preperiodic part of expansion (e.g. we can count the partial denominators in this part, including the zeroth b_0). Then $|x_{k+1}| \ge 1$, $|x_k| \ge 2$ and, in general, $|x_i| \ge k + 2 - i$ for all $1 \le i \le k+1$.

From $x_{i+2} = (x_i - b_i x_{i+1})/p$, we obtain

$$\begin{aligned} |x_{i+2}| &\le \frac{|x_i| + b_i |x_{i+1}|}{p} \le \frac{p - 1 + b_i(p-1)}{p} \\ &= b_i + \frac{p - 1 - b_i}{p} < b_i + 1 \end{aligned}$$

and it follows that $|x_{i+2}| \le b_i$. For $0 \le i \le k$, we have

$$k - i = (k+2) - (i+2) \le |x_{i+2}| \le b_i, \text{ i.e. } k \le i + b_i,$$

which is what we wanted to prove.

Let us also note here that the partial denominator $p-1$ can appear in the preperiod only at the beginning. If $b_i = p-1$ for $i \ge 1$, then x_{i+2} being integer implies

$$0 \equiv x_i - (p-1)x_{i+1} \equiv x_i + x_{i+1} \pmod{p},$$

so $x_i x_{i+1} < 0$ and $|x_i|, |x_{i+1}| < p$ imply $x_i + x_{i+1} = 0$, i.e. $x_i/x_{i+1} = -1$ from which $i \ge k+1$. □

Recall that a variant of Sylvester's sequence [5] is defined by $s_0 = 1$, $s_1 = 2$, and

$$s_n = s_{n-1}^2 - s_{n-1} + 1 \text{ for } n > 1.$$

Thus, the first few elements in the sequence are

$$s_0 = 1,\ s_1 = 2,\ s_2 = 3,\ s_3 = 7, s_4 = 43.$$

It is easily shown that $s_n = 1 + \prod_{k=0}^{n-1} s_k$ holds for $n \geq 1$.

Theorem 4. *If, for some $n \geq 1$, we have $p = s_{n+1}$, then*

$$\Big[p-1, \big(p : \tfrac{p-1}{s_i}, p : p - s_i^2\big)_{i=1,\dots,n-1}, \\ p : \tfrac{p-1}{s_n}, p : p - s_n + 1, p : -1\Big]$$

is the p-adic continued fraction expansion of $\frac{1}{p-1}$. Here, for continued fraction as in (1),

$$\begin{aligned} a_i &= 1 && \text{for} \quad 1 \leq i \leq 2n+1, \\ b_{2i-1} &= \frac{p-1}{s_i} && \text{for} \quad 1 \leq i \leq n, \\ b_{2i} &= p - s_i^2 && \text{for} \quad 0 \leq i \leq n-1, \\ b_{2n} &= p - s_n + 1. \end{aligned} \tag{5}$$

Proof. Starting with the fraction $u_0/u_1 = 1/(p-1)$, we show that the successive complete quotients in its p-adic continued fraction expansion are $u_\ell/u_{\ell+1}$ for $\ell \in \{1, 2, \dots, 2n-1\}$ where

$$u_{2i-1} = \frac{p-1}{s_i - 1} \text{ and } u_{2i} = -\frac{p - s_i}{s_i - 1}$$

for $i \in \{1, 2, \dots, n\}$. We see immediately that

$$u_{2i-1} = \prod_{k=i}^{n} s_k \text{ and } u_{2i} = -u_{2i-1} + 1,$$

so $\gcd(u_\ell, u_{\ell+1}) = \gcd(u_\ell u_{\ell+1}, p) = 1$ for all $\ell \in \{0, 1, \dots, 2n-1\}$.

We only need to check that

$$\frac{u_\ell}{u_{\ell+1}} = b_\ell + \frac{p^{a_{\ell+1}}}{\frac{u_{\ell+1}}{u_{\ell+2}}}, \quad \text{i.e.} \quad u_{\ell+2} = \frac{-b_\ell u_{\ell+1} + u_\ell}{p^{a_{\ell+1}}}$$

holds for $a_{\ell+1}$, b_ℓ defined in (5) when $\ell \in \{0, 1, \dots, 2n-2\}$. For $\ell = 0$ this is easy, while

$$\begin{aligned}
&\frac{-b_{2i}u_{2i+1}+u_{2i}}{p^{a_{2i+1}}}\\
&=\frac{1}{p}\Big(-(p-s_i^2)\frac{p-1}{s_{i+1}-1}-\frac{p-s_i}{s_i-1}\Big)\\
&=-\frac{(p-1)(p-s_i^2)+s_i(p-s_i)}{p(s_{i+1}-1)}\\
&=-\frac{p-s_i^2+s_i-1}{s_{i+1}-1}\\
&=-\frac{p-s_{i+1}}{s_{i+1}-1}=u_{2i+2}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{-b_{2i-1}u_{2i}+u_{2i-1}}{p^{a_{2i}}}\\
&=\frac{1}{p}\Big(-\frac{p-1}{s_i}\cdot\big(-\frac{p-s_i}{s_i-1}\big)+\frac{p-1}{s_i-1}\Big)\\
&=\frac{p-1}{(s_i-1)s_i}=\frac{p-1}{s_{i+1}-1}=u_{2i+1}
\end{aligned}$$

for $i\in\{1,2,\dots,n-1\}$.

Thus, we have shown that u_{2n-1}/u_{2n} is indeed a complete quotient of $\frac{1}{p-1}$. Now, using $p=s_{n+1}=s_n^2-s_n+1$, we obtain

$$\begin{aligned}
\frac{u_{2n-1}}{u_{2n}}&=\frac{\frac{p-1}{s_n-1}}{-\frac{p-s_n}{s_n-1}}=\frac{s_n}{-s_n+1}\\
&=\frac{s_n^2}{s_n(-s_n+1)}=\frac{p-1}{s_n}+\frac{p}{-s_n+1}\\
&=\frac{p-1}{s_n}+\frac{p}{p-s_n+1+\frac{p}{-1}},
\end{aligned}$$

so the tail of the continued fraction expansion of $\frac{1}{p-1}$ is also as claimed in this theorem. □

The largest currently known prime in the Sylvester sequence [1] is $s_6 = 3263443$ for which we obtain

$$\begin{aligned}
\frac{1}{p-1}=\big[&p-1,p:\tfrac{p-1}{2},p:p-4,p:\tfrac{p-1}{3},p:p-9,\\
&p:\tfrac{p-1}{7},p:p-49,p:\tfrac{p-1}{43},p:p-1849,\\
&p:\tfrac{p-1}{1807},p:p-1806,p:-1\big].
\end{aligned}$$

It is conjectured that there are no larger primes in this sequence, but already the status of s_{33} is currently unknown [1].

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