Schneider's p-adic continued fractions of rational numbers

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Abstract

Unlike the usual simple continued fraction, Schneider's p-adic continued fraction expansion of a rational number can be infinite. Hirsh and Washington conjectured that rational numbers with nonterminating expansion are more common than those with terminating expansion. We prove this conjecture and give upper and lower bounds on the number of reduced fractions, with bounded numerator and denominator, that have terminating p-adic continued fraction expansion. We also present examples of sets containing rationals with nonterminating expansion.

1 Introduction

For a prime number p, diverse continued fraction algorithms in the field of p-adic numbers have been proposed over the years. A recent survey by Romeo [14] shows the similarities and differences between various continued fraction algorithms as well as the properties they have in common with the usual simple continued fraction expansion of real numbers.

In this paper, we study Schneider's p-adic continued fractions [15, 16], concentrating on questions concerning the expansion of rational numbers.

For a list of the most important papers on this topic as well as a short introduction to the subject, the reader can consult the first section and the references in the paper [12]. Here, we restrict ourselves to presenting only the essential notation and results used in the remainder of the paper.

Throughout this text, p denotes a prime number and $\mathbb{Q}_p$ is the field of p-adic numbers equipped with the p-adic absolute value $|\cdot|_p$ normalized in such a way that $|p|_p = p^{-1}$ (see [7, 10] for more on p-adic numbers).

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Schneider's p-adic continued fraction is an expression of the form

$$b_0 + \cfrac{p^{a_1}}{b_1 + \cfrac{p^{a_2}}{b_2 + \cfrac{p^{a_3}}{\ddots}}} \tag{1.1}$$

written more compactly as $[b_0, p^{a_1} : b_1, p^{a_2} : b_2, p^{a_3} : \ldots]$, where all a_i are positive integers and $b_i \in \{1, 2, \ldots, p-1\}$.

The numbers b_i are called partial denominators, while p^{a_i} are partial numerators. By splitting the expression (1.1) at any point, we obtain the convergents $\frac{P_k}{Q_k} = [b_0, p^{a_1} : b_1, \ldots, p^{a_k} : b_k]$ and the complete quotients $[b_k, p^{a_{k+1}} : b_{k+1}, \ldots]$.

By simplifying the finite continued fraction $[b_0, p^{a_1} : b_1, \ldots, p^{a_k} : b_k]$, we see that the convergents are indeed rational numbers and the integers P_k and Q_k satisfy the recurrence

$$P_n = b_n P_{n-1} + p^{a_n} P_{n-2}, \quad Q_n = b_n Q_{n-1} + p^{a_n} Q_{n-2}, \quad \text{for } n \geqslant 0, \tag{1.2}$$

where the sequences were extended by including the initial values

$$a_0 = 0, \quad P_{-2} = 0,\ P_{-1} = 1, \quad Q_{-2} = 1,\ Q_{-1} = 0. \tag{1.3}$$

It follows easily that $\gcd(P_n Q_n, p) = \gcd(P_n, P_{n-1}) = \gcd(Q_n, Q_{n-1}) = \gcd(P_n, Q_n) = 1$ for all $n \geqslant 0$.

If the initial continued fraction (1.1) is infinite, it can be shown that the sequence of convergents $(P_n/Q_n)_{n\geqslant 0}$ actually converges in the p-adic absolute value to a p-adic unit which is the value assigned to (1.1). In the other direction, after multiplying a given p-adic number by an appropriate power of p so that it becomes a p-adic unit, we can expand this number into a p-adic continued fraction.

Although a finite Schneider's p-adic continued fraction always represents a positive rational number, the converse is not true. This is a stark difference from the usual simple continued fraction algorithm in the reals.

Bundschuh [5] showed that if a rational number has an infinite Schneider's p-adic continued fraction expansion, then -1 must appear as its complete quotient. Since $-1 = [p-1, p : p-1, p : p-1, \ldots]$, this is the same as saying that if the continued fraction expansion of a rational number does not terminate, then from some point onwards the block $p-1, p$ repeats indefinitely. In [12] an upper bound was given for the required number of steps in the expansion of a rational number before the expansion either terminates or reaches -1 as a complete quotient.

This paper is organized as follows. In Section 2, we prove that there are more (positive) rational numbers with infinite p-adic continued fractions than rational numbers whose expansion terminates. Thus we prove a conjecture made by Hirsh and Washington [9, §§ 2, 6]. In Section 3, we show results in the opposite direction, there is still a sufficient number of rational numbers with finite expansions. The precise meaning of these statements will be provided in the relevant theorems. Section 4 gathers diverse results on rational numbers with nonterminating continued fraction expansions. Some of our results generalize to any prime p results proved for $p = 2$ by Hirsh and Washington [9, § 4] or extend to more general sets of rationals.

2 Upper bound on the number of fractions with terminating expansion

For a prime p and a real number $x > 1$, denote by

$$N_p(x) = \{u/v \,:\, u, v \in \mathbb{Z},\ 1 \leqslant u \leqslant x,\ 1 \leqslant v \leqslant x,\ \gcd(u, v) = \gcd(uv, p) = 1\}.$$

Let d be a positive integer not divisible by p. When x tends to infinity, the number of pairs (u, v) of positive integers both bounded by x and both divisible by d such that p does not divide uv is asymptotically $(\frac{p-1}{p}\frac{x}{d})^2$. Therefore, the inclusion-exclusion principle suggests that

$$\begin{aligned}\operatorname{card} N_p(x) &\sim \Big(\frac{p-1}{p}x\Big)^2 \sum_{\substack{d=1\\ p\nmid d}}^{\infty} \frac{\mu(d)}{d^2} = x^2\,\frac{(p-1)^2}{p^2}\,\Big(1-\frac{1}{p^2}\Big)^{-1} \sum_{d=1}^{\infty}\frac{\mu(d)}{d^2}\\ &= x^2\,\frac{(p-1)^2}{p^2}\,\frac{p^2}{p^2-1}\,\Big(\sum_{d=1}^{\infty}\frac{1}{d^2}\Big)^{-1} = \frac{p-1}{p+1}\,\frac{6}{\pi^2}\,x^2.\end{aligned}$$

Regardless of this, since the number of coprime pairs (u, v) with $1 \leqslant u \leqslant x$ and $1 \leqslant v \leqslant x$ is $\frac{6}{\pi^2}x^2 + O(x \log x)$ (see e.g. [1, Theorem 3.9] or [6, Proposition 6.6.3]), by excluding pairs of the form (pm, n) or (n, pm), where $\gcd(n, p) = 1$ and pm, $n \leqslant x$, we can crudely bound

$$\operatorname{card} N_p(x) > \Big(\frac{6}{\pi^2} - \frac{1}{200} - 2\,\frac{1}{p}\,\frac{p-1}{p}\Big)x^2 > \frac{1}{10}\,x^2, \tag{2.1}$$

for x large enough. This will be sufficient for our purposes. Let us also note that because of symmetry taking (u, v) to (v, u), we have

$$\operatorname{card}\big(N_p(x) \cap (1, +\infty)\big) = \frac{1}{2}(\operatorname{card} N_p(x) - 1) \tag{2.2}$$

Now we state in a precise manner the result confirming a conjecture by Hirsh and Washington, for which they gave numerical support and heuristic arguments [9, § 6], that rational numbers with nonterminating expansion are more common than those with terminating expansion.

Theorem 1. *For a prime p and a positive number x, the number of elements in $N_p(x)$ with terminating p-adic continued fraction expansion is $\ll x^{2(1-p^{-8})}$. The number of elements in $N_p(x) \cap (1, +\infty)$ with nonterminating expansion is $\gg x^2$.*

Note that throughout this paper the constants implicit in Vinogradov $\gg$, $\ll$ and Landau $O(\cdot)$ notation depend at most on p.

Proof. Let $x = p^D$ for some $D > 0$ large enough. Consider any fraction from $N_p(p^D)$ with terminating p-adic continued fraction expansion

$$\frac{P_n}{Q_n} = [b_0, p^{a_1} : b_1, p^{a_2} : b_2, \ldots, p^{a_n} : b_n],$$

where we denoted our fraction P_n/Q_n in order to be consistent with the notation for its convergents P_k/Q_k. We can assume that P_n is large enough in terms of p.

Since $P_n \geqslant P_{n-1}$ and $Q_n \geqslant Q_{n-1}$ for $n \geqslant 0$ by (1.2) and the inequalities are strict for $n \geqslant 2$, we have from the same basic recurrences (1.2) that $P_n \geqslant (p^{a_n} + 1)P_{n-2}$ and $Q_n \geqslant (p^{a_n} + 1)Q_{n-2}$. It follows that

$$\begin{aligned} P_nQ_n &\geqslant P_nQ_{n-1} \geqslant (p^{a_n} + 1)(p^{a_{n-1}} + 1)P_{n-2}Q_{n-3} \geqslant \ldots \geqslant \\ &\geqslant (p^{a_n} + 1)(p^{a_{n-1}} + 1) \cdots (p^{a_2} + 1)P_1Q_0 \\ &\geqslant (p^{a_n} + 1)(p^{a_{n-1}} + 1) \cdots (p^{a_2} + 1)(p^{a_1} + 1)P_{-1}Q_0 \end{aligned}$$

for n odd and

$$\begin{aligned} P_nQ_n &\geqslant P_{n-1}Q_n \geqslant (p^{a_n} + 1)(p^{a_{n-1}} + 1)P_{n-3}Q_{n-2} \geqslant \ldots \geqslant \\ &\geqslant (p^{a_n} + 1)(p^{a_{n-1}} + 1) \cdots (p^{a_1} + 1)P_{-1}Q_0 \end{aligned}$$

for n even, so the same lower bound holds for P_nQ_n for all positive integers n.

Since $P_n/Q_n \in N_p(p^D)$, this implies

$$\begin{aligned} &p^{2D} \geqslant (p^{a_1} + 1)(p^{a_2} + 1) \cdots (p^{a_n} + 1), \quad \text{so} \\ &2D \geqslant \sum_{i=1}^{n} \log_p(p^{a_i} + 1) = \sum_{i=1}^{n} \left(a_i + \log_p\left(1 + \frac{1}{p^{a_i}}\right)\right) > \sum_{i=1}^{n} \left(a_i + \frac{1}{p^{a_i+1}}\right), \end{aligned}$$

where the last inequality follows from Bernoulli inequality saying

$$\Big(1+\frac{1}{p^{a_i}}\Big)^{p^{a_i+1}} > p.$$

Multiplying by p^2, we obtain

$$2Dp^2 > \sum_{i=1}^{n}\Big(p^2a_i+\frac{1}{p^{a_i-1}}\Big) \geqslant \sum_{i=1}^{n}(p^2a_i+\varepsilon_i), \tag{2.3}$$

where $\varepsilon_i = 1$ for $a_i = 1$ and $\varepsilon_i = 0$ for $a_i > 1$.

Therefore, the number of pairs of sequences $\big((a_i)_{1\leqslant i\leqslant n},(b_i)_{0\leqslant i\leqslant n}\big)$ such that the value of the continued fraction $[b_0, p^{a_1} : b_1, \dots, p^{a_n} : b_n]$ is in $N_p(p^D)$ is less than the number of solutions of inequality (2.3) (for $(a_i)_i$) multiplied by $(p-1)^{n+1}$ (for $(b_i)_i$).

Now, employing generating functions (refer e.g. to [17]), we see that the number of fractions in $N_p(p^D)$ with terminating expansion is smaller than the coefficient of $z^{\lfloor 2Dp^2\rfloor}$ in the power series

$$\begin{aligned}
&(1+z+z^2+\cdots)\sum_{n=1}^{\infty}(p-1)\big((p-1)z^{p^2+1}+(p-1)z^{2p^2}+(p-1)z^{3p^2}+\cdots\big)^n\\
&\quad=\frac{1}{1-z}\;\frac{(p-1)^2\big(z^{p^2+1}+\frac{z^{2p^2}}{1-z^{p^2}}\big)}{1-(p-1)\big(z^{p^2+1}+\frac{z^{2p^2}}{1-z^{p^2}}\big)}\\
&\quad=\frac{(p-1)^2(z^{p^2+1}+z^{2p^2}-z^{2p^2+1})}{(1-z)\big(1-z^{p^2}-(p-1)(z^{p^2+1}+z^{2p^2}-z^{2p^2+1})\big)}.
\end{aligned} \tag{2.4}$$

From this, we obtain that the number of such fractions is

$$\ll r^{(2D-1)p^2-1}, \tag{2.5}$$

where r is an upper bound on the (standard) absolute value of all roots in complex numbers of the polynomial reciprocal to the polynomial in the denominator of the rational function (2.4). Thus, we have to find an upper bound for the roots of the polynomial

$$\begin{aligned}
R(z) &= z^{2p^2+1}-z^{p^2+1}-(p-1)z^{p^2}-(p-1)z+(p-1)\\
&= (z^{p^2+1}-p+1)(z^{p^2}-1)-(p-1)z\\
&= \big(z^{2p^2}-z^{p^2}-(p-1)\big)z-(p-1)(z^{p^2}-1).
\end{aligned}$$

Let us show that we can take

$$r = p^{\frac{1}{p^2}}-\frac{1}{p^8},$$

so let r be defined in this way. We have to show that it is indeed the required upper bound on the absolute values of the roots of $R(z)$.

First note that $r > 1$ since

$$\Big(1+\frac{1}{p^8}\Big)^{p^2} < e^{\frac{1}{p^8}\,p^2} = e^{\frac{1}{p^6}} < 2 \leqslant p.$$

Furthermore, from Bernoulli inequality, for $m \geqslant 1$, we get

$$\begin{aligned} r^m &= \Big(p^{\frac{1}{p^2}} - \frac{1}{p^8}\Big)^m = p^{\frac{m}{p^2}}\big(1-p^{-8-\frac{1}{p^2}}\big)^m \\ &> p^{\frac{m}{p^2}}\big(1-mp^{-8-\frac{1}{p^2}}\big) > p^{\frac{m}{p^2}}\Big(1-\frac{m}{p^8}\Big). \end{aligned}$$

Thus, we have the lower bounds

$$\begin{aligned} r^{p^2} &> p^{\frac{p^2}{p^2}}\Big(1-\frac{p^2}{p^8}\Big) = p\Big(1-\frac{1}{p^6}\Big) > p-1, \\ r^{2p^2} &> p^{\frac{2p^2}{p^2}}\Big(1-\frac{2p^2}{p^8}\Big) \geqslant p^2\Big(1-\frac{1}{p^5}\Big), \end{aligned} \tag{2.6}$$

as well as the upper bound

$$\begin{aligned} r^{p^2} &= p\big(1-p^{-8-\frac{1}{p^2}}\big)^{p^2} < \frac{p}{\big(1+p^{-8-\frac{1}{p^2}}\big)^{p^2}} \\ &< \frac{p}{1+p^{-6-\frac{1}{p^2}}} < \frac{p}{1+p^{-7}} < p \end{aligned} \tag{2.7}$$

For a complex number z such that $|z| \geqslant r$, the triangle inequality and (2.6) imply

$$\begin{aligned} |R(z)| &= \big|(z^{p^2+1}-p+1)(z^{p^2}-1)-(p-1)z\big| \\ &\geqslant \big|z^{p^2+1}-p+1\big|\cdot\big|z^{p^2}-1\big| - (p-1)|z| \\ &\geqslant \big(|z|^{p^2+1}-p+1\big)\big(|z|^{p^2}-1\big) - (p-1)|z| = R(|z|), \end{aligned}$$

so it suffices to check that for real $z \geqslant r$, $R(z) \neq 0$ holds.

Because of (2.6), for $z \geqslant r$, we have

$$\begin{aligned} R'(z) &= (2p^2+1)z^{2p^2} - (p^2+1)z^{p^2} - (p-1)p^2z^{p^2-1} - (p-1) \\ &> \big((2p^2+1)(p-1)-(p^2+1)\big)z^{p^2} - (p-1)p^2z^{p^2-1} - (p-1) \\ &\geqslant (p-1)p^2\big(z^{p^2}-z^{p^2-1}\big) + (p-2)(p^2+1)z^{p^2} - (p-1). \end{aligned} \tag{2.8}$$

For $p \geqslant 3$, (2.8) is positive since $z > 1$ and $(p-2)(p^2+1) > p-1$. For $p = 2$, (2.8) is $4z^4 - 4z^3 - 1$ and substituting $z = r > 1.18$, we check directly that the value is positive for all $z \geqslant r$. Therefore, $R'(z) > 0$ for all $z \geqslant r$, so there only remains to verify that $R(r) > 0$.

We have

$$\begin{aligned} R(r) &= \bigl(r^{2p^2} - r^{p^2} - p + 1\bigr)r - (p-1)\bigl(r^{p^2} - 1\bigr) \\ &> \Bigl(p^2 - \frac{1}{p^3} - p - p + 1\Bigr)r - (p-1)^2 = (p-1)^2(r-1) - \frac{1}{p^3}\,r, \end{aligned}$$

so $R(r) > 0$ will follow from

$$\frac{r-1}{r} > \frac{1}{p^3(p-1)^2}, \quad \text{i.e.}$$

$$r > 1 + \frac{1}{p^3(p-1)^2 - 1}. \tag{2.9}$$

Since

$$r = e^{\frac{\log p}{p^2}} - \frac{1}{p^8} > 1 + \frac{\log p}{p^2} + \frac{1}{2}\left(\frac{\log p}{p^2}\right)^2 - \frac{1}{p^8} > 1 + \frac{\log p}{p^2},$$

(2.9) will certainly hold if

$$\bigl(p^3(p-1)^2 - 1\bigr)\log p > p^2,$$

and, as can easily be checked, this inequality is true for all $p \geqslant 2$. This concludes the proof that $r = p^{\frac{1}{p^2}} - \frac{1}{p^8}$ is a valid bound for the roots of $R(z)$.

Bernoulli inequality implies that $p < (1 + p^{-7})^{p^8}$. This is equivalent to

$$\frac{p}{1 + p^{-7}} < p^{1-p^{-8}}. \tag{2.10}$$

Now, (2.5), (2.6), (2.7), and (2.10) show that the number of elements in $N_p(x)$ with terminating p-adic continued fractions is

$$\ll \bigl(r^{p^2}\bigr)^{2D} \frac{1}{r^{p^2+1}} \ll \left(\frac{p}{1+p^{-7}}\right)^{2D} \ll \bigl(p^{1-p^{-8}}\bigr)^{2D} = x^{2(1-p^{-8})}.$$

The last claim in the theorem follows by comparing the previous inequality with (2.1) and (2.2). □

While the conjecture and the discussion in [9, § 6] are made for fractions u/v with bounded $|u| + |v|$ and our results are given for u/v with bound on $\max\{|u|, |v|\}$ (see the definition of $N_p(x)$), this poses no difficulty since

$$\max\{|u|, |v|\} \leqslant |u| + |v| \leqslant 2\max\{|u|, |v|\}$$

so the asymptotic bounds in our results are still valid in the setting of Hirsh and Washington.

3 Lower bound on the number of fractions with terminating expansion

Although rational numbers with nonterminating p-adic continued fractions are more common than those with terminating expansions, the next theorem shows that the latter are still fairly common.

Theorem 2. *For a prime p and a positive number x, the number of elements in $N_p(x)$ with terminating p-adic continued fraction expansion is $\gg x^{1+\varepsilon}$, where ε is a small positive number depending only on p.*

For $p > 7$, even if we restrict ourselves to elements in $N_p(x)$ with terminating expansion whose every partial numerator is equal to p, the same bound still holds.

Proof. Let

$$\frac{P_n}{Q_n} = [b_0, p^{a_1} : b_1, p^{a_2} : b_2, \ldots, p^{a_n} : b_n]$$

be a rational number with a finite p-adic continued fraction and, as before, we denote its convergents by P_k/Q_k for $0 \leqslant k \leqslant n$.

From (1.2), it can easily be proven by induction that $P_k \geqslant Q_k$ and $P_k \geqslant P_{k-1}$ for $k \geqslant 0$ with strict inequalities unless $k = 0$ and $b_0 = 1$.

We have

$$\begin{aligned} P_n &= b_n P_{n-1} + p^{a_n} P_{n-2} = b_n(b_{n-1}P_{n-2} + p^{a_{n-1}}P_{n-3}) + p^{a_n}P_{n-2} \\ &< (b_n b_{n-1} + b_n p^{a_{n-1}} + p^{a_n})P_{n-2} < \ldots < \\ &< (b_n b_{n-1} + b_n p^{a_{n-1}} + p^{a_n})(b_{n-2}b_{n-3} + b_{n-2}p^{a_{n-3}} + p^{a_{n-2}}) \cdots \\ &\qquad \cdots (b_2 b_1 + b_2 p^{a_1} + p^{a_2})P_0 \end{aligned} \tag{3.1}$$

for n even and a similar inequality for n odd.

Let $x = p^D$ for some $D > 0$ large enough. From (3.1), it now follows that the number of fractions in $N_p(x)$ with terminating p-adic continued fraction expansion is at least as large as the number of sequences $(b_i)_{0 \leqslant i \leqslant n} \in \{1, 2, \ldots, p-1\}^{n+1}$, $n \geqslant 0$, (we set all $a_i = 1$) such that

$$(b_n b_{n-1} + b_n p + p)(b_{n-2}b_{n-3} + b_{n-2}p + p) \cdots < p^{D-1}, \tag{3.2}$$

where the product on the left hand side ends with either $b_2 b_1 + b_2 p + p$ or $b_1 b_0 + b_1 p + p$ depending on the parity of n.

Clearly, we have $b_n b_{n-1} + b_n p + p < 3pb_n$ and we consider the size of this expression depending on which of the following intervals contains b_n:

$$\left[1, \frac{p^{1/3}}{3}\right], \ \left(\frac{p^{1/3}}{3}, \frac{p^{1/2}}{3}\right], \ \left(\frac{p^{1/2}}{3}, \frac{p^{2/3}}{3}\right], \ \left(\frac{p^{2/3}}{3}, \frac{p}{3}\right], \ \left(\frac{p}{3}, \frac{p^{4/3}}{3}\right]. \tag{3.3}$$

Note that for $p > 27$, the union of these intervals covers the entire segment $[1, p-1]$. If, for example, we take b_n from the interval $\big(p^{1/3}/3, p^{1/2}/3\big]$, then there are at least $\big(\frac{1}{3}(p^{1/2} - p^{1/3}) - 1\big)(p-1)$ pairs (b_n, b_{n-1}) and for these pairs we have $b_n b_{n-1} + b_n p + p < p^{3/2}$. Similarly, we observe that for b_n from the intervals in (3.3), the upper bound on $b_n b_{n-1} + b_n p + p$ is equal to

$$p^{4/3},\ p^{3/2},\ p^{5/3},\ p^2,\ p^{7/3},$$

respectively. Here we have to ensure that the length of each interval is at least 1, which can easily be confirmed for $p \geqslant 6^3 = 216$.

If we take the sixth power of inequality (3.2) and substitute the just observed bounds, we see that the number of the required sequences $(b_i)_{0 \leqslant i \leqslant n}$ is at least as large as the coefficient of z^{6D-6} in the power series

$$(1 + z + z^2 + \cdots)\big(L(z) + (L(z))^2 + (L(z))^3 + \cdots\big) \tag{3.4}$$

$$= \frac{1}{1-z}\,\frac{L(z)}{1-L(z)}, \quad \text{where} \tag{3.5}$$

$$L(z) = \frac{1}{3}(p-1)\big((p^{1/3} - 3)z^8 + (p^{1/2} - p^{1/3} - 3)z^9 + \\ + (p^{2/3} - p^{1/2} - 3)z^{10} + (p - p^{2/3} - 3)z^{12} + (3p - 3 - p)z^{14}\big).$$

For an asymptotic bound, it is sufficient to show that for the polynomial $K(z) = z^{14}\big(1 - L(\frac{1}{z})\big)$, its dominant root (in standard absolute value) in complex numbers is strictly greater than $p^{1/6}$ and that all of its other roots are of strictly smaller absolute value than the dominant root. Combined with the fact that the coefficients of $L(z)$ are nonnegative, this guarantees that for fixed p and for D large enough, the number of the required sequences $(b_i)_{0 \leqslant i \leqslant n}$, and thus also the number of elements of $N_p(p^D)$ with terminating p-adic continued fraction expansion, is greater than $p^{D(1+\varepsilon)}$, for some small $\varepsilon > 0$ depending only on p.

Let us check these bounds for the roots of $K(z)$. If we substitute $q = p^{1/6}$ into

$$K(z) = z^{14} - \frac{1}{3}(p-1)\big((p^{1/3} - 3)z^6 + (p^{1/2} - p^{1/3} - 3)z^5 + \\ + (p^{2/3} - p^{1/2} - 3)z^4 + (p - p^{2/3} - 3)z^2 + (2p - 3)\big),$$

we obtain

$$3K(q) = -q^{14} + 2q^{13} + 2q^{12} + 3q^{11} + 3q^{10} + 7q^8 \\ - 2q^7 + q^6 - 3q^5 - 3q^4 - 3q^2 - 3.$$

Writing this as

$$
\begin{aligned}
3K(q) = -q^{14}\Big(1 - \frac{2}{q} - \frac{2}{q^2} - \frac{3}{q^3} - \frac{3}{q^4} - \frac{7}{q^6}\Big) \\
- q^7\Big(2 - \frac{1}{q}\Big) - (3q^5 + 3q^4 + 3q^2 + 3),
\end{aligned}
$$

where functions in each pair of parentheses are strictly increasing in $q > 0$, we easily verify that for $q \geqslant 3.09$, i.e. $p \geqslant 871$, the value of this polynomial in q is negative.

Since the leading term of $K(z)$ is equal to z^{14}, we have $\lim_{z\to+\infty} K(z) = +\infty$, which implies that, for $p \geqslant 871$, $K(z)$ has a real root ζ in the interval $(p^{1/6}, +\infty)$. As the function $K(z)/z^{14}$ is strictly increasing on this interval, there are no other roots of $K(z)$ in this interval.

Assume that a complex number η, $|\eta| \geqslant \zeta$, is some other root of $K(z)$. Then from the triangle inequality and the fact that all the coefficients of $L(z)$ are nonnegative, it follows

$$
1 = L\Big(\frac{1}{\eta}\Big) = \Big|L\Big(\frac{1}{\eta}\Big)\Big| \leqslant L\Big(\frac{1}{|\eta|}\Big) \leqslant L\Big(\frac{1}{\zeta}\Big) = 1,
$$

so η has to be a positive real number of absolute value equal to ζ and thus $\eta = \zeta$.

The coefficients of $L'(z)$ are nonnegative, which immediately gives $L'(\frac{1}{\zeta}) > 0$, so $\frac{1}{\zeta}$ is a simple root of $1 - L(z)$ and ζ is a simple root of $K(z)$.

The fraction $\frac{c}{1-\zeta z}$ appears in the partial fraction decomposition of (3.5) with a coefficient $c \neq 0$, otherwise $\frac{1}{\zeta}$ would not be a pole of the rational function (3.5). Finally, the coefficients in (3.4) are positive for terms z^k of large enough degree (actually, for $k \geqslant 8$) and are also asymptotically equal to $c\zeta^k$ by the preceding discussion. Hence, we conclude that $c > 0$. This completes the proof for $p \geqslant 871$.

For every prime p, $11 \leqslant p < 871$, using a computer we count for how many pairs $(b_n, b_{n-1}) \in \{1, 2, \ldots, p-1\}^2$ the value of $b_n b_{n-1} + b_n p + p$ lies in each of the intervals

$$
(p, p^{4/3}],\ (p^{4/3}, p^{3/2}],\ (p^{3/2}, p^{5/3}],\ (p^{5/3}, p^2],\ (p^2, p^{7/3}].
$$

If we denote these counts of pairs by c_1, c_2, c_3, c_4, c_5 respectively, then the procedure using generating functions is the same as before. For each of these p, we check with a computer algebra system (we used Wolfram Mathematica) that the root with the smallest absolute value of

$$
1 - L(z) = 1 - (c_1 z^8 + c_2 z^9 + c_3 z^{10} + c_4 z^{12} + c_5 z^{14})
$$

is strictly smaller than $p^{-1/6}$.

The remaining cases to verify are those where $p \in \{2, 3, 5, 7\}$. Here, for $\max\{a_n, a_{n-1}\} \leqslant 4$, we replace in (3.1)

$$t = b_n b_{n-1} + b_n p^{a_{n-1}} + p^{a_n} \quad \text{with} \quad p^{\frac{1}{6}\lceil 6 \log_p(t) \rceil},$$

where $\lceil \cdot \rceil$ is the ceiling function. The next steps are similar as in the previous case. For example, in the case $p = 2$, we obtain the polynomial

$$\begin{aligned} 1 - L(z) = 1 - z^{14} - 2z^{17} - z^{20} - 2z^{21} - 2z^{23} \\ - z^{25} - 2z^{26} - 2z^{27} - 2z^{28} - z^{31} \end{aligned}$$

which has a root smaller than $2^{-1/6} - 0.009$.

Analogously, we check the cases $p \in \{3, 5, 7\}$. □

Corollary 3. *Let p be any prime number. For any $k \geqslant 0$, any sequences $(a_i)_{1 \leqslant i \leqslant k}$ of positive integers and $(b_i)_{0 \leqslant i \leqslant k}$ of elements in $\{1, \ldots, p-1\}$, and any positive number C, there are infinitely many positive integers u such that there are at least C reduced positive rational numbers u/v with nonterminating p-adic continued fraction expansion, all of them having initial part*

$$[b_0, p^{a_1} : b_1, p^{a_2} : b_2, \ldots, p^{a_k} : b_k, \ldots].$$

Proof. In the proof of Theorem 2, we can fix the initial partial numerators and denominators in the expansion of the rational number P_n/Q_n which we considered. The only thing changing is that the right hand side of (3.2) needs to be divided by a large enough constant depending on the terms that we fixed in the continued fraction expansion. This, however, changes neither the rest of the proof nor its conclusion. We obtain the same asymptotic lower bound with constants in $\gg$ now depending only on p and $(a_i)_{1 \leqslant i \leqslant k}$.

If, for some C, there were only finitely many positive integers u with at least C rational numbers u/v having nonterminating expansion with a given initial part, then the number of such rational numbers belonging to $N_p(x)$ would be less than $(C+1)x$ for x large enough. This contradicts the lower bound from Theorem 2. Note that we also used here the fact that for a fixed positive integer u, there are less than u fractions u/v with terminating p-adic continued fraction since for $u/v < 0$ and $0 < u/v < 1$, i.e. for $v < 0$ and $v > u$, the p-adic continued fraction expansion of u/v is clearly infinite. □

Using the simple identity

$$b + \frac{p^a}{\frac{u}{v}} = \frac{bu + p^a v}{u},$$

we see that a result analogous to the previous corollary holds if we look at denominators instead of numerators, i.e. changing only "there are infinitely many positive integers u" in the statement of Corollary 3 with "there are infinitely many positive integers v", the claim still holds.

4 Various results on expansion of rational numbers

For a positive integer n, we slightly modify and generalize the notation from [9] and denote by $S_p(n)$ the set of all positive rationals of the form u/v, where $u+v=n$ and $\gcd(uv,p)=\gcd(u,v)=1$. Let $T_p(n)$ be the set of elements in $S_p(n)$ with terminating p-adic continued fraction expansion.

The next proposition was proved for $p=2$ in [9, Proposition 1].

Proposition 4. *If p divides n, then* $\operatorname{card} T_p(pn) = \operatorname{card} T_p(n)$.

Proof. Let $u/v \in T_p(pn)$. Then $1=\gcd(u,v)=\gcd(pn-v,v)=\gcd(pn,v)$ and

$$\frac{u}{v}-(p-1)=\frac{pn-v}{v}-p+1=p\Big(\frac{n}{v}-1\Big)=\frac{p}{\frac{v}{n-v}}.$$

Since $v+(n-v)=n$ and p divides n, we immediately see that $\gcd(n-v,p)=\gcd(v,p)=1$ and $\gcd(v,n-v)=\gcd(v,n)=1$.

From $\frac{u}{v}\in T_p(pn)$, we conclude that $\frac{v}{n-v}\in T_p(n)$. Thus we have a map $T_p(pn)\to T_p(n)$ given by $\frac{u}{v}\mapsto\frac{v}{n-v}$ with the inverse $\frac{w}{z}\mapsto\frac{pn-w}{w}$. Note that while the first rule does not give a map from $S_p(pn)$ to $S_p(n)$ since $n-v$ can be negative, the second one does give a map from $S_p(n)$ to $S_p(pn)$. □

In the next proposition, we generalize to any prime p the result from [9, Corollary 2] which was proved for $p=2$.

Proposition 5. *For a positive integer n, the set $T_p(p^n)$ contains only the rational number u_n/u_{n-1}, where*

$$u_k=\frac{1}{p+1}\big(p^{k+1}+(-1)^k\big),\quad k\geqslant 0.$$

Proof. Note that u_k is indeed a positive integer for every nonnegative integer k.

We prove the claim by induction.

For $n=1$, if $u/v\in T_p(p)$, then $u+v=p$, which is equivalent to

$$\frac{u}{v}=p-1+\frac{p}{\frac{v}{1-v}}$$

if $v > 1$. However, $\frac{v}{1-v} < 0$ has nonterminating p-adic continued fraction expansion, so we must have $v = 1$, which gives $u/v = p - 1 = u_1/u_0$ as the only fraction with a terminating expansion.

Supposing that the statement of the proposition holds for $n - 1 \geqslant 1$, let $u/v \in T_p(p^n)$. Then

$$\frac{u}{v} = \frac{p^n - v}{v} = p - 1 + \frac{p^n}{v} - p = p - 1 + \frac{p}{\frac{v}{p^{n-1}-v}}.$$

Now, $v+(p^{n-1}-v) = p^{n-1}$ and the induction assumption imply that $v = u_{n-1}$, so that $u = p^n - v = p^n - u_{n-1} = u_n$. □

Notice that

$$\left|\frac{u_n}{u_{n-1}} - (-1)\right|_p = \left|\frac{p^{n+1} + p^n}{p^n + (-1)^{n-1}}\right|_p = p^{-n}$$

which is in line with the fact that $u_n/u_{n-1} = [p - 1, p : p - 1, \dots, p : p - 1]$ is a convergent of the p-adic continued fraction expansion of -1.

For $p > 2$, the number -1 has rational approximations which are equally close to it in p-adic distance, but have smaller height (maximum of the standard absolute value of numerator and denominator), such as $\frac{p^n+1}{2}/\frac{p^n-1}{2}$. However, according to the previous proposition, these approximations must have infinite p-adic continued fractions. Some results connecting the quality of rational approximations with the finiteness of their Schneider's continued fraction expansion can be found in [12, § 3].

Let us also mention that because of the mirror formula for continued fractions, it was to be expected that the p-adic continued fraction expansion of u_n/u_{n-1} is symmetric (i.e. palindromic). Namely, if

$$\frac{P_n}{Q_n} = [b_0, p^{a_1} : b_1, \dots, p^{a_n} : b_n] \tag{4.1}$$

and P_{n-1}/Q_{n-1} is its penultimate convergent, then it can easily be shown [13, § I.4] that $P_n/P_{n-1} = [b_n, p^{a_n} : b_{n-1}, \dots, p^{a_1} : b_0]$ (reversing the order of all elements). Here we have $Q_n = u_{n-1} = P_{n-1}$. Hence, $P_n/Q_n = P_n/P_{n-1}$ and the continued fraction expansion of P_n/Q_n is necessarily symmetric.

In a similar vein, if b_0 and b_n are not 1 and the expansion (4.1) is not symmetric, then

$$\frac{P_n - Q_n}{Q_n} \quad \text{and} \quad \frac{P_n - P_{n-1}}{P_{n-1}}$$

are different elements of $T_p(P_n)$ and, therefore, $\operatorname{card} T_p(P_n) \geqslant 2$.

Proposition 6. *For every prime p,* $\limsup_{n\to+\infty} \operatorname{card} T_p(n) = +\infty$ *holds.*
For primes $p > 2$,

$$\limsup_{\substack{n\to+\infty \\ p\nmid n}} \operatorname{card} T_p(n) = +\infty,$$

where lim sup *is taken over positive integers not divisible by p.*

Proof. Let $p \geqslant 2$. Take any positive number C, fix only $b_0 = 1$ and apply Corollary 3. If u is any of the positive integers guaranteed to exist by that corollary and u/v has a terminating p-adic continued fraction with the initial partial denominator equal to $b_0 = 1$, then $p\frac{u}{v} - 1 = \frac{pu-v}{v}$ lies in $T_p(pu)$, so that $\operatorname{card} T_p(pu) \geqslant C$. Since C was arbitrary, the first assertion of the proposition holds.

Now, let $p > 2$. Take any positive number C, fix only $b_0 = 2$ and again apply Corollary 3. If u is any of the positive integers guaranteed to exist by that corollary and u/v has a terminating p-adic continued fraction with the initial partial denominator equal to $b_0 = 2$, then u is not divisible by p and $\frac{u}{v} - 1 = \frac{u-v}{v}$ lies in $T_p(u)$, so that $\operatorname{card} T_p(u) \geqslant C$. Since C was arbitrary, the second assertion of the proposition holds. □

Proposition 7. *For any prime p, there exists a positive integer n divisible by p such that $T_p(n)$ is an empty set while $S_p(n)$ is not empty.*

Note that we are not interested in the cases like $p = 3$ and $n = 10$ where already $S_p(n)$ is empty which tells us nothing about continued fraction expansions.

Proof. For $p = 2$ we can take $n = 30 = 2 \cdot 15$, while for $p = 3$, we can take $n = 66 = 3 \cdot 22$. The statement follows by checking expansions of elements in $S_p(n)$, where we only need to consider elements larger than 1, since those smaller than 1 obviously have nonterminating continued fraction expansions. Thus we verify expansions of 4 elements in $S_2(30)$ and 10 elements in $S_3(66)$.

Now, suppose $p \geqslant 5$ and let k be an odd number, $1 \leqslant k \leqslant p-1$, such that $2p + k$ is a prime number. Although, for a fixed k, it remains an open question whether there exist infinitely many prime numbers p such that $2p+k$ is also a prime (e.g. for $k = 1$ such p are called Sophie Germain primes), still, we know that there is always a prime number in the interval $(2\ell, 3\ell)$ for any integer $\ell > 1$ (see e.g. [2], but note that a stronger result was proved earlier in [8, Lemma 2]).

We will show that for $n = p(3p + k)$, the set $T_p(n)$ is empty. Let $u/v \in S_p\big(p(3p + k)\big)$. We have

$$\frac{u}{v} - (p-1) = p\,\frac{3p+k-v}{v}. \tag{4.2}$$

For $v = 3p + k$, $\gcd(u, v) = 3p + k > 1$, so $u/v \notin S_p(n)$.

If $v > 3p + k$, (4.2) shows that $u/v \notin T_p(n)$.

Thus, when $v \equiv k \pmod{p}$, we only need to check $v \in \{k, 2p + k\}$ since for $v = p + k$, both u and v are even.

For $v = 2p + k$, we have

$$\frac{3p + k - v}{v} = \frac{p}{2p + k} = \frac{p}{k + \frac{p}{\frac{1}{2}}},$$

which, together with (4.2), shows that the expansion of u/v does not terminate.

For $v = k$, we have

$$\frac{3p + k - v}{v} = \frac{3p}{k} = \frac{p}{\frac{k}{3}} = \frac{p}{\ell + \frac{k-3\ell}{3}},$$

where we take $\ell \in \{1, \ldots, p - 1\}$ such that p divides $k - 3\ell$, which, since $k - 3\ell < k < p$, implies that $k - 3\ell \leqslant 0$. If k is not divisible by 3, then $k - 3\ell < 0$ and the expansion of u/v is nonterminating. If k is divisible by 3, then 3 divides both $v = k$ and $n = p(3p + k)$ and thus also $u = n - v$, in contradiction with $\gcd(u, v) = 1$.

In what follows, we assume that $v < 3p + k$ and $v \not\equiv k \pmod{p}$. Then (4.2) shows that the beginning of p-adic expansion of u/v is

$$\frac{u}{v} = p - 1 + \frac{p}{\frac{v}{3p+k-v}},$$

where $\gcd(v, 3p + k - v) = \gcd(v, 3p + k) = 1$ and $\gcd(v(3p + k - v), p) = 1$. Denote $w = 3p + k - v$. We want to show that $v/w \notin T_p(3p + k)$.

Let $b \in \{1, \ldots, p - 1\}$ such that $p \mid (v - wb)$. Then

$$\frac{v}{w} - b = \frac{v - wb}{w} = \frac{3p + k - w(b + 1)}{w}$$

and from $3p + k - w(b + 1) < 3p + k < 4p$, we have the following five possible cases.

Case 1. If $3p + k - w(b + 1) < 0$, then v/w obviously has infinite p-adic continued fraction expansion.

Case 2. If $3p + k - w(b + 1) = 0$, then $v = wb$, so $\gcd(v, w) = 1$ implies $w = 1$, $b = 3p + k - 1 \geqslant 3p$ which is impossible.

Case 3. If $3p + k - w(b + 1) = p$, then $w(b + 1) = 2p + k$, which is prime by assumption, so either $w = 1$ and we get a contradiction as in Case 2, or $w = 2p + k$ and $b + 1 = 1$ which is again impossible.

Case 4. If $3p + k - w(b+1) = 2p$, then $w(b+1) = p + k$ while

$$\frac{v}{w} - b = \frac{2p}{w} = \frac{p}{\frac{w}{2}}.$$

Since $\gcd(v, w) = 1$ and $v + w$ is even, both v and w are odd and $w/2$ is not an integer. Taking $b' \in \{1, \ldots, p-1\}$ such that $w - 2b'$ is divisible by p, we see that $w - 2b'$ is odd, $w - 2b' < w = \frac{p+k}{b+1} < \frac{2p}{2} = p$ and conclude $w - 2b' < 0$, so that

$$\frac{v}{w} = b + \frac{p}{b' + \frac{w-2b'}{2}}$$

is not in $T_p(3p + k)$.

Case 5. Finally, if $3p + k - w(b+1) = 3p$, then $w(b+1) = k$ and

$$\frac{v}{w} - b = \frac{3p}{w} = \frac{p}{\frac{w}{3}}.$$

If 3 divides w, from $w(b+1) = k$, we get that 3 divides k and thus also $v + w = 3p + k$, implying $\gcd(v, w) > 1$ which is not possible.

Thus $w/3$ is not an integer and we take $b' \in \{1, \ldots, p-1\}$ such that p divides $w - 3b'$. Now, $2w \leqslant w(b+1) = k \leqslant p-1$ shows that $w - 3b' < w \leqslant \frac{p-1}{2}$ and, since $w - 3b' \neq 0$, we have $w - 3b' < 0$. This means that

$$\frac{v}{w} = b + \frac{p}{b' + \frac{w-3b'}{3}}$$

has nonterminating expansion and we are finished.

The set $S_p(3p + k)$ is not empty since for $k > 1$ it contains $\frac{3p+k-1}{1}$, while for $k = 1$ and $p > 3$ it contains $\frac{3p-2}{3}$. Therefore, $S_p\big(p(3p + k)\big)$ is also not empty having

$$\frac{p(3p+k) - (3p+k-1)}{3p+k-1} \quad \text{as an element if } k > 1 \text{ and}$$

$$\frac{3p^2 - 2p + 2}{3p-2} \quad \text{if } k = 1,\ p > 3.$$

□

Corollary 8. *For any prime p, there are infinitely many positive integers n such that $S_p(n)$ is nonempty while $T_p(n)$ is empty.*

Proof. The statement follows immediately from Propositions 4 and 7. The only detail left to mention is that from $u/v \in S_p(n)$, we get $p - 1 + \frac{p}{u/v} \in S_p(pn)$, so $S_p(n)$ being nonempty implies that $S_p(pn)$ is nonempty as well. □

Proposition 9. *Let p be an odd prime and $n > 3$ an odd positive integer such that $n(n-1)$ is not divisible by p and $(p, n) \notin \{(3,5), (3,11), (5,9)\}$. Then there is a fraction $u/v \in S_p(n) \setminus T_p(n)$ such that $u/v > 1$.*

Proof. For odd $n > 3$, the fraction

$$\frac{\frac{n+1}{2}}{\frac{n-1}{2}} = 1 + \frac{1}{\frac{n-1}{2}}$$

is obviously reduced and lies in the interval $(1, 2)$. If both its numerator and denominator are not divisible by p, then this fraction is in $S_p(n)$. However, in its p-adic continued fraction expansion, the first partial denominator (b_0 as in (1.1)) cannot be 1, since $\frac{n+1}{2} \not\equiv \frac{n-1}{2} \pmod{p}$, so this fraction cannot be in $T_p(n)$ since its next complete quotient is negative. Therefore, we are left to consider cases when p divides $\frac{n-1}{2}$ or $\frac{n+1}{2}$. The first case is excluded by the statement of the proposition.

Let $n = 2kp - 1$ for a positive integer k. The fraction

$$\frac{\frac{n+p}{2}}{\frac{n-p}{2}} = 1 + \frac{p}{\frac{p-1}{2} + (k-1)p}$$

is in $S_p(n)$ since p does not divide n. It is in $T_p(n)$ if and only if $k = 1$ or $k = p^t + 1$ for a nonnegative integer t.

On the other hand,

$$\frac{n-1}{1} = p - 2 + (2k-1)p$$

is also in $S_p(n)$ since p does not divide $n - 1$. It is in $T_p(n)$ if and only if $2k = 1$ or $2k = p^{t'} + 1$ for a nonnegative integer t'.

Thus, if both of these fractions are in $T_p(n)$, we can only have $2 = p^{t'} + 1$ or $2(p^t + 1) = p^{t'} + 1$. The first equation gives $k = 1$, $n = 2p - 1$, while the second one implies $t' \geqslant 1$ and reducing modulo p gives $2 \equiv 1 \pmod{p}$ or $4 \equiv 1 \pmod{p}$ which holds only for $p = 3$, $t = 0$, $k = 2$, $n = 11$ which was not allowed.

The only case left to study is $n = 2p - 1$. Let ℓ be a positive integer such that $2^\ell < p < 2^{\ell+1}$. Observe the fraction

$$\frac{n - 2^\ell}{2^\ell} = \frac{2p - 2^\ell - 1}{2^\ell}.$$

It is easily seen that this fraction lies in the open interval $(1, 3)$. Hence, for it to have a finite p-adic continued fraction expansion, the first partial denominator b_0 has to be 1 or 2.

If the first partial denominator is 1, the numerator of

$$\frac{n-2^\ell}{2^\ell}-1=\frac{2p-2^{\ell+1}-1}{2^\ell},$$

i.e. $2p-2^{\ell+1}-1$ has to be divisible by p, which means that p divides $2^{\ell+1}+1$. However,

$$1=\frac{2^{\ell+1}}{2^{\ell+1}}<\frac{2^{\ell+1}+1}{p}<\frac{2^{\ell+1}+1}{2^\ell}<3,$$

implying $2^{\ell+1}+1=2p$ and this is not possible because of the difference in the parity.

If the first partial denominator is 2, the numerator of

$$\frac{n-2^\ell}{2^\ell}-2=\frac{2p-3\cdot 2^\ell-1}{2^\ell},$$

i.e. $2p-3\cdot 2^\ell-1$ must be divisible by p, which means that p divides $3\cdot 2^\ell+1$. Since

$$1<\frac{3\cdot 2^\ell+1}{2^{\ell+1}}<\frac{3\cdot 2^\ell+1}{p}<\frac{3\cdot 2^\ell+1}{2^\ell}<4,$$

this would imply $3\cdot 2^\ell+1$ is either $2p$ or $3p$, which is impossible because 1 is not divisible by 2 and 3.

There remains only the possibility that $(n-2^\ell)/2^\ell\notin S_p(n)$. Since $\gcd(n-2^\ell,2^\ell)=\gcd(2p-1,2^\ell)=1$ and p does not divide 2^ℓ, this can only happen if p divides $n-2^\ell=2p-2^\ell-1$, i.e. p divides $2^\ell+1$. Because $2^\ell+1\leqslant p$, this implies $p=2^\ell+1$, so that p is a Fermat prime of the form $p=2^{2^r}+1$ for some integer $r\geqslant 0$ (see e.g. [6, Example 2.8]).

For $r\in\{0,1\}$, we get $(p,n)\in\{(3,5),(5,9)\}$, which was excluded in the proposition. For $r\geqslant 2$, we have $p\geqslant 17$ and

$$p\equiv 2^{2^r}+1\equiv\left(2^{2^2}\right)^{2^{r-2}}+1\equiv 1^{2^{r-2}}+1\equiv 2\pmod 5,$$

so we have

$$\frac{p+2}{p-3}\in S_p(n)\quad\text{and}\quad\frac{p+2}{p-3}=1+\frac{5}{p-3}$$

is in the interval $(1,2)$. Just as before, because p does not divide 5, we see that this fraction is not in $T_p(n)$.

For every n given in the statement of this proposition, we found a fraction greater than 1 lying in the set $S_p(n)\setminus T_p(n)$. This completes the proof of the proposition. □

Note that we have

$$S_3(5) = \left\{\frac{1}{4}, \frac{4}{1}\right\}, \quad S_3(11) = \left\{\frac{1}{10}, \frac{4}{7}, \frac{7}{4}, \frac{10}{1}\right\}, \quad S_5(9) = \left\{\frac{1}{8}, \frac{2}{7}, \frac{7}{2}, \frac{8}{1}\right\},$$
$$4 = [1, 3 : 1], \quad \frac{7}{4} = [1, 3 : 1, 3 : 1], \quad 10 = [1, 3^2 : 1],$$
$$\frac{7}{2} = [1, 5 : 2], \quad \frac{8}{1} = [3, 5 : 1],$$

so all fractions $u/v \in S_p(n) \cap (1, +\infty)$ are in $T_p(n)$ if $(p, n) \in \{(3, 5), (3, 11), (5, 9)\}$ and thus this pairs had to be excluded in the statement of the previous proposition.

For a prime $p \geqslant 5$, let $n = (p-2)!$. Then the fractions

$$\frac{1}{n-1}, \frac{2}{n-2}, \ldots, \frac{p+1}{n-(p+1)}$$

and their reciprocals are not in $S_p(n)$. For $k \in \{2, 3, \ldots, p-2\}$, $\gcd(k, n-k) = \gcd(k, n) = k > 1$. Since n is even, for $k \in \{p-1, p+1\}$, both the numerator and the denominator of $\frac{k}{n-k}$ are divisible by 2. In $\frac{p}{n-p}$ the numerator is divisible by p. Finally, Wilson's theorem (see e.g. [6, § 3.6]) shows that p divides the denominator $n-1$ in the fraction $\frac{1}{n-1}$.

Similarly, it is not hard to show that for any positive integer ℓ and

$$n = \prod_{\substack{1 \leqslant m \leqslant 2\ell p \\ p \nmid m}} m,$$

we have $\frac{k}{n-k}, \frac{n-k}{k} \notin S_p(n)$ for every k in the interval $1 \leqslant k \leqslant 2\ell p$.

The examples show that we cannot simply go through numbers of the form $\frac{k}{n-k}$ for fixed $k \geqslant 1$ eliminating n such that $\frac{k}{n-k} \in S_p(n) \setminus T_p(n)$ hoping to confirm that this set difference is nonempty for all but finitely many positive integers n. Nevertheless, the next theorem confirms just this fact and shows much more. The downside is we need to use significantly stronger arguments than those used thus far in this paper.

Theorem 10. *For an odd prime p, there exist positive numbers c_1 and c_2 depending only on p such that for all integers $n \geqslant c_1$, we have*

$$\operatorname{card}\big(S_p(n) \setminus T_p(n)\big) > c_2 \frac{n}{\log n}. \tag{4.3}$$

For $p = 2$, the same statement holds if we further require that such n be even.

For $p > 3$, the claim remains true if instead of the inequality (4.3), *we set*

$$\operatorname{card}\Big(\big(S_p(n) \setminus T_p(n)\big) \cap (1, +\infty)\Big) > c_2 \frac{n}{\log n}. \tag{4.4}$$

Proof. According to the prime number theorem for arithmetic progressions, for a prime p, an integer s not divisible by p, and $x > 0$, the number $\pi(x; p, s)$ of primes $q < x$, $q \equiv s \pmod p$ is asymptotically equal to $\frac{x}{(p-1)\log x}$, where $\log x$ is the natural logarithm of x. More precisely, the Siegel-Walfisz theorem (see e.g. [11, pp. 5,382]) or the related results with explicit constants [4, Theorem 1.3] say that there exist constants c_3 and c_4 depending only on p such that

$$\left|\pi(x; p, s) - \frac{x}{(p-1)\log x}\right| < c_3 \frac{x}{(\log x)^2} \quad \text{for all } x \geqslant c_4. \tag{4.5}$$

Let k_0 be an integer satisfying

$$\begin{aligned} k_0 &\not\equiv 0 \pmod p, \\ k_0 &\not\equiv n \pmod p, \\ \text{and if } p > 3, \quad 2k_0 &\not\equiv n \pmod p. \end{aligned}$$

Recall that for $p = 2$, n should be even, so such an integer k_0 always exists.

We look at the prime numbers k in the interval $(n/3, n/2)$ such that $k \equiv k_0 \pmod p$ and obtain from (4.5) that the number of such k is greater than

$$\begin{aligned} \pi\Big(\frac{n}{2}; p, k_0\Big) - \pi\Big(\frac{n+1}{3}; p, k_0\Big) &> \frac{1}{p-1}\left(\frac{\frac{n}{2}}{\log \frac{n}{2}} - \frac{\frac{n+1}{3}}{\log \frac{n+1}{3}}\right) - c_5 \frac{n}{(\log n)^2} \\ &> \frac{1}{p-1}\left(\frac{\frac{n}{2}}{\log n} - \frac{\frac{2n}{5}}{\log n}\right) - c_5 \frac{n}{(\log n)^2} > c_6 \frac{n}{\log n} \end{aligned}$$

if $n > c_7$ for some positive constants c_5, c_6, c_7 depending only on p.

For such k, the fractions $\frac{k}{n-k}$, $\frac{n-k}{k}$ are in $S_p(n)$ since $\gcd(k(n-k), p) = \gcd(k_0(n-k_0), p) = 1$ and from $\frac{n}{3} < k < \frac{n}{2}$, we get $1 < \frac{n-k}{k} < 2$, so that $\gcd(n-k, k) = 1$ as well. The numbers $\frac{k}{n-k} < 1$ are obviously not in $T_p(n)$, while for $p > 3$, the numbers $\frac{n-k}{k} > 1$ are also not in $T_p(n)$ since $1 > \frac{n-k}{k} - 1 = \frac{n-2k}{k}$ and $n - 2k$ is not divisible by p. $\square$

Note that $\operatorname{card}(S_p(n)) \leqslant \varphi(n)$, where φ is Euler's totient function. Since

$$\varphi(n) > \frac{n}{2\log\log n} \text{ for } n > 10^{14} \quad \text{and} \quad \varphi(n) < \frac{n}{\log\log n} \text{ for infinitely many } n$$

as can be seen in [3, p. 72], perhaps the bound in (4.3) can be improved by sieve methods.

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