Schneider's p-adic continued fractions

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Abstract

We study Schneider's version of p-adic continued fractions. We are interested in the finiteness of rational number expansion, the quality of approximation by convergents, the irrationality exponent of a number with a given continued fraction expansion, and the convergence of Schneider's continued fractions in the field of real numbers. The main requirement for all of these problems is a good estimate of growth for the sequences of numerators and denominators of convergents.

For a prime p, several types of p-adic continued fractions have been introduced, but none of them have all the useful properties that the usual simple continued fractions of real numbers possess. In this paper, we study Schneider's p-adic continued fractions [21, 23]. This type of p-adic continued fractions has been analysed with respect to finiteness and periodicity of expansion [10, 11, 12, 1, 22, 23, 19, 16] as well as the distribution of "digits" [16, 15, 17, 14]. However, it has also proved useful in constructing p-adic numbers with required Diophantine approximation properties [4, 6, 7, 9]. It is exactly this aspect of Schneider's p-adic continued fractions that we focus on.

The paper is organized as follows. The first section is of introductory character, gathering the definitions and properties that will be needed. In Section 2 we refine some results on deciding when the p-adic continued fraction expansion of a rational number is finite. The next section connects this question with the quality of approximation by convergents. In Section 4 we give several examples of Schneider's continued fractions and analyse the rate of growth for numerators and denominators of convergents in these continued fractions. The bounds that we prove are then used to obtain results on whether the convergents are the best rational approximations (Section 4), the irrationality exponent of numbers constructed in this way (Section 5) and the question of convergence of these continued fractions in the field of real numbers (Section 6).

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1 Introduction

Throughout this text, p denotes a prime number and $\mathbb{Q}_p$ is the field of p-adic numbers equipped with the p-adic absolute value $|\cdot|_p$ normalized in such a way that $|p|_p = p^{-1}$.

Let $\mathbf{a} = (a_n)_{n\geqslant 1}$ be a sequence of positive integers and $\mathbf{b} = (b_n)_{n\geqslant 0}$ a sequence of integers such that $1 \leqslant b_n \leqslant p-1$ for $n \geqslant 0$. Set

$$\begin{pmatrix} P_{-1} & P_{-2} \\ Q_{-1} & Q_{-2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{aligned} P_n &= b_n P_{n-1} + p^{a_n} P_{n-2} \\ Q_n &= b_n Q_{n-1} + p^{a_n} Q_{n-2} \end{aligned} \quad \text{for } n \geqslant 0, \tag{1.1}$$

where we put $a_0 = 0$ for completeness. This implies

$$\begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} = \begin{pmatrix} P_{n-1} & P_{n-2} \\ Q_{n-1} & Q_{n-2} \end{pmatrix} \begin{pmatrix} b_n & 1 \\ p^{a_n} & 0 \end{pmatrix} \text{ for } n \geqslant 0,$$

and by induction

$$\begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} = \prod_{i=0}^{n} \begin{pmatrix} b_i & 1 \\ p^{a_i} & 0 \end{pmatrix} \quad (n \geqslant 0). \tag{1.2}$$

Writing the last equality as

$$\begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} = \begin{pmatrix} b_0 & 1 \\ p^0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p^{a_1} \end{pmatrix} \begin{pmatrix} b_1 & 1 \\ p^0 & 0 \end{pmatrix} \prod_{i=2}^{n} \begin{pmatrix} b_i & 1 \\ p^{a_i} & 0 \end{pmatrix} \quad (n \geqslant 1),$$

we see that

$$\begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} = \begin{pmatrix} b_0 & p^{a_1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} P'_{n-1} & P'_{n-2} \\ Q'_{n-1} & Q'_{n-2} \end{pmatrix},$$

where the sequences $(P'_n)_{n\geqslant -2}$ and $(Q'_n)_{n\geqslant -2}$ are defined using the same initial conditions and recurrence equations as for (P_n) and (Q_n), but substituting for $\mathbf{a}$ and $\mathbf{b}$ the shifted sequences $\mathbf{a}' = (a_{n+1})_{n\geqslant 1}$ and $\mathbf{b}' = (b_{n+1})_{n\geqslant 0}$.

Now we have

$$\frac{P_n}{Q_n} = \frac{b_0 P'_{n-1} + p^{a_1} Q'_{n-1}}{P'_{n-1}} = b_0 + \frac{p^{a_1}}{\frac{P'_{n-1}}{Q'_{n-1}}}$$

and continuing, we obtain by induction a finite Schneider's p-adic continued fraction

$$\frac{P_n}{Q_n} = b_0 + \cfrac{p^{a_1}}{b_1 + \cfrac{p^{a_2}}{b_2 + \cfrac{p^{a_3}}{\ddots + \cfrac{p^{a_n}}{b_n}}}} \quad (n \geqslant 0) \tag{1.3}$$

which we write as $[b_0, p^{a_1} : b_1, p^{a_2} : b_2, \ldots, p^{a_n} : b_n]$.

Taking the determinant of (1.2), we get

$$P_n Q_{n-1} - P_{n-1} Q_n = (-1)^{n+1} p^{\sum_{i=1}^{n} a_i} \quad (n \geqslant 1), \tag{1.4}$$

so that

$$\begin{aligned} \frac{P_n}{Q_n} - \frac{P_{n-1}}{Q_{n-1}} &= \frac{(-1)^{n+1}}{Q_{n-1}Q_n} p^{\sum_{i=1}^{n} a_i}, \\ \frac{P_n}{Q_n} - \frac{P_m}{Q_m} &= \sum_{k=m+1}^{n} \left(\frac{(-1)^{k+1}}{Q_{k-1}Q_k} p^{\sum_{i=1}^{k} a_i} \right) \quad (n > m \geqslant 0). \end{aligned} \tag{1.5}$$

It follows easily from (1.1) that p divides neither P_n nor Q_n for any $n \geqslant 0$. Thus (1.4) implies that $\gcd(P_n, P_{n-1}) = \gcd(Q_n, Q_{n-1}) = \gcd(P_n, Q_n) = 1$ and the fractions P_n/Q_n are already reduced. Taking into account that p-adic absolute value is non-Archimedean [18], (1.5) gives

$$\left| \frac{P_n}{Q_n} - \frac{P_m}{Q_m} \right|_p = p^{-\sum_{i=1}^{m+1} a_i} \quad (n > m \geqslant 0). \tag{1.6}$$

This shows that $(P_n/Q_n)_{n \geqslant 0}$ is a Cauchy sequence and therefore converges to some $\xi_{\mathbf{a},\mathbf{b}} \in \mathbb{Q}_p$. Actually, $\xi_{\mathbf{a},\mathbf{b}}$ is a p-adic unit, i.e. $|\xi_{\mathbf{a},\mathbf{b}}|_p = 1$. Now (1.6) implies that

$$\left| \xi_{\mathbf{a},\mathbf{b}} - \frac{P_m}{Q_m} \right|_p = p^{-\sum_{i=1}^{m+1} a_i} \quad (m \geqslant 0), \tag{1.7}$$

so that we can write

$$\xi_{\mathbf{a},\mathbf{b}} = b_0 + \cfrac{p^{a_1}}{b_1 + \cfrac{p^{a_2}}{b_2 + \cfrac{p^{a_3}}{\ddots}}} = [b_0, p^{a_1} : b_1, p^{a_2} : b_2, p^{a_3} \ldots]. \tag{1.8}$$

From the other direction, we take any $\xi_0 \in \mathbb{Q}_p$ and, dividing it by $|\xi_0|_p$, we can assume that $|\xi_0|_p = 1$. Then $|\xi_0 - b_0|_p < 1$ for exactly one $b_0 \in \{1, \ldots, p-1\}$, so that defining $p^{a_1} = |\xi_0 - b_0|_p^{-1}$ and $\xi_1 = p^{a_1}/(\xi_0 - b_0)$, we see that $|\xi_1|_p = 1$ and $\xi_0 = b_0 + p^{a_1}/\xi_1$. This process can be continued

$$\xi_0 = [b_0, p^{a_1} : b_1, \ldots, p^{a_n} : \xi_n] \tag{1.9}$$

and we obtain (finite or infinite) sequences $\mathbf{a} = (a_n)_{n \geqslant 1}$ and $\mathbf{b} = (b_n)_{n \geqslant 0}$. It is then easily shown that $\xi_0 = \xi_{\mathbf{a},\mathbf{b}}$ for these sequences $\mathbf{a}$ and $\mathbf{b}$.

As is conventional, we will call P_n/Q_n from (1.3) a convergent of $\xi_{\mathbf{a},\mathbf{b}}$, and ξ_n in (1.9) a complete quotient. Note that analogously to the relation between (1.1) and (1.3), we can see that

$$\xi_0 = \frac{\xi_n P_{n-1} + p^{a_n} P_{n-2}}{\xi_n Q_{n-1} + p^{a_n} Q_{n-2}} \quad (n \geqslant 0). \tag{1.10}$$

We can immediately read off how close two p-adic numbers are looking at the initial parts of their continued fraction expansions that coincide. Let

$$\alpha = [b_0, p^{a_1} : b_1, p^{a_2} : b_2, \ldots] \quad \text{and} \quad \beta = [b'_0, p^{a'_1} : b'_1, p^{a'_2} : b'_2, \ldots]$$

be two p-adic numbers with their (finite or infinite) expansions. As usual, we suppose $|\alpha|_p = |\beta|_p = 1$. Let $(\alpha_n)_{n\geqslant 0}$ and $(\beta_n)_{n\geqslant 0}$ be the complete quotients of α and β, respectively, and $(P_n/Q_n)_{n\geqslant 0}$ the sequence of convergents to α. Depending on the position of the first difference between the continued fraction expansions of α and β, we have two cases.

When $a_i = a'_i$ for $1 \leqslant i \leqslant k-1$ and $b_i = b'_i$ for $0 \leqslant i \leqslant k-1$, but $a_k \neq a'_k$, then (1.10) implies

$$\begin{aligned} \alpha - \beta &= \frac{\alpha_k P_{k-1} + p^{a_k} P_{k-2}}{\alpha_k Q_{k-1} + p^{a_k} Q_{k-2}} - \frac{\beta_k P_{k-1} + p^{a'_k} P_{k-2}}{\beta_k Q_{k-1} + p^{a'_k} Q_{k-2}} \\ &= \frac{(\alpha_k p^{a'_k} - \beta_k p^{a_k})(P_{k-1}Q_{k-2} - P_{k-2}Q_{k-1})}{(\alpha_k Q_{k-1} + p^{a_k} Q_{k-2})(\beta_k Q_{k-1} + p^{a'_k} Q_{k-2})} \end{aligned}$$

from which, using (1.4) and the fact that $|\alpha_k|_p = |\beta_k|_p = |Q_{k-1}|_p = |Q_{k-2}|_p = 1$, we obtain

$$|\alpha - \beta|_p = p^{-\sum_{i=1}^{k-1} a_i - \min\{a_k, a'_k\}}. \tag{1.11}$$

Similarly, if $a_i = a'_i$ for $1 \leqslant i \leqslant k$ and $b_i = b'_i$ for $0 \leqslant i \leqslant k-1$, but $b_k \neq b'_k$, we get

$$|\alpha - \beta|_p = p^{-\sum_{i=1}^{k} a_i}. \tag{1.12}$$

2 Expansion of rational numbers

Although a finite Schneider's p-adic continued fraction always represents a rational number, the converse is not true. For example, $-1 = [p-1, p : p-1, p : p-1, \ldots]$ has an infinite periodic expansion. For brevity, we will put $(B)_n$ and $(B)_\infty$ if a block B in a continued fraction repeats n times or indefinitely, respectively. Thus, we can write $-1 = [p-1, (p : p-1)_\infty]$. Bundschuh [10] showed that if a rational number has an infinite p-adic continued fraction expansion, then this expansion has the same periodic tail, i.e. it has the form $[\ldots, p-1, (p : p-1)_\infty]$.

There is no complete characterization of rational numbers with infinite Schneider's p-adic continued fraction expansion, although Hirsh and Washington [16] tackled some special cases.

We modify Bundschuh's result in order to obtain an upper bound on the required number of steps in the expansion before it terminates or reaches -1 as a complete quotient.

We use Vinogradov notation $\ll$, $\gg$, and $\asymp$ as well as Landau big-$\mathcal{O}$ notation. The implied constants, which can be computed explicitly, always depend on the prime p and possibly also on a parameter used and mentioned in the specific example. Denote by $\mathrm{H}(u/v) = \max\{|u|, |v|\}$ the (naive) height of the reduced rational number u/v.

Expanding a rational number $\xi = x_0/x_1$ into a continued fraction (1.8) (either terminating or non-terminating), denote one complete quotient $x_i/x_{i+1} = [b_i, p^{a_{i+1}} : b_{i+1}, \ldots]$, where $x_i, x_{i+1} \in \mathbb{Z}$ and $\gcd(x_i, x_{i+1}) = 1$. Then the next complete quotient is

$$\frac{p^{a_{i+1}}}{\frac{x_i}{x_{i+1}} - b_i} = \frac{x_{i+1}}{\frac{x_i - b_i x_{i+1}}{p^{a_{i+1}}}} = \frac{x_{i+1}}{x_{i+2}},$$

where $x_{i+2} = (x_i - b_i x_{i+1})/p^{a_{i+1}}$ is an integer and $\gcd(x_{i+1}, x_{i+2}) = 1$. In this way we obtain a sequence $(x_i)_{i\geqslant 0}$ of integers which terminates if $x_i/x_{i+1} \in \{1, 2, \ldots, p-1\}$ for some i. Otherwise, we reach some pair $(x_i, x_{i+1}) \in \{(-1, 1), (1, -1)\}$ which means the complete quotient is $x_i/x_{i+1} = -1$ and the sequence is periodic from that place onward.

Theorem 1. *The expansion of a rational number ξ into a p-adic continued fraction either terminates or the complete quotient -1 is reached. The number of steps required, i.e. the number of complete quotients that need to be computed before either of the cases occur is $\mathcal{O}\big((\log \mathrm{H}(\xi))^2\big)$, where the implied constant depends only on p.*

Proof. We have

$$x_{i+2} = \frac{x_i - b_i x_{i+1}}{p^{a_{i+1}}}, \quad x_{i+3} = \frac{x_{i+1} - b_{i+1} x_{i+2}}{p^{a_{i+2}}}. \tag{2.1}$$

Note that $|x_i| \neq |x_{i+1}|$ since otherwise $x_i/x_{i+1} = 1$ or $x_i/x_{i+1} = -1$ which means we are already finished.

Compare $\mathrm{H}(x_i/x_{i+1})$ with $\mathrm{H}(x_{i+2}/x_{i+3})$. From (2.1),

$$\begin{aligned} |x_{i+2}| &\leqslant \frac{|x_i| + (p-1)|x_{i+1}|}{p} < \frac{1+p-1}{p}\,\mathrm{H}(x_i/x_{i+1}) = \mathrm{H}(x_i/x_{i+1}), \\ |x_{i+3}| &< \mathrm{H}(x_{i+1}/x_{i+2}) \leqslant \mathrm{H}(x_i/x_{i+1}), \end{aligned}$$

unless $x_{i+1}/x_{i+2} \in \{-1, 1\}$. Therefore, taking into account that x_{i+2} and x_{i+3} are coprime integers,

$$\mathrm{H}(x_{i+2}/x_{i+3}) \leqslant \mathrm{H}(x_i/x_{i+1}) - 1. \tag{2.2}$$

If $(a_{i+1}, b_i) \neq (1, p-1)$, a better bound can be obtained

$$|x_{i+2}| \leqslant \max\left\{\frac{|x_i| + (p-2)|x_{i+1}|}{p}, \frac{|x_i| + (p-1)|x_{i+1}|}{p^2}\right\} \leqslant \frac{p-1}{p}\,\mathrm{H}(x_i/x_{i+1}),$$

$$|x_{i+3}| \leqslant \frac{|x_{i+1}| + (p-1)|x_{i+2}|}{p} \leqslant \frac{1}{p}|x_{i+1}| + \left(\frac{p-1}{p}\right)^2 \mathrm{H}(x_i/x_{i+1})$$

$$\leqslant \frac{p^2-p+1}{p^2}\,\mathrm{H}(x_i/x_{i+1}).$$

Thus

$$\mathrm{H}(x_{i+2}/x_{i+3}) \leqslant \frac{p^2-p+1}{p^2}\,\mathrm{H}(x_i/x_{i+1}). \tag{2.3}$$

If $(a_{i+1}, b_i) = (1, p-1)$, we have

$$|x_{i+1} + x_{i+2}| = \left|x_{i+1} + \frac{x_i - (p-1)x_{i+1}}{p}\right| = \frac{1}{p}|x_i + x_{i+1}|.$$

However, if $|x_i + x_{i+1}| < 1$, then $x_i + x_{i+1} = 0$, i.e. $x_i/x_{i+1} = -1$. Hence, if $(a_{i+1}, b_i) = (a_{i+2}, b_{i+1}) = \ldots = (a_{i+k}, b_{i+k-1}) = (1, p-1)$, from

$$1 \leqslant |x_{i+k} + x_{i+k+1}| = \frac{1}{p^k}|x_i + x_{i+1}| \leqslant \frac{2}{p^k}\,\mathrm{H}(x_i/x_{i+1}),$$

we get

$$k \leqslant \frac{\log \mathrm{H}(x_i/x_{i+1})}{\log p} + 1. \tag{2.4}$$

Taking into account (2.2), (2.3), and (2.4), we see that

$$\mathrm{H}(x_{i+\kappa}/x_{i+\kappa+1}) \leqslant \frac{p^2-p+1}{p^2}\,\mathrm{H}(x_i/x_{i+1}) \quad \text{for} \quad \kappa = \left\lfloor \frac{\log \mathrm{H}(x_i/x_{i+1})}{\log p} \right\rfloor + 3.$$

This implies that in κ steps $\log \mathrm{H}(x_i/x_{i+1})$ will decrease by at least $-\log \frac{p^2-p+1}{p^2} > 0$. If $\mathrm{H}(x_i/x_{i+1}) \leqslant 1$, i.e. $\log \mathrm{H}(x_i, x_{i+1}) \leqslant 0$, we are finished. This shows that the number of steps required to obtain from ξ a complete quotient from the set $\{-1, 1, 2, \ldots, p-1\}$ is $\mathcal{O}\big((\log \mathrm{H}(\xi))^2\big)$ with the implied constant depending only on p. □

When computing the expansion of a particular rational number with an infinite p-adic continued fraction, it is natural to ask when the first appearance of a negative complete quotient might occur. At that moment, we know that -1 will be encountered as a complete quotient and the expansion will not terminate. The next theorem describes the worst case.

Theorem 2. *Let $\lambda = (1+\sqrt{1+4p})/2$. If the p-adic continued fraction expansion of a rational number ξ is infinite, then within the first*

$$\left\lfloor \frac{\log \mathrm{H}(\xi)}{\log \lambda} \right\rfloor + 3$$

complete quotients, at least one has to be negative.

This bound is in general asymptotically best possible. More precisely, there exists an infinite sequence of rational numbers $(\nu_n)_{n\geqslant 1}$ such that $\lim_{n\to\infty} \mathrm{H}(\nu_n) = +\infty$ and the first negative complete quotient of ν_n is encountered after n steps in its p-adic continued fraction expansion while

$$\lim_{n\to\infty} n \left(\left\lfloor \frac{\log \mathrm{H}(\nu_n)}{\log \lambda} \right\rfloor + 3 \right)^{-1} = 1.$$

Proof. The case $\xi < 0$ being trivial, we assume $\xi = x_0/x_1 > 0$ so that x_0 and x_1 are positive relatively prime integers. As before, let $(x_i/x_{i+1})_{i\geqslant 0}$ be the sequence of complete quotients of $\xi = [b_0, p^{a_1} : b_1, p^{a_2} : b_2, \ldots]$, where the continued fraction expansion does not terminate.

Let k be the smallest integer such that $x_{k+1} < 0$. This implies that x_k/x_{k+1} is the first negative complete quotient in the expansion of ξ. We set

$$y_i = \frac{\lambda}{p} x_i + x_{i+1} \quad \text{for } i \geqslant 0.$$

If $x_{i+1} > x_i > 0$, then $x_{i+2} < 0$, so $i = k-1$. If $x_i \geqslant x_{i+1} > 0$, then

$$\begin{aligned} y_{i+1} &= \frac{\lambda}{p} x_{i+1} + x_{i+2} = \frac{\lambda}{p} x_{i+1} + \frac{x_i - b_i x_{i+1}}{p^{a_{i+1}}} \\ &\leqslant \frac{\lambda}{p} x_{i+1} + \frac{x_i - x_{i+1}}{p} = \frac{1}{p} x_i + \frac{-1+\sqrt{1+4p}}{2p} x_{i+1} \\ &= \frac{1}{\lambda}\Big(\frac{\lambda}{p} x_i + x_{i+1}\Big) = \frac{1}{\lambda} y_i. \end{aligned}$$

For $0 \leqslant i < k-1$, we have $x_i \geqslant x_{i+1} > 0$, so that

$$\begin{aligned} y_{k-1} &\leqslant \frac{1}{\lambda} y_{k-2} \leqslant \frac{1}{\lambda^2} y_{k-3} \leqslant \ldots \leqslant \frac{1}{\lambda^{k-1}} y_0, \\ y_{k-1} &\geqslant \frac{\lambda}{p} + 1 > 1, \end{aligned}$$

which combines into

$$\lambda^{k-1} < y_0 \leqslant \Big(\frac{\lambda}{p} + 1\Big) \operatorname{H}(x_0/x_1) \leqslant 2 \operatorname{H}(\xi),$$

where the last inequality becomes an equality for $p = 2$. Hence,

$$k < \frac{\log(2 \operatorname{H}(\xi))}{\log \lambda} + 1 \leqslant \frac{\log \operatorname{H}(\xi)}{\log \lambda} + 2.$$

This proves the first part of the theorem.

Define $P_n/Q_n = [1, (p : 1)_n]$ for $n \geqslant 0$. These are obviously convergents of the infinite continued fraction $[1, (p : 1)_\infty]$, so by (1.1), we have

$$P_n = P_{n-1} + pP_{n-2}, \quad Q_n = Q_{n-1} + pQ_{n-2} \quad \text{for } n \geqslant 2$$

with $P_0 = Q_0 = Q_1 = 1$, $P_1 = p + 1$. These linear recurrence relations with constant coefficients give

$$Q_n = P_{n-1} = \frac{1}{\sqrt{1+4p}} \big(\lambda^{n+1} - (-p/\lambda)^{n+1}\big), \quad n \geqslant 1. \tag{2.5}$$

Set now

$$\nu_n = [1, (p : 1)_n, p^2 : -1] = \frac{-P_n + p^2 P_{n-1}}{-Q_n + p^2 Q_{n-1}}.$$

The complete quotients of ν_n, starting from the last one and moving backwards, are

$$-1 < 0, \quad [1, p^2 : -1] = 1 - p^2 < 0, \quad [1, p : 1, p^2 : -1] = \frac{1 + p - p^2}{1 - p^2} > 0.$$

This implies that in the continued fraction expansion of ν_n, the first negative complete quotient is obtained in the n-th step. Since $\lambda > \sqrt{p} > 1$, which implies $|-p/\lambda| < \lambda$, (2.5) shows that $P_{n-1} = Q_n \sim \lambda^{n+1}/\sqrt{1+4p}$ when $n \to \infty$, so that

$$\operatorname{H}(\nu_n) \sim -\frac{\lambda^{n+2}}{\sqrt{1+4p}} + \frac{p^2 \lambda^{n+1}}{\sqrt{1+4p}} = \frac{p^2 - \lambda}{\sqrt{1+4p}} \lambda^{n+1}.$$

Thus,

$$\left\lfloor \frac{\log \operatorname{H}(\nu_n)}{\log \lambda} \right\rfloor + 3 \sim n,$$

and we conclude that the bound we obtained is indeed asymptotically best possible. □

In view of Theorem 2, the bound in Theorem 1 might not be best possible. A sharper bound could be obtained if a better estimate is found for the length of the preperiodic part of p-adic continued fraction expansion of negative rational numbers.

3 Approximation by rational numbers

As is well known (see e.g. [13, Theorem 8.29]), for the standard continued fractions of real numbers, convergents are the best rational approximations. This means that for the sequence $(P_n/Q_n)_{n\geqslant 0}$ of convergents to $\zeta \in \mathbb{R}$, we have $|Q_0\zeta - P_0| > |Q_1\zeta - P_1| > |Q_2\zeta - P_2| > \dots$ and if $n \geqslant 1$, $1 \leqslant B \leqslant Q_n$ and $(A, B) \neq (P_{n-1}, Q_{n-1}), (P_n, Q_n)$, then $|B\zeta - A| > |Q_{n-1}\zeta - P_{n-1}|$.

For approximation in $\mathbb{Q}_p$, we have to bound both the numerator and the denominator of the rational approximation. For example, let ξ be a p-adic integer. Then $|B\xi - A|_p$ can be as small as we like if we only bound the size of B, e.g. set $B = 1$ and $A \equiv \xi \pmod{p^k}$ for k as large as wanted. However, even with this restriction, the convergents in Schneider's continued fraction expansion of $\xi \in \mathbb{Q}_p$ are not necessarily the best rational approximations of ξ as shown by the next example.

Immediately from the pigeonhole principle, we obtain the following result. For any $\xi \in \mathbb{Z}_p$ and any nonnegative integer k, there are $u_k, v_k \in \mathbb{Z}$ not both zero such that

$$\max\{|u_k|, |v_k|\} \leqslant p^k \quad \text{and} \quad |v_k\xi - u_k|_p \leqslant p^{-2k}, \tag{3.1}$$

which implies

$$\max\{|u_{k+1}|, |v_{k+1}|\} \ll |v_k\xi - u_k|_p^{-1/2}. \tag{3.2}$$

If ξ is not a rational number, then $|v_k\xi - u_k|_p > 0$ for every k, so the sequence $(\max\{|u_k|, |v_k|\})_{k\geqslant 0}$ is unbounded and can be assumed to be non-decreasing.

Set $\xi = [1, (p : 1)_\infty] \in \mathbb{Q}_p \setminus \mathbb{Q}$ for $p > 2$ and let $(u_k)_k$ and $(v_k)_k$ be the associated sequences satisfying (3.1). Using the results from the proof of Theorem 2, we have by (2.5) that the convergents P_n/Q_n of ξ satisfy $Q_n = P_{n-1} \sim \lambda^{n+1}/\sqrt{1+4p}$ when $n \to \infty$, where $\lambda = (1 + \sqrt{1+4p})/2$. From (1.7), we see that

$$|Q_n\xi - P_n|_p = \left|\xi - \frac{P_n}{Q_n}\right|_p = p^{-n-1} \quad \text{for } n \geqslant 0.$$

For some n large enough, let k be an integer such that

$$\max\{|u_k|, |v_k|\} \leqslant \max\{P_n, Q_n\} < \max\{|u_{k+1}|, |v_{k+1}|\}.$$

Then

$$\begin{aligned} |Q_n\xi - P_n|_p = p^{-n-1} &\asymp \max\{P_n, Q_n\}^{-\log p/\log\lambda} \\ &\gg \max\{|u_{k+1}|, |v_{k+1}|\}^{-\log p/\log\lambda} \gg |v_k\xi - u_k|_p^{\log p/(2\log\lambda)}, \end{aligned}$$

where the last inequality follows from (3.2). Since $\log p/(2\log\lambda) < 1$, u_k/v_k is certainly a much better rational approximation of ξ than P_n/Q_n. We can easily modify the preceding argument and apply it to $\xi = [1, (p^a : 1)_\infty]$ for $a \geqslant 2$ which is then an example suitable also for $p = 2$.

Let $\xi \in \mathbb{Z}_p^\times$ be a p-adic unit and A/B a rational number written as a reduced fraction. Slightly changing the previous terminology, we say that the rational number with the reduced fraction u/v is a *better rational approximation* of ξ than A/B if

$$|v\xi - u|_p \leqslant |B\xi - A|_p \quad \text{while} \quad |u| \leqslant |A|,\ |v| \leqslant |B|, \tag{3.3}$$

with at least one of the bounds on $|u|, |v|$ being strict.

There is a simple connection between the finiteness of the continued fraction expansion of u/v and how well u/v approximates some p-adic number.

Theorem 3. *A rational number with the reduced fraction $u/v \in \mathbb{Z}_p$ has an infinite p-adic Schneider's continued fraction expansion if and only if it is a better rational approximation of some $\xi \in \mathbb{Z}_p$ than some convergent of ξ.*

Proof. If $u/v \in \mathbb{Q}$ has a non-terminating continued fraction expansion, then it is certainly a better rational approximation of itself than its convergent with numerator or denominator larger than $\mathrm{H}(u/v)$. Recall that the sequences of numerators and denominators of the convergents of an infinite continued fraction are strictly increasing sequences of positive integers which, therefore, tend to $+\infty$.

On the other hand, suppose that (3.3) holds for a convergent A/B of ξ while u/v has a finite continued fraction expansion. From (3.3) we see that

$$\left|\xi - \frac{u}{v}\right|_p = |v\xi - u|_p \leqslant |B\xi - A|_p = \left|\xi - \frac{A}{B}\right|_p.$$

According to (1.11) and (1.12) compared with (1.7), this implies that the continued fraction expansions of ξ and u/v coincide in an initial segment not shorter than the one in which the expansions of ξ and A/B coincide. Since A/B is a convergent of ξ, this means that A/B is also a convergent of u/v. However, since u/v has a terminating expansion, it is its own last convergent, different from A/B. Therefore, $u > A$ and $v > B$ which contradicts (3.3). □

Suppose now that u/v is a better rational approximation of $\xi = [b_0, p^{a_1} : b_1, \ldots]$ than one of its convergents $(P_n/Q_n)_{n \geqslant 0}$, say P_{k-1}/Q_{k-1}. Again, as in the previous proof, looking at (1.7), (1.11) and (1.12), we see that

$$\frac{u}{v} = \frac{\frac{c}{d}P_{k-1} + p^a P_{k-2}}{\frac{c}{d}Q_{k-1} + p^a Q_{k-2}},$$

where $a \geqslant a_k$ and c/d is a reduced rational number with $|c/d|_p = 1$. Noting that $\gcd(P_{k-1}, p^a P_{k-2}) = \gcd(Q_{k-1}, p^a Q_{k-2}) = \gcd(P_{k-1}, Q_{k-1}) = 1$, we get

$$u = cP_{k-1} + dp^a P_{k-2},$$
$$v = cQ_{k-1} + dp^a Q_{k-2}.$$

The inequalities $|u| \leqslant P_{k-1}$, $|v| \leqslant Q_{k-1}$ are equivalent to

$$-P_{k-1} \leqslant cP_{k-1} + dp^a P_{k-2} \leqslant P_{k-1}, \quad -Q_{k-1} \leqslant cQ_{k-1} + dp^a Q_{k-2} \leqslant Q_{k-1},$$

or

$$c \in \Big[-\frac{dp^a P_{k-2}}{P_{k-1}} - 1, -\frac{dp^a P_{k-2}}{P_{k-1}} + 1\Big] \cap \Big[-\frac{dp^a Q_{k-2}}{Q_{k-1}} - 1, -\frac{dp^a Q_{k-2}}{Q_{k-1}} + 1\Big]. \quad (3.4)$$

The two intervals in (3.4) are both of length 2, so they have a nonempty intersection if and only if the distance of their midpoints is at most 2. If this distance is at most 1, the intersection is of length at least 1, so it has to contain an integer. Unfortunately, this integer can be divisible by p which can not be the case for c. This obstruction can sometimes be circumvented as we will soon show.

We see from (1.4) that

$$\Big|\frac{-dp^a P_{k-2}}{P_{k-1}} - \frac{-dp^a Q_{k-2}}{Q_{k-1}}\Big| \leqslant 2$$

is equivalent to

$$|d| p^{a + \sum_{i=1}^{k-1} a_i} = |d| p^a \cdot |P_{k-1} Q_{k-2} - P_{k-2} Q_{k-1}| \leqslant 2 P_{k-1} Q_{k-1}.$$

Thus a necessary condition for a better rational approximation than P_{k-1}/Q_{k-1} to exist is

$$p^{\sum_{i=1}^{k} a_i} \leqslant 2 P_{k-1} Q_{k-1}.$$

In other words, a better rational approximation (in our terminology) than P_{k-1}/Q_{k-1} does not exist if

$$p^{\sum_{i=1}^{k} a_i} > 2 P_{k-1} Q_{k-1}, \quad (3.5)$$

so the sequence $(a_n)_{n \geqslant 1}$ has to grow pretty rapidly.

One sufficient condition for the existence of such an approximation is e.g.

$$p^{\sum_{i=1}^{k} a_i} \leqslant P_{k-1} Q_{k-1} \quad \text{and} \quad p^{a_k} P_{k-2}/P_{k-1} \in (1, p-1). \quad (3.6)$$

Note that the first condition ensures the existence of some integer c for $d = 1$ and the second condition guarantees that p does not divide c.

As an example, we take again $\xi = [1, (p : 1)_\infty]$ for $p > 2$ and (2.5) then gives

$$Q_n = P_{n-1} \sim \frac{(1 + \sqrt{1 + 4p})^{n+1}}{2^{n+1}\sqrt{1 + 4p}},$$

so that

$$p^{\sum_{i=1}^k a_i} = p^k, \quad P_{k-1}Q_{k-1} \sim \frac{(1 + \sqrt{1 + 4p})^{2k+1}}{2^{2k+1}(1 + 4p)}$$

and (3.6) holds for all k large enough since $(1+\sqrt{1+4p})^2/4 > p$ and $2p/(1+\sqrt{1+4p}) \in (1, p-1)$. Thus $[1, (p : 1)_\infty]$ has better rational approximations than convergents with index large enough.

We could have obtained a slightly more general result for ξ having again $(a_n)_{n\geqslant 1}$ a constant sequence $a_n = 1$, but letting $(b_n)_{n\geqslant 0}$ be an arbitrary sequence with elements in $\{1, 2, \ldots, p-1\}$, by using the inequality

$$P_nQ_n > \prod_{i=1}^{n}(1 + p^{a_i}) \tag{3.7}$$

which holds for any continued fraction and every $n \geqslant 2$ (for $n = 1$ the inequality also holds, but is not necessarily strict). To prove (3.7), we first show by induction using (1.1) that

$$\begin{aligned} P_n &> (1 + p^{a_n})(1 + p^{a_{n-2}})(1 + p^{a_{n-4}}) \cdots \quad \text{for all } n \geqslant 2 \quad \text{and} \\ Q_n &\geqslant (1 + p^{a_n})(1 + p^{a_{n-2}})(1 + p^{a_{n-4}}) \cdots \quad \text{for all even } n \geqslant 2, \end{aligned} \tag{3.8}$$

where both products stop with the term $1+p^{a_1}$ or $1+p^{a_2}$ depending on n being odd or even. From (3.8) and $P_nQ_n \geqslant P_{n-1}Q_n$ for n even and $P_nQ_n \geqslant P_nQ_{n-1}$ for n odd, we obtain (3.7).

Now, in our example where $p > 2$ and $a_n = 1$ for all n, (3.7) immediately implies the first part of (3.6), while the second part holds for all k such that $b_{k-1} \leqslant p/3$ since from (1.1) we have $P_n/P_{n-1} = b_n + p^{a_n}(P_{n-1}/P_{n-2})^{-1}$ which implies

$$\begin{aligned} pP_{k-2}/P_{k-1} &\geqslant p \cdot [p/3, p : 1, p : p-1, p : 1]^{-1} = (9p-3)/(9p-4) > 1, \\ pP_{k-2}/P_{k-1} &\leqslant p \cdot [1, p : p-1, p : 1]^{-1} = p(2p-1)/(3p-1) < p-1. \end{aligned}$$

The previous discussion shows that it is of interest to find good estimates of size for numerators and denominators of convergents given sequences $(a_n)_n$ with different rates of growth.

4 Bounds on height of convergents for some examples

The main purpose of this section is to obtain good estimates of size for the height of convergents in several examples of p-adic continued fractions. These examples have different rates of growth of the sequence $(a_n)_{n\geqslant 1}$ of exponents in the partial numerators of continued fractions. The obtained estimates will give us in Theorem 4 some partial results on the quality of approximation by convergents. The same bounds will be used in Section 5 to determine the irrationality exponent of the constructed continued fractions and in Section 6 to study the convergence of the sequence of convergents $(P_n/Q_n)_{n\geqslant 0}$ in the field of real numbers.

We are interested in the lower and upper bounds of the sequence

$$X_0 = X_1 = 1, \quad X_n = cX_{n-1} + p^{a_n} X_{n-2} \text{ for } n \geqslant 2, \tag{4.1}$$

where $c \in \{1, \ldots, p-1\}$ is fixed. Comparing the initial values, we get $P_n \gg X_n$ and $Q_n \gg X_n$ if $c = 1$ is chosen, while $P_n \ll p^{a_1} X_n$, $Q_n \ll X_n$ if $c = p-1$ is taken. In these bounds, the implied constants depend only on p. For all the examples in the rest of this paper, the choice of sequence $(b_n)_{n\geqslant 0}$ in $\{1, \ldots, p-1\}$ will be irrelevant in the estimates of size and we can disregard the value of c.

Define

$$T_n = \prod_{k=1}^{n} p^{a_k} = p^{\sum_{k=1}^{n} a_k} \quad \text{and} \quad Y_n = \frac{X_n}{\sqrt{T_{n+1}}} \qquad (n \geqslant 0), \tag{4.2}$$

so that, after substitution, (4.1) becomes

$$Y_n = cp^{-a_{n+1}/2} Y_{n-1} + p^{(a_n - a_{n+1})/2} Y_{n-2}. \tag{4.3}$$

Let $g(n) = p^{(a_n - a_{n+1})/2} + cp^{-a_{n+1}/2}$ for $n \geqslant 1$. Then

$$\begin{aligned} Y_n &\geqslant g(n) \min\{Y_{n-1}, Y_{n-2}\} && (n \geqslant 2), \\ Y_{n-1} &\geqslant g(n-1) \min\{Y_{n-2}, Y_{n-3}\} && (n \geqslant 3), \end{aligned}$$

so that

$$\min\{Y_n, Y_{n-1}\} \geqslant \min\{g(n), g(n-1), g(n)g(n-1)\} \min\{Y_{n-2}, Y_{n-3}\}. \tag{4.4}$$

Using $Y_n \geqslant \min\{Y_n, Y_{n-1}\}$ and iterating (4.4), we obtain

$$Y_n \geqslant \min\{Y_0, Y_1, Y_2\} \prod_{\substack{3\leqslant k\leqslant n \\ 2|(n-k)}} \min\{g(k), g(k-1), g(k)g(k-1)\}. \tag{4.5}$$

Completely analogously, we show

$$Y_n \leqslant \max\{Y_0, Y_1, Y_2\} \prod_{\substack{3\leqslant k\leqslant n\\ 2|(n-k)}} \max\{g(k), g(k-1), g(k)g(k-1)\}. \tag{4.6}$$

Now we continue the analysis for particular classes of sequences $(a_n)_{n\geqslant 1}$.

4.1 $a_n = \lfloor \alpha \log_p n \rfloor$

We use log for logarithm with base e and $\log_p$ for logarithm with base p. Let α be a positive real number and define $a_n = \lfloor \alpha \log_p n \rfloor$ for $n \geqslant 1$. Then $p^{a_n} \asymp n^\alpha$ and we want to bound T_n and Y_n in this case.

Note that

$$T_n \geqslant \prod_{k=1}^{n} p^{\alpha \log_p(n)-1} = p^{-n}(n!)^\alpha \quad \text{and} \quad T_n \leqslant (n!)^\alpha,$$

hence, by Stirling's formula,

$$T_n = e^{\alpha(n\log n + \mathcal{O}(n))}. \tag{4.7}$$

We look only at $n > (p^{1/\alpha}-1)^{-1}$ or $(n+1)/n < p^{1/\alpha}$ so that

$$n \leqslant p^{\frac{k}{\alpha}} < p^{\frac{k+1}{\alpha}} \leqslant n+1$$

cannot happen for an integer k.

If $n = \lceil p^{k/\alpha} \rceil - 1$ for some positive integer k, then $a_{n+1} = k = a_n + 1$, so

$$\begin{aligned} g(n) &= p^{-1/2} + cp^{-\lfloor \alpha \log_p(n+1)\rfloor/2},\\ p^{-1/2} + cp^{(-\alpha\log_p(n+1))/2} \leqslant g(n) &\leqslant p^{-1/2} + cp^{(-\alpha\log_p(n+1)+1)/2},\\ p^{-1/2} + c(n+1)^{-\alpha/2} \leqslant g(n) &\leqslant p^{-1/2} + cp^{1/2}(n+1)^{-\alpha/2}. \end{aligned} \tag{4.8}$$

Otherwise, $a_{n+1} = a_n$ and

$$\begin{aligned} g(n) &= 1 + cp^{-\lfloor \alpha\log_p(n+1)\rfloor/2},\\ 1 + c(n+1)^{-\alpha/2} \leqslant g(n) &\leqslant 1 + cp^{1/2}(n+1)^{-\alpha/2}. \end{aligned} \tag{4.9}$$

Now, using (4.8), (4.9) and the estimates

$$
\begin{aligned}
e^{x/2} \leqslant 1 + x \leqslant e^{x} \quad \text{for } 0 \leqslant x \leqslant 1, \qquad (4.10)\\
\sum_{k=1}^{n} k^{-\alpha/2} = \int_1^n x^{-\alpha/2}\mathrm{d}x + \mathcal{O}(1) = \frac{2}{2-\alpha}(n^{1-\alpha/2} + \mathcal{O}(1)) \quad (\alpha \neq 2),\\
\sum_{\substack{1\leqslant k\leqslant n\\ 2|(k-\varepsilon)}} k^{-\alpha/2} = \frac{1}{2-\alpha}(n^{1-\alpha/2} + \mathcal{O}(1)),\ \varepsilon \in \{0,1\},\\
\sum_{\substack{1\leqslant k\leqslant n\\ k=\lceil p^{\ell/\alpha}\rceil,\ \ell\in\mathbb{Z}}} k^{-\alpha/2} < \sum_{\ell=0}^{\infty} p^{-\ell/2} = 1 + \frac{1}{\sqrt{p}-1} < 4,\\
p^{-\frac{1}{2}\operatorname{card}\{k\in\mathbb{Z}\,:\,1\leqslant k\leqslant n,\ k=\lceil p^{\ell/\alpha}\rceil,\ \ell\in\mathbb{Z}\}} = e^{-\frac{1}{2}\alpha(\log n + \mathcal{O}(1))}
\end{aligned}
$$

with (4.5) and (4.6), we obtain

$$
\begin{aligned}
Y_n &\gg_\alpha e^{\frac{1}{2(2-\alpha)}n^{1-\frac{\alpha}{2}} - \frac{1}{2}\alpha\log n},\\
Y_n &\ll_\alpha e^{p^{\frac{3}{2}}\frac{2}{2-\alpha}n^{1-\frac{\alpha}{2}}}.
\end{aligned} \qquad (4.11)
$$

It follows that

$$
\begin{aligned}
\lim_{n\to\infty} Y_n &= +\infty \quad \text{for } \alpha < 2,\\
Y_n &= \mathcal{O}_\alpha(1) \quad \text{for } \alpha > 2.
\end{aligned} \qquad (4.12)
$$

Unfortunately, for $\alpha > 2$ we cannot in general conclude that $\lim_{n\to\infty} Y_n = 0$. The difference between the upper and lower bound in (4.11) comes from the fact that the decrease in the sequence $(Y_n)_n$ happens at $n = \lceil p^{k/\alpha}\rceil - 1$ for positive integers k. Let D be the infinite set of all these indices n where the descent occurs. The sequence of Y_n with odd (even) indices will tend to 0 if and only if there are infinitely many odd (even) numbers in D.

The claim in one direction is obvious. If, for example, there are only finitely many odd numbers in D, then (4.3) shows that for odd n which are large enough, we have $Y_n > Y_{n-2}$ so that $Y_n \gg 1$ for odd n. This happens for example if $p^{1/\alpha}$ is an odd integer.

Suppose now that there are infinitely many odd numbers in D. Let $(n_i)_{i\geqslant 0}$ be the strictly increasing infinite sequence of all odd integers in D starting from some n_0 large enough. From (4.11) we know that there is some upper bound M for $(cp^{1/2}Y_n)_{n\geqslant 0}$, so that by (4.3), for $n \geqslant n_0$ it holds

$$
\begin{aligned}
Y_{n-2} < Y_n &\leqslant \frac{M}{(n+1)^{\alpha/2}} + Y_{n-2} &&\text{if } n \notin D,\\
Y_n &\leqslant \frac{M}{(n+1)^{\alpha/2}} + p^{-1/2}Y_{n-2} &&\text{if } n \in D.
\end{aligned} \qquad (4.13)
$$

For a large enough odd integer n, let i be such that $n_i \leqslant n < n_{i+1}$. Then we see by (4.13) that

$$Y_n \leqslant Y_{n_{i+1}-2} \leqslant M\big((n_{i+1}-1)^{-\alpha/2}+(n_{i+1}-3)^{-\alpha/2}+\cdots+(n_i+1)^{-\alpha/2}\big)+p^{-1/2}Y_{n_i-2}. \tag{4.14}$$

From the monotonicity of the function $n \mapsto n^{-\alpha/2}$, we have

$$\begin{aligned}(n_{i+1}-1)^{-\alpha/2} &+ (n_{i+1}-3)^{-\alpha/2} + \cdots + (n_i+1)^{-\alpha/2} \\ &< \frac{1}{\alpha-2}\big((n_i-1)^{1-\alpha/2} - (n_{i+1}-1)^{1-\alpha/2}\big),\end{aligned}$$

so that iterating (4.14), we get

$$\begin{aligned}Y_{n_{i+1}-2} &< \frac{M}{\alpha-2}\sum_{j=0}^{i}\big((n_j-1)^{1-\alpha/2} - (n_{j+1}-1)^{1-\alpha/2}\big)p^{-(i-j)/2} + p^{-(i+1)/2}Y_{n_0-2} \\ &< \frac{M}{\alpha-2}\left(\sum_{j=1}^{i}\big((n_j-1)^{1-\alpha/2}p^{-(i-j)/2}(1-p^{-1/2})\big) + (n_0-1)^{1-\alpha/2}p^{-i/2}\right) + Mp^{-i/2}.\end{aligned}$$

Using $n_j - 1 > \frac{1}{2}p^{j/\alpha}$ and $1-\alpha/2 < 0$, this gives

$$Y_{n_{i+1}-2} \ll \sum_{j=1}^{i}\big(p^{j\cdot\frac{2-\alpha}{2\alpha}+\frac{j-i}{2}}\big) + p^{-\frac{i}{2}} \ll p^{-i/2}\Big(1+\sum_{j=1}^{i}p^{j/\alpha}\Big) \ll p^{-i/2}\cdot p^{i/\alpha}.$$

Finally,

$$Y_n \leqslant Y_{n_{i+1}-2} \ll_\alpha p^{(\frac{1}{\alpha}-\frac{1}{2})i}, \tag{4.15}$$

so that

$$\lim_{\substack{n\to\infty \\ n \text{ odd}}} Y_n = 0.$$

Note that by choosing $p^{1/\alpha}$ to be an even (odd) integer, we get that $\lceil p^{k/\alpha}\rceil - 1$ is odd (even) for every positive integer k, so that D contains only odd (even) numbers.

On the other hand, since $\lceil p^{k/\alpha}\rceil$ is odd if and only if the fractional part $\{\frac{1}{2}p^{k/\alpha}\}$ is in $(0, \frac{1}{2}]$, we see that this question is closely related to the problem of distribution modulo 1 of powers of a real number. From a result by Koksma (see e.g. [5, Theorem 1.10]), we know that for almost all real numbers $r > 1$ (in the sense of Lebesgue measure), the sequence $(\{\frac{1}{2}r^k\})_{k\geqslant 0}$ is uniformly distributed in the interval $[0, 1)$. This shows that for almost all $\alpha > 2$ the sequence $(\{\frac{1}{2}p^{k/\alpha}\})_{k\geqslant 0}$ is uniformly distributed in $[0, 1)$ and thus the set D contains infinitely many odd and infinitely many even numbers, so that $\lim_{n\to\infty} Y_n = 0$ really holds.

4.2 $a_n = \lfloor n^{1/r} \rfloor$

Let $r > 1$ be an integer and set $a_n = \lfloor n^{1/r} \rfloor$ for $n \geqslant 1$. Note that $(n+1)^{1/r} - n^{1/r} < n^{-1+1/r}/r < 1$, so $0 \leqslant a_{n+1} - a_n \leqslant 1$ holds for all n. Here we have

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \lfloor k^{1/r} \rfloor \in \Big(-n + \sum_{k=1}^{n} k^{1/r}, \sum_{k=1}^{n} k^{1/r} \Big],$$
$$\sum_{k=1}^{n} k^{1/r} = \int_0^n x^{1/r} \mathrm{d}x + \mathcal{O}(n^{1/r}) = \frac{r}{r+1} n^{1+1/r} + \mathcal{O}(n^{1/r})$$

and thus

$$T_n = p^{\frac{r}{r+1} n^{1+1/r} + \mathcal{O}(n)}. \tag{4.16}$$

If $n = k^r - 1$ for some positive integer k, then $a_{n+1} = k = a_n + 1$, so

$$g(n) = p^{-1/2} + cp^{-k/2} = p^{-1/2} + cp^{-(n+1)^{1/r}/2}.$$

Otherwise, $a_{n+1} = a_n$ and

$$g(n) = 1 + cp^{-\lfloor (n+1)^{1/r} \rfloor /2}$$
$$1 + cp^{-(n+1)^{1/r}/2} \leqslant g(n) \leqslant 1 + cp^{1/2} p^{-(n+1)^{1/r}/2}.$$

As before, using (4.10) and

$$\sum_{k=1}^{n} p^{-(k+1)^{1/r}/2} < \int_0^{+\infty} e^{-1/2 \cdot \log p \cdot x^{1/r}} \mathrm{d}x = (2/\log p)^r r!,$$

we obtain from (4.5) and (4.6) that

$$Y_n \gg p^{-n^{1/r}/2}, \qquad Y_n \ll 1, \tag{4.17}$$

where the implied constants depend on p and r.

In order to improve the upper bound on Y_n, we follow the same strategy as before. Let M be some upper bound for $(cp^{1/2} Y_n)_{n \geqslant 0}$ which is finite by (4.17). For n large enough, let k be a positive integer such that

$$(k-2)^r - 1 \leqslant n < k^r - 1 \quad \text{and} \quad n \equiv (k-2)^r - 1 \equiv k - 1 \pmod 2.$$

Then by (4.3), we have

$$\begin{aligned} Y_n \leqslant Y_{k^r-3} &\leqslant M\big(p^{-(k^r-2)^{1/r}/2} + p^{-(k^r-4)^{1/r}/2} + \cdots + p^{-((k-1)^r+1)^{1/r}/2} + \cdots \\ &\qquad + p^{-((k-2)^r+2)^{1/r}/2} + p^{-(k-2)/2}\big) + p^{-1/2} Y_{(k-2)^r-3} \\ &\leqslant M \frac{k^r - (k-2)^r}{2} p^{-(k-2)/2} + p^{-1/2} Y_{(k-2)^r-3} \\ &\leqslant Mrk^{r-1} p^{-(k-2)/2} + p^{-1/2} Y_{(k-2)^r-3} \\ &\leqslant Mrp\big(k^{r-1} p^{-k/2} + (k-2)^{r-1} p^{-(k-2)/2-1/2} + (k-4)^{r-1} p^{-(k-4)/2-2/2} + \cdots \\ &\qquad + \ell^{r-1} p^{-\ell/2-(k-\ell)/4}\big) + p^{-(k-\ell+2)/4} Y_{(\ell-2)^r-3}, \end{aligned}$$

where $\ell \equiv k \pmod 2$ is large enough but fixed and depends only on p, r and the parity of k.

Noting that $x^{r-1}p^{-x/2} \ll p^{-x/4}p^{-x/5}$ and $Y_{(\ell-2)^r-3} \leqslant M$, we obtain

$$Y_n \ll p^{-k/4}(1 + p^{-1/5} + p^{-2/5} + \cdots) \ll p^{-k/4} \ll p^{-n^{1/r}/4}, \tag{4.18}$$

which gives with (4.17)

$$p^{-n^{1/r}/2} \ll Y_n \ll p^{-n^{1/r}/4}. \tag{4.19}$$

4.3 $a_n = n^r$

Let r be a positive integer and set $a_n = n^r$ for $n \geqslant 1$. For this sequence, we have

$$\begin{aligned} \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} k^r &= \frac{1}{r+1} n^{r+1} + \mathcal{O}(n^r), \\ T_n &= p^{\frac{1}{r+1} n^{r+1} + \mathcal{O}(n^r)}. \end{aligned} \tag{4.20}$$

Since

$$g(n) = p^{(n^r-(n+1)^r)/2} + cp^{-(n+1)^r/2},$$

the bounds

$$p^{(n^r-(n+1)^r)/2} < g(n) < p^{(n^r-(n+1)^r)/2+1/8}$$

hold for n large enough. Thus (4.5) implies

$$Y_n > p^{-(n+1)^r/2+\mathcal{O}(1)} = p^{-n^r/2+\mathcal{O}(n^{r-1})}. \tag{4.21}$$

From the other side, since $n \mapsto (n+1)^r - n^r$ is non-decreasing and

$$\sum_{k=1}^{n} (-1)^{n-k} k^r > r\big((n-1)^{r-1} + (n-3)^{r-1} + \cdots\big) = n^r/2 + \mathcal{O}(n^{r-1}),$$

(4.6) implies

$$Y_n < p^{-n^r/4+n/8+\mathcal{O}(n^{r-1})}. \tag{4.22}$$

4.4 $a_n = \lfloor \beta^n \rfloor$

As a final example, we take $a_n = \lfloor \beta^n \rfloor$ for $n \geqslant 1$, where $\beta > 1$ is a real number. Then

$$\begin{aligned} \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \beta^k + \mathcal{O}(n) &= \frac{\beta^{n+1}}{\beta-1} + \mathcal{O}(n), \\ T_n &= p^{\beta^{n+1}/(\beta-1)+\mathcal{O}(n)}. \end{aligned} \tag{4.23}$$

Now the standard procedure we employed in the previous examples using (4.5) and (4.6) would give

$$p^{-\beta^{n+1}/2+\mathcal{O}(n)} < Y_n < p^{-\beta^{n+1}/(2\beta+2)+\mathcal{O}(n)}. \tag{4.24}$$

However, comparing (4.24) and (4.23), we see that the bounds are not tight enough. Therefore, instead of going through the substitution (4.2), we start directly with (4.1) and set

$$Z_n = X_n p^{-\beta^{n+2}/(\beta^2-1)} \qquad (n \geqslant 0).$$

Then (4.1) becomes

$$Z_n = cp^{-\beta^{n+1}/(\beta+1)} Z_{n-1} + p^{-\beta^n+\lfloor\beta^n\rfloor} Z_{n-2}.$$

Since

$$\lim_{n\to\infty} p^{-\beta^{n+1}/(\beta+1)} = 0 \quad \text{and} \quad -\beta^n + \lfloor\beta^n\rfloor \in (-1, 0],$$

we can easily show by induction that

$$p^{-n} \ll Z_n \ll p^{2n},$$

which implies

$$X_n = p^{\beta^{n+2}/(\beta^2-1)+\mathcal{O}(n)}, \quad Y_n = p^{-\beta^{n+2}/(2\beta+2)+\mathcal{O}(n)}. \tag{4.25}$$

We can now describe the quality of approximation by convergents for all our examples.

Theorem 4. *For a positive real number α, let $a_n = \lfloor \alpha \log_p n \rfloor$, $n \geqslant 1$. If $\alpha > 2$, better rational approximations do not exist for all convergents with large enough odd indices or for all convergents with large enough even indices. For almost all $\alpha > 2$, better rational approximations exist for at most finitely many convergents.*

If $a_n = \lfloor n^{1/r} \rfloor$ $(n \geqslant 1)$ or $a_n = n^r$ $(n \geqslant 1)$ or $a_n = \lfloor \beta^n \rfloor$ $(n \geqslant 1)$, where r is any positive integer and $\beta > 1$ any real number, then for all but finitely many convergents there are no better rational approximations.

Proof. Taking into account (4.1), (4.2) and the discussion there, we see that

$$P_n Q_n / p^{\sum_{i=1}^{n+1} a_i}$$

satisfies the same lower and upper bounds as those obtained for Y_n^2. Now the conclusion follows from (3.5) and the bounds on $(Y_n)_n$ from this section. □

In the previous theorem, we claim no complementary result stating that better rational approximations than some convergents do exist for certain $\alpha > 2$. For this to be proven, not only the first inequality in (3.6) has to hold, but also the second condition in (3.6), or a similar one, must be satisfied. Unlike the simple examples given at the end of Section 3, the way to verify such conditions for continued fractions given in Theorem 4 is not clear since our bounds on P_n are not tight enough.

5 Irrationality exponent

The irrationality exponent $\mu(\xi)$ of an irrational p-adic number ξ is the supremum of the real numbers μ such that

$$\left|\xi - \frac{a}{b}\right|_p < \mathrm{H}(a/b)^{-\mu} \tag{5.1}$$

has infinitely many solutions in rational numbers a/b.

It is easily seen that (5.1) can be replaced by

$$|b\xi - a|_p < \mathrm{H}(a/b)^{-\mu}. \tag{5.2}$$

The lower bound $\mu(\xi) \geqslant 2$ always holds, see for example Section 9.3 of [3]. In order to determine the irrationality exponent of numbers introduced in the previous section, we use the following immediate consequence of Lemma 2 from [8].

Lemma 5. *For $\xi \in \mathbb{Q}_p$, let $(\vartheta_k)_{k\geqslant 0}$ be a sequence of real numbers such that $\liminf_{k\to\infty} \vartheta_k > 1$ and let $(P_k/Q_k)_{k\geqslant 0}$ be a sequence of distinct rational numbers such that*

$$\left|\xi - \frac{P_k}{Q_k}\right|_p = \mathrm{H}(P_k/Q_k)^{-\vartheta_k}$$

holds for $k \geqslant 0$. If

$$\limsup_{k\to\infty} \vartheta_k \geqslant 1 + \limsup_{k\to\infty} \frac{\log \mathrm{H}(P_{k+1}/Q_{k+1})}{(\vartheta_k - 1)\log \mathrm{H}(P_k/Q_k)}, \tag{5.3}$$

then $\mu(\xi) = \limsup_{k\to\infty} \vartheta_k$.

Note that $\mu(\xi) \geqslant \limsup_{k\to\infty} \vartheta_k$ follows trivially from the existence of the sequence $(P_k/Q_k)_{k\geqslant 0}$ of good rational approximations to ξ. The condition (5.3) is needed to prove the inequality in the other direction.

Using the convergents of the p-adic continued fraction as a sequence of rational approximations $(P_k/Q_k)_{k\geqslant 0}$ from the previous Lemma, we obtain the following result.

Theorem 6. *Let $\alpha > 0$ and $\beta > 1$ be real numbers and r a positive integer.*

If $a_n = \lfloor \alpha \log_p n \rfloor$ $(n \geqslant 1)$ or $a_n = \lfloor n^{1/r} \rfloor$ $(n \geqslant 1)$ or $a_n = n^r$ $(n \geqslant 1)$, the irrationality exponent of ξ defined in (1.8) *is* 2.

If $a_n = \lfloor \beta^n \rfloor$ $(n \geqslant 1)$, we have $\mu(\xi) = \beta + 1$.

Proof. Keeping notation as in Lemma 5 and taking $(P_k/Q_k)_{k\geqslant 0}$ to be the sequence of convergents to ξ in (1.8), we get

$$\begin{aligned} \log \mathrm{H}(P_k/Q_k) &\sim \log X_k \sim \frac{1}{2}\log T_{k+1} + \log Y_k, \\ \vartheta_k = \frac{-\log|\xi - P_k/Q_k|_p}{\log \mathrm{H}(P_k/Q_k)} &\sim \frac{\log T_{k+1}}{\frac{1}{2}\log T_{k+1} + \log Y_k}, \end{aligned}$$

where we used (1.7) and (4.2).

If $a_n = \lfloor \alpha \log_p n \rfloor$, then from (4.7) and (4.11), we have

$$\begin{aligned} \log T_{k+1} &= \alpha k \log k + \mathcal{O}(k), \\ \log Y_k &\geqslant \frac{1}{2(2-\alpha)} k^{1-\frac{\alpha}{2}} - \frac{1}{2}\alpha \log k + \mathcal{O}(1), \\ \log Y_k &\leqslant p^{\frac{3}{2}} \frac{2}{2-\alpha} k^{1-\frac{\alpha}{2}} + \mathcal{O}(1), \end{aligned}$$

so that

$$\lim_{k\to\infty} \vartheta_k = 2, \qquad \lim_{k\to\infty} \frac{\log \mathrm{H}(P_{k+1}/Q_{k+1})}{(\vartheta_k - 1)\log \mathrm{H}(P_k/Q_k)} = 1 \tag{5.4}$$

and Lemma 5 implies $\mu(\xi) = 2$.

For sequences $a_n = \lfloor n^{1/r} \rfloor$ $(n \geqslant 1)$ and $a_n = n^r$ $(n \geqslant 1)$, we show that (5.4) holds using (4.16), (4.17), (4.18), and (4.20), (4.21), (4.22). Thus, we have $\mu(\xi) = 2$ in both cases.

If $a_n = \lfloor \beta^n \rfloor$ $(n \geqslant 1)$, estimates in (4.23) and (4.25) give

$$\begin{aligned} \log T_{k+1} &= (\log p)\beta^{k+2}/(\beta - 1) + \mathcal{O}(k), \\ \log Y_k &= -(\log p)\beta^{k+2}/(2\beta + 2) + \mathcal{O}(k), \end{aligned}$$

so that

$$\begin{aligned} \lim_{k\to\infty} \vartheta_k &= \lim_{k\to\infty} \frac{\beta^{k+2}/(\beta-1)}{\beta^{k+2}/(2\beta-2) - \beta^{k+2}/(2\beta+2)} = \beta + 1, \\ \lim_{k\to\infty} \frac{\log \mathrm{H}(P_{k+1}/Q_{k+1})}{(\vartheta_k - 1)\log \mathrm{H}(P_k/Q_k)} &= \lim_{k\to\infty} \frac{\beta^{k+3}/(2\beta-2) - \beta^{k+3}/(2\beta+2)}{\beta(\beta^{k+2}/(2\beta-2) - \beta^{k+2}/(2\beta+2))} = 1. \end{aligned}$$

Since $\beta + 1 > 1 + 1$, we conclude from Lemma 5 that $\mu(\xi) = \beta + 1$. □

6 Convergence in the reals

Finally, we briefly examine the question of convergence of Schneider's p-adic continued fractions in the field of real numbers. It follows from (1.5) that the sequence of convergents $(P_n/Q_n)_n$ of a p-adic number ξ has a limit in $\mathbb{R}$ if and only if the series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{Q_{k-1}Q_k} p^{\sum_{i=1}^{k} a_i} = \sum_{k=1}^{\infty} (-1)^{k+1} (Q_{k-1}Q_k)^{-1} T_k \tag{6.1}$$

converges in $\mathbb{R}$.

Multiplying the second recurrence equation in (1.1) with Q_{n-1}/T_n, where T_n is as in (4.2), we obtain

$$Q_{n-1}Q_nT_n^{-1} = b_nQ_{n-1}^2T_n^{-1} + Q_{n-2}Q_{n-1}T_{n-1}^{-1}. \tag{6.2}$$

From (6.2), we see that

$$|Q_{n-1}Q_nT_n^{-1}| > |Q_{n-2}Q_{n-1}T_{n-1}^{-1}|, \text{ i.e.}$$

$$|(-1)^{n+1}(Q_{n-1}Q_n)^{-1}T_n| < |(-1)^n(Q_{n-2}Q_{n-1})^{-1}T_{n-1}|. \tag{6.3}$$

If (6.1) converges in $\mathbb{R}$, then $\lim_{k\to\infty}(Q_{k-1}Q_k)^{-1}T_k = 0$. In the other direction, if $\lim_{k\to\infty}(Q_{k-1}Q_k)^{-1}T_k = 0$, then (6.3) and the alternating series test imply that (6.1) converges in $\mathbb{R}$. This condition is equivalent to $\lim_{k\to\infty} Q_{k-1}Q_kT_k^{-1} = +\infty$. However, iterating (6.2), we obtain

$$Q_{n-1}Q_nT_n^{-1} = \sum_{k=2}^{n} b_kQ_{k-1}^2T_k^{-1} + Q_0Q_1T_1^{-1}$$

which shows that the condition for convergence of (6.1) can be given as $\sum_{k=1}^{\infty} b_kQ_{k-1}^2T_k^{-1} = +\infty$.

As explained at the beginning of Section 4, $Q_nT_{n+1}^{-1/2} \gg Y_n$ (for $c = 1$) and $Q_nT_{n+1}^{-1/2} \ll Y_n$ (for $c = p - 1$). Therefore, if in an example $\sum_{k=1}^{\infty} Y_k^2$ converges or diverges regardless of $c \in \{1, \ldots, p-1\}$, then (6.1) converges in the reals if and only if

$$\sum_{k=1}^{\infty} Y_k^2 = +\infty. \tag{6.4}$$

For $a_n = \lfloor \alpha \log_p n \rfloor$ with $\alpha < 2$, the lower bound in (4.11) implies that the respective continued fraction converges in the reals.

For sequences $a_n = \lfloor n^{1/r} \rfloor$ and $a_n = n^r$, where r is a positive integer, the upper bounds in (4.19) and (4.22) show that (6.4) does not hold and the continued fractions they determine do not converge in $\mathbb{R}$. The same conclusion is obtained if $a_n = \lfloor \beta^n \rfloor$ for a real number $\beta > 1$, as follows from (4.25). These results partially answer a question posed in Conjecture 2.1 of [2].

If $a_n = \lfloor \alpha \log_p n \rfloor$ and $\alpha > 2$, the situation is more complicated. As discussed before, the rate of decrease of the associated sequence $(Y_n)_n$ depends on the distribution of $(\{p^{k/\alpha}/2\})_k$ in $[0, 1)$. As shown in Section 4.1, for some $\alpha > 2$, the subsequences of $(Y_n)_n$ obtained by choosing only even n or only odd n are non-decreasing and thus (6.4) certainly holds in those cases.

However, for almost all $\alpha > 2$, the sequence $(\{p^{k/\alpha}/2\})_{k \geqslant 0}$ is uniformly distributed in $[0, 1)$. Thus, for any $\varepsilon \in (0, 1/2)$, for all K large enough, there are between $(\frac{1}{2} - \varepsilon)K$ and $(\frac{1}{2} + \varepsilon)K$ integers k in $(0, K)$ satisfying $\{p^{k/\alpha}/2\} \in (0, \frac{1}{2}]$, or equivalently, $\lceil p^{k/\alpha} \rceil - 1$ is even. This shows that, starting with a large enough number, the sequence $(n_i)_i$ of even integers in D (see the discussion between (4.12) and (4.13)) satisfies

$$p^{\frac{2i}{(1+2\varepsilon)\alpha}} < n_i < p^{\frac{2i}{(1-2\varepsilon)\alpha}} \tag{6.5}$$

or, equivalently,

$$\Big(\frac{1}{2} - \varepsilon\Big)\alpha \log_p n_i < i < \Big(\frac{1}{2} + \varepsilon\Big)\alpha \log_p n_i$$

for i large enough.

For any large enough even integer n, we have $n_i \leqslant n < n_{i+1}$ for some i and now (4.15) and (6.5) give

$$\begin{aligned} Y_n &\ll p^{(\frac{1}{\alpha} - \frac{1}{2})i} \ll p^{(\frac{1}{\alpha} - \frac{1}{2})(i+1)} \ll p^{(\frac{1}{\alpha} - \frac{1}{2})(\frac{1}{2} - \varepsilon)\alpha \log_p n_{i+1}} \\ &\ll n_{i+1}^{(1-\frac{\alpha}{2})(\frac{1}{2}-\varepsilon)} \ll n^{(1-\frac{\alpha}{2})(\frac{1}{2}-\varepsilon)}. \end{aligned} \tag{6.6}$$

For $\alpha > 4$, choose $\varepsilon > 0$ such that

$$\varepsilon < \frac{\alpha - 4}{2\alpha}.$$

Then (6.6) implies

$$Y_n \ll n^{(1-\frac{\alpha}{2})(\frac{1}{2}-\varepsilon)} \ll n^{-\frac{1}{2}-\varepsilon}.$$

Analogously, we prove that the same bound holds for all odd numbers n which are large enough. This shows that (6.4) does not hold for such α.

We summarize all these results in one theorem.

Theorem 7. *First, let* $a_n = \lfloor \alpha \log_p n \rfloor$ *for some positive real number* α *and all positive integers* n*. If* $\alpha < 2$*, the continued fraction* (1.8) *converges in the field of real numbers. For* $p > 2$*, there exist* $\alpha > 2$ *such that* (1.8) *converges in* $\mathbb{R}$*. For almost all real numbers* $\alpha > 4$*,* (1.8) *does not converge in* $\mathbb{R}$*.*

If $a_n = \lfloor n^{1/r} \rfloor$ $(n \geqslant 1)$ *or* $a_n = n^r$ $(n \geqslant 1)$ *or* $a_n = \lfloor \beta^n \rfloor$ $(n \geqslant 1)$*, where* r *is any positive integer and* $\beta > 1$ *real number, then* (1.8) *does not converge in* $\mathbb{R}$*.*

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