An inequality for values of Koksma's functions of two algebraically dependent p-adic numbers

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Abstract

We establish a p-adic version of the inequalities linking values of Koksma's functions of two algebraically dependent transcendental numbers and show that in a very special, but nontrivial case these inequalities become equalities.

1 Introduction

We will be using the most natural measure for the size of a polynomial or an algebraic number. The notation $\mathrm{H}(P)$ stands for naive height of polynomial P, i.e. the maximum of the absolute values of its coefficients. The height $\mathrm{H}(\alpha)$ of a number α algebraic over $\mathbb{Q}$ is that of its minimal polynomial over $\mathbb{Z}$.

Let p be a rational prime number. We denote by $\mathbb{Q}_p$ the completion of the field of rational numbers $\mathbb{Q}$ with respect to p-adic absolute value $|\cdot|_p$ which is normalised in such a way that $|p|_p = p^{-1}$. By $\mathbb{Z}_p$ we denote the ring of p-adic integers. Basic facts about p-adic theory will be tacitly used, interested reader can consult e.g. [5, 6].

Mahler [9] introduced in 1932 a classification of complex transcendental numbers according to how small the value of an integer polynomial at the given number can be with regards to the the height and degree of this polynomial. In 1939 Koksma [7] devised another classification which looks

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at how closely the complex transcendental number can be approximated by algebraic numbers of bounded height and degree. Koksma proved that the two classifications are identical. See [2] for all references.

In analogy with his classification of complex numbers, Mahler proposed a classification of p-adic numbers. Let $\xi \in \mathbb{Q}_p$ and given $n \geq 1$, $H \geq 1$, define the quantity

$$w_n(\xi, H) := \min\{|P(\xi)|_p \,:\, P(X) \in \mathbb{Z}[X],\ \deg(P) \leq n,\ \mathrm{H}(P) \leq H,\ P(\xi) \neq 0\}.$$

We set

$$w_n(\xi) := \limsup_{H\to\infty} \frac{-\log(H w_n(\xi, H))}{\log H},$$

and thus $w_n(\xi)$ is the upper limit of the real numbers w for which there exist infinitely many integer polynomials $P(X)$ of degree at most n satisfying

$$0 < |P(\xi)|_p \leq \mathrm{H}(P)^{-w-1}.$$

In analogy with Koksma's classification of complex numbers, for $\xi \in \mathbb{Q}_p$ and given $n \geq 1$, $H \geq 1$, we define the quantity

$$w_n^*(\xi, H) := \min\{|\xi - \alpha|_p \,:\, \alpha \text{ algebraic in } \mathbb{Q}_p,\ \deg(\alpha) \leq n,\ \mathrm{H}(\alpha) \leq H,\ \alpha \neq \xi\}.$$

We set

$$w_n^*(\xi) := \limsup_{H\to\infty} \frac{-\log(H w_n^*(\xi, H))}{\log H},$$

and thus $w_n^*(\xi)$ is the upper limit of the real numbers w for which there exist infinitely many algebraic numbers α in $\mathbb{Q}_p$ of degree at most n satisfying

$$0 < |\xi - \alpha|_p \leq \mathrm{H}(\alpha)^{-w-1}.$$

Mahler proved in [9] that his classification of real numbers has the property that every two algebraically dependent numbers belong to the same class. In order to prove this basic property he showed that if ξ and η are transcendental real numbers such that $P(\xi, \eta) = 0$ for an irreducible polynomial $P(x, y) \in \mathbb{Z}[x, y]$ of degree M in x and N in y, then the inequalities

$$w_n(\xi) + 1 \leq M(w_{nN}(\eta) + 1) \quad \text{and} \quad w_n(\eta) + 1 \leq N(w_{nM}(\xi) + 1) \tag{1}$$

are valid for every positive integer n. Schmidt [12, (4), p. 276] showed that these conditions also imply inequalities

$$w_n^*(\xi) + 1 \leq M(w_{nN}^*(\eta) + 1) \quad \text{and} \quad w_n^*(\eta) + 1 \leq N(w_{nM}^*(\xi) + 1), \tag{2}$$

i.e. the analogous inequalities we get when Mahler's function w_k is replaced with Koksma's function w_k^*. Note that the definitions of functions w_k and w_k^* on real numbers are very similar to the ones given above for the p-adic numbers. Full explanations including a slight difference between the settings can be found in [2].

Mahler himself [10] proved the inequalities (1) under analogous conditions in the p-adic setting. We will establish in this paper a p-adic version of (2). Let us mention that our proof is in different fashion from what Mahler did in [10] and is more in vein with [12].

Theorem 1. *Let $\xi, \eta \in \mathbb{Q}_p$ be two transcendental numbers which are algebraically dependent. Suppose $P(x, y) \in \mathbb{Z}[x, y]$ is a non-zero polynomial irreducible over $\mathbb{Q}$, of degree M in x and degree N in y such that $P(\xi, \eta) = 0$. Then for every positive integer n, it holds*

$$w_n^*(\xi) + 1 \leq M(w_{nN}^*(\eta) + 1) \quad \textit{and} \quad w_n^*(\eta) + 1 \leq N(w_{nM}^*(\xi) + 1).$$

We will exhibit a case when one of these inequalities becomes an equality in Proposition 2.

2 Proof of the main result

The following lemma and its proof holds whether we take the algebraic number to be in $\mathbb{C}$ or in $\mathbb{C}_p$.

Lemma 1. *Let α be a non-zero algebraic number of degree n. Let a, b and c be integers with $c \neq 0$. We then have*

$$\mathrm{H}\left(\frac{a\alpha + b}{c}\right) \leq 2^{n+1}\,\mathrm{H}(\alpha) \max\{|a|, |b|, |c|\}^n.$$

Proof. See [11, Lemma 2]. ∎

Our next lemma deals with the standard representation of symmetric polynomials through elementary symmetric polynomials but with an important observation that will later be required.

Lemma 2. *Let $P(t_1, \ldots, t_k) \in \mathbb{Z}[t_1, \ldots, t_k]$ be a homogeneous symmetric polynomial. Denote*

$$\deg_{t_1} P = \cdots = \deg_{t_k} P = d.$$

There exists a unique polynomial $Q(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$ such that

$$P(t_1, \ldots, t_k) = Q(s_1, \ldots, s_k),$$

where $s_i = s_i(t_1, \ldots, t_k)$ $(1 \leq i \leq k)$ are elementary symmetric polynomials of $t_1, \ldots, t_k$. For every monomial $s_1^{i_1} \cdots s_k^{i_k}$ in $Q(s_1, \ldots, s_k)$, we have $i_1 + \cdots + i_k \leq d$.

Proof. See [8, Theorem 6.1, §IV.6, p. 191], [4, Theorem 3.3.1, p. 25], [3, Exercise 13, §7.1, p. 326]. ■

Now we prove the announced result.

Proof of Theorem 1. Of course, it is enough to prove only the first inequality since the second follows by interchanging ξ and η.

Fix a positive integer n. It is not hard to see from Lemma 1 that for any integer $l \neq 0$ we have $w_n^*(\xi) = w_n^*(l\xi)$ and $w_n^*(\eta) = w_n^*(l\eta)$. Hence, by taking l to be a large power of p and multiplying the polynomial $P(x, y)$ by the appropriate power of p, we see that without loss of generality we can suppose $\xi, \eta \in \mathbb{Z}_p$.

The partial derivatives of $P(x, y)$ do not vanish at (ξ, η).

Suppose to the contrary that $\frac{\partial}{\partial y} P(x, y)$ vanishes at (ξ, η). By considering $P(\xi, y)$ as a polynomial in y with coefficients in $\mathbb{Q}(\xi)$, one sees that $P(\xi, y) = P_1^*(\xi, y) P_2^*(\xi, y)$, where $P_1^*(\xi, y)$, $P_2^*(\xi, y)$ are polynomials of positive degree in y, with coefficients in $\mathbb{Q}(\xi)$. Since ξ is transcendental, $\mathbb{Q}(\xi)$ is isomorphic to $\mathbb{Q}(x)$, so in fact we have $P(x, y) = P_1^*(x, y) P_2^*(x, y)$, where $P_1^*(x, y), P_2^*(x, y) \in \mathbb{Q}(x)[y]$. But $\mathbb{Q}(x)$ is the fraction field of $Q[x]$, so by Gauss's Lemma [8, Theorem 2.1, Corollary 2.2, §IV.2, p. 181] we can find polynomials $P_1(x, y), P_2(x, y) \in \mathbb{Q}[x, y]$ of positive degree in y and with $P(x, y) = P_1(x, y) P_2(x, y)$. This contradicts the irreducibility of $P(x, y)$.

Let $H > 1$ and suppose $\beta \in \mathbb{Q}_p$ is an algebraic number with $\deg(\beta) \leq n$, $\mathrm{H}(\beta) \leq H$ such that $w_n^*(\xi, H) = |\xi - \beta|_p$. Obviously, if H is large enough, $w_n^*(\xi, H)$ becomes as small as we want, and since $\xi \in \mathbb{Z}_p$, we can assume $\beta \in \mathbb{Z}_p$ as well. Since $P(x, y)$ and $\frac{\partial}{\partial y} P(x, y)$ are polynomials, there exist ε, c_1, c_2 all positive real numbers depending only on $P(x, y)$ (in other words, only on ξ and η) such that for any $u \in \mathbb{Q}_p$

$$|u - \xi|_p < \varepsilon \ \Rightarrow \begin{cases} |P(u, \eta)|_p = |P(u, \eta) - P(\xi, \eta)|_p < c_1 |u - \xi|_p, \\ \left| \dfrac{\partial}{\partial y} P(u, \eta) - \dfrac{\partial}{\partial y} P(\xi, \eta) \right|_p < \dfrac{1}{2} \left| \dfrac{\partial}{\partial y} P(\xi, \eta) \right|_p = c_2 > 0. \end{cases}$$

If we take H large enough, we get

$$|\xi - \beta|_p < \min \left\{ \varepsilon, \frac{c_2^2}{2c_1} \right\},$$

which implies

$$|P(\beta,\eta)|_p < c_1|\beta-\xi|_p < \frac{c_2^2}{2} \qquad \text{and} \qquad \left|\frac{\partial}{\partial y}P(\beta,\eta)\right|_p > c_2.$$

Therefore,

$$\left|\frac{P(\beta,\eta)}{\left(\frac{\partial}{\partial y}P(\beta,\eta)\right)^2}\right|_p < \frac{1}{2}$$

and if we look at $P(\beta,y)$ as a polynomial in y, we see that the conditions of general form of Hensel's Lemma (see [8, Proposition 7.6, §XII.7, p. 493]) are fulfilled. This lemma implies there is a $\beta' \in \mathbb{Z}_p$ such that $P(\beta,\beta')=0$ and

$$|\beta'-\eta|_p \le \left|\frac{P(\beta,\eta)}{\left(\frac{\partial}{\partial y}P(\beta,\eta)\right)^2}\right|_p < \frac{c_1}{c_2^2}|\beta-\xi|_p \ll |\beta-\xi|_p = w_n^*(\xi,H),$$

where the implied constants in $\ll$ and $\gg$ everywhere they appear in this proof depend at most on ξ, η and n.

Let $Q(x)=a_k(x-\beta_1)\cdots(x-\beta_k)$ $(k\le n)$ be the minimal polynomial of $\beta=\beta_1$ over $\mathbb{Z}$. The number β' is a root of the polynomial $P(\beta,y)$ in y, hence a root of the polynomial

$$R(y)=a_k^M P(\beta_1,y)P(\beta_2,y)\cdots P(\beta_k,y).$$

The polynomial $P(\beta,y)$ is not identically zero, since $P(x,y)$ would otherwise be divisible by the minimal polynomial of β. Thus $R(y)$ is not identically zero. The coefficients of $R(y)$ are linear combinations with rational integer coefficients of terms of the type

$$a_k^M \sum_\sigma \beta_{\sigma(1)}^{i_1}\cdots\beta_{\sigma(k)}^{i_k},$$

where the sum is taken over all permutations σ of $\{1,\dots,k\}$ while $0\le i_j\le M$ for $1\le j\le k$. But, because of Lemma 2 and Vietè's formulas for the polynomial $Q(x)$, such terms are rational integers themselves and $\ll \mathrm{H}(Q)^M \le H^M$. Therefore, $R(y)\in\mathbb{Z}[y]$ and $\mathrm{H}(R)\ll H^M$. Since $R(\beta')=0$, we see that β' is algebraic and its minimal polynomial over $\mathbb{Z}$ is a factor of $R(y)$. Using Gauss's Lemma [8, Theorem 2.1, Corollary 2.2, §IV.2, p. 181] and Gelfond's Lemma (see e.g. [2, Lemma A.3, p. 221] or [1, Lemma 1.6.11, p. 27]), we get that this minimal polynomial also has coefficients $\ll H^M$.

Hence $\mathrm{H}(\beta') \ll H^M$, say $\mathrm{H}(\beta') \leq cH^M$. Thus

$$\begin{aligned} w_{nN}^*(\eta, cH^M) \leq w_{kN}^*(\eta, cH^M) \leq |\eta - \beta'|_p \ll w_n^*(\xi, H) \Rightarrow \\ w_{nN}^*(\xi) + 1 = \limsup_{H\to\infty} \frac{-\log(w_{nN}^*(\xi, cH^M))}{\log(cH^M)} \\ \geq \frac{1}{M} \limsup_{H\to\infty} \frac{-\log(w_n^*(\xi, H))}{\log H} = \frac{1}{M}(w_n^*(\xi) + 1). \end{aligned}$$

■

3 An example

If $w > \frac{1+\sqrt{5}}{2}$, we can give a simple construction of $\xi \in \mathbb{Q}_p$ such that $w_1^*(\xi) = w$.

Proposition 1. *Let $w > \frac{1+\sqrt{5}}{2}$ and*

$$\xi = \sum_{i=1}^{\infty} a_i p^{\lfloor (w+1)^i \rfloor} \in \mathbb{Q}_p,$$

where $a_i \in \{1, \ldots, p-1\}$ for all $i \geq 1$. Then $w_1(\xi) = w_1^*(\xi) = w$.

Proof. Take any $w > \frac{1+\sqrt{5}}{2}$ and let $\xi_k = \sum_{i=1}^k a_i p^{\lfloor (w+1)^i \rfloor}$. All the implicit constants in $\ll$ and $\asymp$ in this proof depend at most on p. Since

$$|\xi - \xi_k|_p = p^{-\lfloor (w+1)^{k+1} \rfloor} \asymp \xi_k^{-w-1}$$

for any $k \geq 1$, we have $w_1^*(\xi) \geq w$.

For a reduced fraction $a/b \in \mathbb{Q}$ whose height is large enough, let $l \geq 1$ be such that

$$\xi_l \leq \mathrm{H}(a/b) < \xi_{l+1}.$$

Suppose that

$$\left|\xi - \frac{a}{b}\right|_p = \mathrm{H}\left(\frac{a}{b}\right)^{-\nu},$$

where $\nu > w + 1$. We have

$$\begin{aligned} |\xi - \xi_l|_p = |\xi_{l+1} - \xi_l|_p = p^{-\lfloor (w+1)^{l+1} \rfloor} \asymp \xi_l^{-w-1}, \\ |\xi - \xi_{l+1}|_p = p^{-\lfloor (w+1)^{l+2} \rfloor} \asymp \xi_{l+1}^{-w-1} \asymp \xi_l^{-(w+1)^2}. \end{aligned}$$

Therefore,

$$\frac{1}{|b|\xi_l + |a|} \le \left|\frac{b\xi_l - a}{b}\right|_p = \left|\xi_l - \frac{a}{b}\right|_p \le \max\{|\xi - \xi_l|_p, \left|\xi - \frac{a}{b}\right|_p\} \tag{3}$$

$$\ll \max\{\xi_l^{-w-1}, \mathrm{H}(a/b)^{-\nu}\} = \xi_l^{-w-1} \quad \text{and}$$

$$\frac{1}{|b|\xi_{l+1} + |a|} \le \left|\xi_{l+1} - \frac{a}{b}\right|_p \le \max\{|\xi - \xi_{l+1}|_p, \left|\xi - \frac{a}{b}\right|_p\} \tag{4}$$

$$\ll \max\{\xi_{l+1}^{-w-1}, \mathrm{H}(a/b)^{-\nu}\}.$$

From (3) we get

$$\xi_l^{w+1} \ll |b|\xi_l + |a| \ll \mathrm{H}(a/b)(\xi_l + 1) \ll \mathrm{H}(a/b)\xi_l, \qquad \text{i.e.}$$

$$\xi_l^w \ll \mathrm{H}(a/b). \tag{5}$$

From (4) we get

$$\min\{\xi_{l+1}^{w+1}, \mathrm{H}(a/b)^{\nu}\} \ll |b|\xi_{l+1} + |a| \ll \mathrm{H}(a/b)\xi_{l+1}.$$

Since $\xi_{l+1}^w \ll \mathrm{H}(a/b)$ does not hold, we must have

$$\mathrm{H}(a/b)^{\nu} \ll \mathrm{H}(a/b)\xi_{l+1}, \qquad \text{i.e.}$$

$$\mathrm{H}(a/b) \ll \xi_{l+1}^{\frac{1}{\nu-1}} \ll \xi_{l+1}^{\frac{1}{w}} \ll \xi_l^{\frac{w+1}{w}}. \tag{6}$$

If there is an infinite sequence $\frac{a_k}{b_k} \in \mathbb{Q}$ such that

$$\limsup_{k\to\infty} \frac{-\log|\xi - \frac{a_k}{b_k}|_p}{\log \mathrm{H}(a_k/b_k)} > w,$$

then $\mathrm{H}(a_k/b_k) \to \infty$ when $k \to \infty$ and we conclude from (5) and (6) that $w \le \frac{w+1}{w}$ which implies $w \le \frac{1+\sqrt{5}}{2}$, contrary to our choice of w.

Hence, it must hold that $w_1^*(\xi) = w$. ■

We are able to show that the inequalities in (2) are sharp at least in a very special situation.

Proposition 2. *Let $k \ge 1$ be an integer, w be a real number such that*

$$w > -1 + k + \frac{k^2 + k\sqrt{k^2+4k}}{2}$$

and

$$\xi = a_0 + \sum_{i=1}^{\infty} a_i p^{\lfloor (w+1)^i \rfloor} \in \mathbb{Q}_p,$$

where $a_i \in \{1, \ldots, p-1\}$ for all $i \ge 0$. Then $w_1^*(\xi) = w$ and $w_1^*(\xi^k) = \frac{w+1}{k} - 1$.

Proof. It has been shown in Proposition 1 that $w_1^*(\xi) = w$. We claim that $w_1^*(\xi^k) = \frac{w+1}{k} - 1$ and, thus,

$$w_1^*(\xi) + 1 = k\big(w_1^*(\xi^k) + 1\big),$$

so that inequality (2.i) becomes an equality for this special choice of $\eta = \xi^k$ and $n = 1$.

Using similar notation as in Proposition 1, we have

$$\begin{aligned} |\xi^k - \xi_l^k|_p &= |(\xi_l + \rho_l p^{\lfloor (w+1)^{l+1} \rfloor})^k - \xi_l^k|_p \\ &\asymp p^{-\lfloor (w+1)^{l+1} \rfloor} \\ &\asymp \xi_l^{-(w+1)} \asymp (\xi_l^k)^{-\frac{w+1}{k}}, \end{aligned}$$

with ρ_l being some element in $\mathbb{Z}_p \setminus p\mathbb{Z}_p$ and constants in $\asymp$ depending only on p and k. Hence,

$$w_1^*(\xi^k) \geq \frac{w+1}{k} - 1.$$

In order to show that the last inequality is actually an equality, we proceed just like in the proof of Proposition 1. Let $a/b \in \mathbb{Q}$ be a reduced fraction such that

$$\xi_l^k \leq \mathrm{H}(a/b) < \xi_{l+1}^k$$

and

$$\left|\xi - \frac{a}{b}\right|_p = \mathrm{H}\left(\frac{a}{b}\right)^{-\nu},$$

where $\nu > \frac{w+1}{k}$.

Instead of (5), we now have

$$(\xi_l^k)^{\frac{w+1}{k} - 1} \leq \mathrm{H}(a/b)$$

and instead of (6),

$$\mathrm{H}(a/b) \leq (\xi_l^k)^{\frac{k(w+1)}{w+1-k}}$$

where we used the fact that $w + 1 > 2k$ which obviously holds if w satisfies the conditions of this proposition.

If we had $w_1^*(\xi^k) > \frac{w+1}{k} - 1$, we could conclude that

$$\frac{w+1}{k} - 1 \leq \frac{k(w+1)}{w+1-k}$$

which contradicts the lower bound on w imposed in the statement of this proposition. ∎

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