Explicit examples of p-adic numbers with prescribed irrationality exponent

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To Jeff Shallit on his sixtieth birthday

Abstract

Let p be a prime number and $\mu \geqslant 2$ a real number. We establish that the irrationality exponent of the p-adic number $\sum_{i=0}^{+\infty} p^{\lfloor \mu^i \rfloor}$ is equal to μ. This provides us with explicit examples of p-adic numbers with any prescribed irrationality exponent.

1 Introduction

The irrationality exponent $\mu(\xi)$ of an irrational real number ξ is the supremum of the real numbers μ such that

$$\left| \xi - \frac{p}{q} \right| < \frac{1}{q^\mu}$$

has infinitely many solutions in rational numbers p/q. It follows from the theory of continued fractions that $\mu(\xi)$ is always at least equal to 2. Furthermore, for any given real number $\mu \geqslant 2$, it is easy to construct explicitly continued fraction expansions of real numbers ξ_μ with irrationality exponent μ. However, if one requires additional properties, like for example ξ_μ being in the middle third Cantor set, this is less straightforward. This question was solved recently in [5], also by means of the theory of continued fractions.

The main goal of this paper is to consider the analogous problem for p-adic numbers (here, and throughout the paper, p denotes a prime number), that is, to construct explicit examples of p-adic numbers with prescribed irrationality exponent and only digits 0 and 1 in their Hensel expansion.

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A natural idea is to translate the argument of [5] to the p-adic setting. Unfortunately, this does not seem to work, since there is no full analogue of the theory of continued fractions in p-adic fields. Admittedly, it is possible to express a p-adic number ξ as a continued fraction, but, unlike in the real case, one cannot read off this expansion all the best rational expansions to ξ. Recall that to determine the exact value of the irrationality exponent of ξ requires, on the one hand, to bound it from below, that is, to find an infinite sequence of very good rational approximations to ξ (this is often an easy step) and, on the other hand, to establish that only finitely many very good rational approximations to ξ are outside this sequence (this is usually much more delicate). In most of the cases in the real setting, this delicate step requires a precise description of the continued fraction expansion of ξ.

For a reduced rational number a/b, let $\mathrm{H}(a/b) = \max\{|a|, |b|\}$ denote its height. The irrationality exponent $\mu(\xi)$ of an irrational p-adic number ξ is the supremum of the real numbers μ such that

$$\left|\xi - \frac{a}{b}\right|_p < \mathrm{H}(a/b)^{-\mu}$$

has infinitely many solutions in rational numbers a/b. As in the real setting, we have $\mu(\xi) \geqslant 2$; see for example Section 9.3 of [4]. Our main result is the following theorem.

Theorem 1. *Let p be a prime number. Let $\mathbf{c} := (c_i)_{i\geqslant 0}$ be a sequence of positive integers such that $c_{i+1} \geqslant 2c_i$ for i large enough. Then the irrationality exponent of the p-adic number*

$$\xi_{\mathbf{c}} := \sum_{i=0}^{+\infty} p^{c_i}$$

is equal to

$$c := \limsup_{i\to+\infty} \frac{c_{i+1}}{c_i}.$$

In particular, for any real number $\mu \geqslant 2$, we have

$$\mu\Big(\sum_{i=0}^{+\infty} p^{\lfloor \mu^i \rfloor}\Big) = \mu.$$

Theorem 1 is the p-adic analogue of a result stated in the Introduction of [6].

It follows from Theorems 1B and 6 of [1], whose proofs rest on a p-adic version of Schmidt Subspace Theorem, that all the p-adic numbers $\xi_{\mathbf{c}}$ defined in Theorem 1 are transcendental, since their Hensel expansions have small block complexity.

For every real number $\mu \geqslant 2$, the Hausdorff dimension of the set of p-adic numbers with irrationality exponent μ is equal to $2/\mu$; see Theorem 6.19 of [2]. This implies the existence of p-adic numbers with any prescribed irrationality exponent. As far as we are aware, explicit examples of p-adic numbers with irrationality exponent equal to μ were not known for $\mu < (3+\sqrt{5})/2$ (for larger values of μ, we can use the argument described in [8] in the real setting). Of course, it is clear that $\sum_{i=0}^{+\infty} p^{\lfloor \mu^i \rfloor}$ is a good candidate for having irrationality exponent μ, but this was proved only for $\mu \geqslant (3+\sqrt{5})/2$.

Our strategy is the following. We work in the field of power series and consider the continued fraction expansion of $\sum_{i=0}^{+\infty} X^{-c_i}$ and its convergents. Replacing X by p^{-1} we deduce good and very good rational approximations to the p-adic number $\xi_{\mathbf{c}}$. We are then in a position to apply Lemma 2 below to complete the proof of Theorem 1.

In the following section, we establish an auxiliary lemma that will be useful for bounding from above the irrationality exponent of a (p-adic) number. In the last section we introduce continued fractions in the field of rational power series and use them to prove Theorem 1. We keep the notation of Theorem 1 throughout this paper.

2 Bounding the irrationality exponent from above

Using the infinite family of rational approximations

$$\sum_{i=0}^{k} p^{c_i} \quad (k \geqslant 0),$$

to $\xi_{\mathbf{c}}$, we easily get that $\mu(\xi_{\mathbf{c}}) \geqslant c$. This proves Theorem 1 when c is infinite. From now on, we assume that c is finite. Establishing that $\mu(\xi_{\mathbf{c}}) \leqslant c$ is more delicate.

The basic idea is the existence of many very good rational approximations to $\xi_{\mathbf{c}}$, which impedes the existence of even better rational approximations. As auxiliary tools, we need the following two elementary lemmas, whose proofs rest on triangle inequalities.

Lemma 1. *Let ξ be in $\mathbb{Q}_p$. Let $\vartheta, \tau \in (1, +\infty)$ be real numbers and P, Q, m, n integers such that Q and n are nonzero, m/n and P/Q are reduced and distinct. If*

$$|\xi - P/Q|_p \leqslant \mathrm{H}(P/Q)^{-\vartheta} \quad \textit{and} \quad |\xi - m/n|_p \leqslant \mathrm{H}(m/n)^{-\tau},$$

then

$$\mathrm{H}(m/n) \leqslant 2^{\frac{1}{\tau-1}}\,\mathrm{H}(P/Q)^{\frac{1}{\tau-1}} \quad or \quad \mathrm{H}(m/n) \geqslant \frac{1}{2}\,\mathrm{H}(P/Q)^{\vartheta-1}. \tag{1}$$

Proof. Using the fact that $|Pn - Qm| \leqslant 2\,\mathrm{H}(P/Q)\,\mathrm{H}(m/n)$ and the non-Archimedean nature of the p-adic absolute value, we get

$$\begin{aligned}\frac{1}{2\,\mathrm{H}(P/Q)\,\mathrm{H}(m/n)} &\leqslant \Big|\frac{Pn-Qm}{Qn}\Big|_p = \Big|\frac{P}{Q} - \frac{m}{n}\Big|_p \\ &\leqslant \max\{|\xi - P/Q|_p, |\xi - m/n|_p\} \\ &\leqslant \max\{\mathrm{H}(P/Q)^{-\vartheta}, \mathrm{H}(m/n)^{-\tau}\},\end{aligned}$$

thus

$$1/2 \leqslant \max\{\mathrm{H}(P/Q)^{1-\vartheta}\,\mathrm{H}(m/n), \mathrm{H}(P/Q)\,\mathrm{H}(m/n)^{1-\tau}\}$$

If we have

$$\mathrm{H}(P/Q)^{1-\vartheta}\,\mathrm{H}(m/n) \leqslant \mathrm{H}(P/Q)\,\mathrm{H}(m/n)^{1-\tau},\ \text{i.e.,}\, \mathrm{H}(m/n) \leqslant \mathrm{H}(P/Q)^{\vartheta/\tau},$$

then

$$\mathrm{H}(m/n) \leqslant 2^{\frac{1}{\tau-1}}\,\mathrm{H}(P/Q)^{\frac{1}{\tau-1}}.$$

If, on the other hand, we have

$$\mathrm{H}(P/Q)^{1-\vartheta}\,\mathrm{H}(m/n) \geqslant \mathrm{H}(P/Q)\,\mathrm{H}(m/n)^{1-\tau},\ \text{i.e.,}\ \mathrm{H}(m/n) \geqslant \mathrm{H}(P/Q)^{\vartheta/\tau},$$

then

$$\mathrm{H}(m/n) \geqslant \frac{1}{2}\,\mathrm{H}(P/Q)^{\vartheta-1}.$$

Therefore, (1) always holds. □

Lemma 2. *For $\xi \in \mathbb{Q}_p$, let $(\vartheta_k)_{k\geqslant 0}$ be a sequence of real numbers such that $\liminf_{k\to+\infty} \vartheta_k > 1$ and let $(P_k/Q_k)_{k\geqslant 0}$ be a sequence of distinct rational numbers such that*

$$\Big|\xi - \frac{P_k}{Q_k}\Big|_p = \mathrm{H}(P_k/Q_k)^{-\vartheta_k}$$

holds for $k \geqslant 0$. Let τ be a real number with

$$\tau > 1 + \limsup_{k\to+\infty} \frac{\log \mathrm{H}(P_{k+1}/Q_{k+1})}{(\vartheta_k - 1)\log \mathrm{H}(P_k/Q_k)}. \tag{2}$$

Then there exist only finitely many rational numbers m/n outside the sequence $(P_k/Q_k)_{k\geqslant 0}$ which satisfy

$$\Big|\xi - \frac{m}{n}\Big|_p \leqslant \mathrm{H}(m/n)^{-\tau}. \tag{3}$$

In particular, we have

$$\mu(\xi) \leqslant \max\Big\{\limsup_{k\to+\infty} \vartheta_k, 1 + \limsup_{k\to+\infty} \frac{\log \mathrm{H}(P_{k+1}/Q_{k+1})}{(\vartheta_k - 1)\log \mathrm{H}(P_k/Q_k)}\Big\}. \tag{4}$$

Proof. Excluding, if necessary, at most finitely many elements from the sequence $(P_k/Q_k)_{k\geqslant 0}$, we can assume without loss of generality that for some $\epsilon > 0$ and all $k \geqslant 0$, we have $\vartheta_k > 1 + \epsilon$. Suppose that (2) is fulfilled and that there are infinitely many rational numbers m/n not in $(P_k/Q_k)_{k\geqslant 0}$ such that (3) holds. For such a rational number m/n with $\mathrm{H}(m/n)$ large enough, let k_0 be the largest positive integer such that

$$2^{\frac{1}{\tau-1}} \mathrm{H}(P_{k_0}/Q_{k_0})^{\frac{1}{\tau-1}} < \mathrm{H}(m/n).$$

Then Lemma 1 implies that

$$\frac{1}{2}\mathrm{H}(P_{k_0}/Q_{k_0})^{\vartheta_{k_0}-1} \leqslant \mathrm{H}(m/n) \leqslant 2^{\frac{1}{\tau-1}} \mathrm{H}(P_{k_0+1}/Q_{k_0+1})^{\frac{1}{\tau-1}},$$

which gives

$$\tau - 1 \leqslant \frac{\tau \log 2 + \log \mathrm{H}(P_{k_0+1}/Q_{k_0+1})}{(\vartheta_{k_0} - 1)\log \mathrm{H}(P_{k_0}/Q_{k_0})}.$$

From our assumption and the fact that $(\mathrm{H}(P_k/Q_k))_{k\geqslant 0}$ is unbounded, the last inequality would have to hold for infinitely many positive integers k_0. This is in contradiction with (2). □

We stress that exact analogues of Lemmas 1 and 2 hold in the real setting.

Example 1. Let $c \geqslant 2$ be a real number. Lemma 2 can be applied to $\xi_c := \sum_{i=0}^{+\infty} p^{\lfloor c^i \rfloor}$ and the p-adic numbers $\beta_k = \sum_{i=0}^{k} p^{\lfloor c^i \rfloor}$, $k \geqslant 2$, to show that $\mu(\xi_c) \leqslant c$ when

$$c > 1 + \limsup_{k\to+\infty} \frac{c^{k+1}}{(c-1)c^k} = 1 + \frac{c}{c-1}.$$

This implies that $\mu(\xi_c) = c$ when $c^2 - 3c + 1 > 0$, that is, for $c > (3+\sqrt{5})/2 = 2.618\ldots$

However, besides $(\beta_k)_{k\geqslant 0}$, we can also take into consideration the following very good rational approximations to ξ_c defined for $k \geqslant 2$ by

$$\begin{aligned}\gamma_k &= \sum_{i=0}^{k-1} p^{\lfloor c^i \rfloor} + p^{\lfloor c^k \rfloor}\big(1 + p^{\lfloor c^{k+1} \rfloor - \lfloor c^k \rfloor} + p^{2(\lfloor c^{k+1} \rfloor - \lfloor c^k \rfloor)} + \cdots\big) \\ &= \sum_{i=0}^{k-1} p^{\lfloor c^i \rfloor} + \frac{p^{\lfloor c^k \rfloor}}{1 - p^{\lfloor c^{k+1} \rfloor - \lfloor c^k \rfloor}}.\end{aligned}$$

It is not difficult to show that

$$\begin{aligned}
\mathrm{H}(\beta_k) = \beta_k \asymp p^{c^k}, \quad \mathrm{H}(\gamma_k) \asymp p^{c^{k+1}-c^k+c^{k-1}},\\
|\xi_c - \beta_k|_p \asymp p^{-c^{k+1}} \asymp \mathrm{H}(\beta_k)^{-c},\\
|\xi_c - \gamma_k|_p \asymp p^{-2c^{k+1}+c^k} \asymp \mathrm{H}(\gamma_k)^{-(2c^2-c)/(c^2-c+1)},\\
\mathrm{H}(\beta_k) < \mathrm{H}(\gamma_k) < \mathrm{H}(\beta_{k+1}) \quad \text{for } k \text{ large enough.}
\end{aligned}$$

Here, all the constants implicit in $\asymp$ can be made explicit and depend only on p and c. If we now apply Lemma 2 with the union of the two sequences $(\beta_k)_{k\geqslant 2}$ and $(\gamma_k)_{k\geqslant 2}$, we get after some calculation that $\mu(\xi_c) \leqslant c$ for $c > 2.325$.

Nevertheless, in order to prove that $\mu(\xi_c) = c$ for all $c \geqslant 2$, we require many more good rational approximations to ξ_c. To find them, looking for repetitions and completing by periodicity is not sufficient. We need an additional idea.

3 Proof of Theorem 1

In order to construct very good rational approximations to our p-adic number, we first need to work with continued fractions in the field of power series. The field of rational power series $\mathbb{Q}((X^{-1}))$ is the completion of $\mathbb{Q}(X)$ with respect to the non-Archimedean absolute value $\|\cdot\|$ defined on $\mathbb{Q}(X)$ by $\|X\| = e$. Instead of e, we can fix any real number greater than 1. If $\hat{\xi} \in \mathbb{Q}((X^{-1}))$ and $\hat{\xi} \neq 0$, then we can write $\hat{\xi} = \sum_{k=k_0}^{+\infty} a_k X^{-k}$, where $k_0 \in \mathbb{Z}$, $a_k \in \mathbb{Q}$ and $a_{k_0} \neq 0$. More information on this subject can be found in [13] and [7].

An element of $\mathbb{Q}(X)$ can be expressed as a finite continued fraction

$$[S_0, S_1, \ldots, S_n] := S_0 + \cfrac{1}{S_1 + \cfrac{1}{\ddots + \cfrac{1}{S_n}}},$$

where each partial quotient S_i is a rational polynomial. This expression is unique if we require that the degree $\deg S_i$ is positive for $i > 0$. Beside some basic facts from the theory of continued fractions (see e.g. [4]), our main tool is the so called Folding Lemma; see [9, 11, 12, 10] and Chapter 6 of [3].

Throughout this paper, for a word $\overrightarrow{w} = a_1, a_2, \ldots, a_h$, we denote by $\overleftarrow{w}$ the word $a_h, a_{h-1}, \ldots, a_1$ and by $-\overleftarrow{w}$ the word $-a_h, -a_{h-1}, \ldots, -a_1$.

Lemma 3 (Folding Lemma)**.** *Let r, s, t be in $\mathbb{Q}[X]$ with t non-constant. Write $r/s = [a_0, a_1, \dots, a_h]$. Then, we have*

$$\frac{r}{s} + \frac{(-1)^h}{ts^2} = [a_0, \overrightarrow{w}, t, -\overleftarrow{w}] = [a_0, \overrightarrow{w}, t-1, 1, -1, 0, \overleftarrow{w}].$$

When using the Folding Lemma, we will usually immediately make the substitution $[\dots, a, 0, b, \dots] = [\dots, a+b, \dots]$, which can be easily checked.

The irrationality exponent of an irrational number does not change when multiplying it or adding to it any nonzero rational number. Thus if necessary, we can omit finitely many terms in $\xi_{\mathbf{c}}$ and multiply this number with p^{-1}, so that $c_{i+1} > 2c_i$ holds for all $i \geqslant 0$. This is no loss of generality and will be assumed during the proof of Theorem 1.

Starting from $X^{-c_0} = [0, X^{c_0} - 1, 1]$ and successively applying Lemma 3 first with $t = X^{c_1 - 2c_0}$ and then with $t = X^{c_2 - 2c_1}$, we obtain

$$\begin{aligned} X^{-c_0} + X^{-c_1} &= X^{-c_0} + (-1)^2 (X^{c_1 - 2c_0} X^{2c_0})^{-1} \\ &= [0, X^{c_0} - 1, 1, X^{c_1 - 2c_0} - 1, X^{c_0}] \end{aligned}$$

and

$$\begin{aligned} X^{-c_0} + X^{-c_1} + X^{-c_2} &= X^{-c_0} + X^{-c_1} + (-1)^4 (X^{c_2 - 2c_1} X^{2c_1})^{-1} \\ &= [0, X^{c_0} - 1, 1, X^{c_1 - 2c_0} - 1, X^{c_0}, X^{c_2 - 2c_1} - 1, \\ &\qquad 1, X^{c_0} - 1, X^{c_1 - 2c_0} - 1, 1, X^{c_0} - 1]. \end{aligned}$$

Continuing this process indefinitely with $t = X^{c_{k+1} - 2c_k}$ for $k \geqslant 1$, we arrive at the continued fraction expansion

$$\begin{aligned} \hat{\xi_{\mathbf{c}}} &= \sum_{i=0}^{+\infty} X^{-c_i} \\ &= [a_0, a_1, a_2, \dots] \\ &= [0, X^{c_0} - 1, 1, X^{c_1 - 2c_0} - 1, X^{c_0}, X^{c_2 - 2c_1} - 1, 1, X^{c_0} - 1, \\ &\quad X^{c_1 - 2c_0} - 1, 1, X^{c_0} - 1, X^{c_3 - 2c_2} - 1, 1, X^{c_0} - 2, \dots] \in \mathbb{Q}((X^{-1})). \end{aligned} \tag{5}$$

We need to justify the convergence of the continued fraction expansion in (5) since there are partial quotients of degree 0, namely those which are equal to 1. The following facts are established by an easy induction and, therefore, we give only an outline of their proofs.

Proposition 1. *Let $\hat{\xi_{\mathbf{c}}} \in \mathbb{Q}((X^{-1}))$ be the sum of the infinite series in (5). Then the following statements hold.*

(i) *No two consecutive elements in the sequence* $(a_i)_{i\geqslant 0}$ *are equal to* 1.

(ii) *For* $i>0$, *if* $a_i \neq 1$, *then* a_i *is a monic integer polynomial and* $\deg a_i \geqslant 1$.

(iii) *If* $i \equiv 3$ *or* $8 \pmod{12}$, *then*

$$a_i = X^{c_1 - 2c_0} - 1.$$

If $i \equiv 3\cdot 2^{k-1} - 1$ *or* $9\cdot 2^{k-1} \pmod{3\cdot 2^{k+1}}$ *for some* $k \geqslant 2$, *then*

$$a_i = X^{c_k - 2c_{k-1}} - 1.$$

If i *does not satisfy any of these congruences, then*

$$a_i \in \{1, X^{c_0}, X^{c_0} - 1, X^{c_0} - 2\}.$$

(iv) *Set* $P_n/Q_n = [a_0, a_1, \ldots, a_n]$, *where* $P_n, Q_n \in \mathbb{Z}[X]$ *for* $n \geqslant 0$. *Then, for any* $n \geqslant 1$,

$$\textit{either } \deg Q_{n-1} < \deg Q_n \textit{ or } \deg Q_{n-1} = \deg Q_n < \deg Q_{n+1}.$$

The polynomials P_n *(for* $n \geqslant 1$*) and* Q_n *are monic, thus* $Q_n(1/p) \neq 0$.

(v)

$$\deg P_n = \sum_{i=2}^{n} \deg a_i \quad \textit{for } n \geqslant 2,$$
$$\deg Q_n = \sum_{i=1}^{n} \deg a_i \quad \textit{for } n \geqslant 1,$$
$$\deg Q_n - \deg P_n = c_0 > 0.$$

(vi)

$$\frac{P_{3\cdot 2^k - 2}}{Q_{3\cdot 2^k - 2}} = \sum_{i=0}^{k} X^{-c_i}, \quad Q_{3\cdot 2^k - 2} = X^{c_k}, \quad \textit{for } k \geqslant 1. \tag{6}$$

(vii) *The sequence* $(P_n/Q_n)_{n\geqslant 0}$ *converges to* $\hat{\xi_{\mathbf{c}}}$ *in* $\mathbb{Q}((X^{-1}))$.

Sketch of the proof. Statements *(i)* and *(ii)* are easy and *(iii)* requires only careful attention to the indices when observing the repeated application of the Folding Lemma.

The claim in (iv) follows from (i) and (ii) using the recurrence relation satisfied by P_n and Q_n, namely

$$\begin{matrix} P_0 = 0, & P_1 = 1, & P_n = a_n P_{n-1} + P_{n-2}, \\ Q_0 = 1, & Q_1 = X^{c_0} - 1, & Q_n = a_n Q_{n-1} + Q_{n-2}, \end{matrix} \quad \text{for } n \geqslant 2 \tag{7}$$

The same identities are used to prove (v), while (vi) follows immediately from the Folding Lemma.

The well-known identity

$$P_{n+1}Q_n - P_nQ_{n+1} = (-1)^n, \ n \geqslant 0, \tag{8}$$

follows inductively from (7). Applying (8) for n consecutive indices gives

$$\frac{P_{n+1}}{Q_{n+1}} = \frac{1}{Q_0Q_1} - \frac{1}{Q_1Q_2} + \cdots + (-1)^n \frac{1}{Q_nQ_{n+1}}. \tag{9}$$

From (iv), we have $\|Q_{i-1}Q_i\| < \|Q_iQ_{i+1}\|$, so the infinite series

$$\sum_{i=0}^{+\infty} (-1)^i (Q_iQ_{i+1})^{-1}$$

converges in $\mathbb{Q}((X^{-1}))$ and the partial sums of this series are convergents of the continued fraction in (5). But (vi) now shows that $\lim_{n\to+\infty} P_n/Q_n = \hat{\xi}_{\mathbf{c}}$ so that (vii) is proved as well. Note that we have also proved (cf. (v)) that

$$\left\| \xi_{\mathbf{c}} - \frac{P_n}{Q_n} \right\| = \|Q_nQ_{n+1}\|^{-1} = e^{-2\sum_{i=1}^n \deg a_i - \deg a_{n+1}}. \tag{10}$$

□

We prove an auxiliary result on the growth of denominators of partial quotients. Recall that

$$c = \limsup_{i\to+\infty} \frac{c_{i+1}}{c_i}.$$

Proposition 2. *The polynomials Q_n defined in Proposition 1 satisfy*

$$\lim_{n\to+\infty} \frac{\log n}{\deg Q_n} = 0, \tag{11}$$

$$\liminf_{n\to+\infty} \frac{\deg Q_{n+1}}{\deg Q_n} = 1, \quad \limsup_{n\to+\infty} \frac{\deg Q_{n+1}}{\deg Q_n} = c - 1. \tag{12}$$

Proof. From (iv) in Proposition 1, we deduce that $\deg Q_n \geqslant \deg Q_{n-2} + 1$ for $n \geqslant 2$. Iterating this gives $\deg Q_n \geqslant \lfloor n/2 \rfloor$ for $n \geqslant 1$ and (11) holds.

The first equality in (12) is obvious if we take into account (v) from Proposition 1 and the fact that $\deg a_i \in \{0, 1\}$ for infinitely many i.

If $n = 3 \cdot 2^k - 2$ for some $k \geqslant 1$, then

$$\frac{\deg Q_{n+1}}{\deg Q_n} = \frac{c_{k+1} - c_k}{c_k} = \frac{c_{k+1}}{c_k} - 1. \tag{13}$$

For an integer $n \geqslant 5$ not of this form, let k be the positive integer such that

$$3 \cdot 2^k - 1 \leqslant n < 3 \cdot 2^{k+1} - 2.$$

Then, by (vi), (v) and (iii) of Proposition 1, we get

$$\deg Q_n \geqslant \deg Q_{3 \cdot 2^k - 1} = c_{k+1} - c_k \geqslant c_k \quad \text{and} \quad \deg Q_{n+1} \leqslant \deg Q_n + d,$$

where

$$d \leqslant \max\{c_i - 2c_{i-1} \,:\, 0 \leqslant i \leqslant k\}, \qquad (c_{-1} := 0).$$

Hence,

$$\frac{\deg Q_{n+1}}{\deg Q_n} \leqslant 1 + \frac{d}{c_k} \leqslant 1 + \frac{c_i/c_{i-1} - 2}{c_k/c_{i-1}} \leqslant 1 + \frac{c_i/c_{i-1} - 2}{2} \leqslant \frac{c_i}{2c_{i-1}}, \tag{14}$$

for some $i \in \{1, \ldots, k\}$. Since $c \geqslant 2$, we get $c/2 \leqslant c - 1$ and the combination of (13) and (14) implies the second equality in (12). □

The following proposition uses the symmetric structure of the continued fraction expansion of $\hat{\xi}_{\mathbf{c}}$ (obtained by means of repeated applications of the Folding Lemma) in order to express the convergents of this continued fraction expansion in terms of certain convergents with smaller indices.

Proposition 3. *Let n and k be positive integers such that*

$$3 \cdot 2^k \leqslant n < 3 \cdot 2^{k+1} - 2.$$

Set $m = 3 \cdot 2^k - 2$ and $l = 3 \cdot 2^{k+1} - 3 - n$. Then,

$$\frac{P_n}{Q_n} = \frac{X^{c_{k+1} - 2c_k} P_m (P_m Q_l - Q_m P_l) - P_l}{X^{c_{k+1} - 2c_k} Q_m (P_m Q_l - Q_m P_l) - Q_l}. \tag{15}$$

Proof. Since $0 \leqslant l \leqslant 3 \cdot 2^k - 3$, we can write for $l < 3 \cdot 2^k - 3$,

$$\frac{P_m}{Q_m} = \frac{(U_l/V_l) P_{l+1} + P_l}{(U_l/V_l) Q_{l+1} + Q_l},$$

where

$$\frac{U_l}{V_l} = [w_l], \qquad w_l = a_{l+2}, \dots, a_m.$$

This gives

$$\frac{U_l}{V_l} = \frac{Q_l P_m - P_l Q_m}{-Q_{l+1} P_m + P_{l+1} Q_m}. \tag{16}$$

Setting

$$\frac{S_l}{T_l} = [0, w_l, X^{c_{k+1}-2c_k}, -\overleftarrow{w_l}],$$

we deduce from the Folding Lemma that

$$\frac{S_l}{T_l} = \frac{V_l}{U_l} + (-1)^{l-1} \frac{1}{X^{c_{k+1}-2c_k} U_l^2} = \frac{X^{c_{k+1}-2c_k} U_l V_l + (-1)^{l-1}}{X^{c_{k+1}-2c_k} U_l^2}. \tag{17}$$

We also have

$$\frac{P_n}{Q_n} = \frac{(T_l/S_l) P_{l+1} + P_l}{(T_l/S_l) Q_{l+1} + Q_l}. \tag{18}$$

Note that the fractions that appear in (16) and (17) are reduced.

Now we substitute (16) into (17) and (17) in (18) and use the fact that $P_{l+1} Q_l - P_l Q_{l+1} = (-1)^l$. Simplifying the expression, we obtain (15). Again, the fractions in (15) are reduced.

If $l = 3 \cdot 2^k - 3 = m - 1$, then $S_l/T_l = [0, X^{c_{k+1}-2c_k}]$ and (18) gives (15) directly. $\square$

Now we make the formal substitution $X \mapsto 1/p$ and define the quantities

$$\xi_{\mathbf{c}} = \sum_{i=0}^{+\infty} p^{c_i} \in \mathbb{Q}_p,$$

$$A_n = p^{\deg Q_n} P_n(1/p), \quad B_n = p^{\deg Q_n} Q_n(1/p), \quad n \geqslant 0.$$

The next proposition summarizes properties of $\xi_{\mathbf{c}}$ and the sequences $(A_n)_{n \geqslant 0}$ and $(B_n)_{n \geqslant 0}$.

Proposition 4. *For the p-adic number $\xi_{\mathbf{c}}$ and the sequences $(A_n)_{n \geqslant 0}$ and $(B_n)_{n \geqslant 0}$ defined above, the following holds.*

(i) The numbers A_n and B_n are integers satisfying

$$p^{c_0} \mid A_n, \quad p^{c_0+1} \nmid A_n, \quad B_n \equiv 1 \pmod{p}.$$

(ii) For every $n \geqslant 0$, the integers A_n and B_n are relatively prime. The sequence $(A_n/B_n)_{n \geqslant 0}$ converges in $\mathbb{Q}_p$ to $\xi_{\mathbf{c}}$ and

$$\left| \xi_{\mathbf{c}} - \frac{A_n}{B_n} \right|_p = p^{-\deg(Q_n Q_{n+1})} = p^{-2 \sum_{i=1}^{n} \deg a_i - \deg a_{n+1}}. \tag{19}$$

(iii) The height of A_n/B_n satisfies

$$\frac{1}{2n^2}p^{\deg Q_n} \leqslant \mathrm{H}(A_n/B_n) \leqslant n^2 p^{\deg Q_n}, \quad n \geqslant 1.$$

Proof. (i) This follows directly from (*iv*) and (*v*) of Proposition 1.

(ii) From (8), we get for $n \geqslant 0$ that

$$A_{n+1}B_n - A_nB_{n+1} = (-1)^n p^{\deg Q_n + \deg Q_{n+1}}, \tag{20}$$

which implies that the sequence $(A_n/B_n)_{n\geqslant 0}$ is a Cauchy sequence and thus converges in $\mathbb{Q}_p$. From

$$\xi_{\mathbf{c}} - \frac{A_{3\cdot 2^k-2}}{B_{3\cdot 2^k-2}} = \sum_{i=k+1}^{+\infty} p^{c_i},$$

we deduce that $\lim_{n\to+\infty} A_n/B_n = \xi_{\mathbf{c}}$. Now (19) follows from

$$\frac{A_{n+1}}{B_{n+1}} = \frac{p^{\deg(Q_0Q_1)}}{B_0B_1} - \frac{p^{\deg(Q_1Q_2)}}{B_1B_2} + \cdots + (-1)^n \frac{p^{\deg(Q_nQ_{n+1})}}{B_nB_{n+1}},$$

similarly as (10) was obtained from (9).

The identity (20) also shows that A_n and B_n are always relatively prime since any common divisor would be a power of p and (*i*) asserts that B_n is never divisible by p.

Note that (19) implies that no two elements of the sequence $(A_n/B_n)_{n\geqslant 0}$ are identical, in view of (*iv*) of Proposition 1.

(iii) We have seen that A_n and B_n are always relatively prime. From the definition of these numbers, it is clearly sufficient to prove that $\max\{|P_n(1/p)|, |Q_n(1/p)|\}$ belongs to the interval $[1/(2n^2), n^2]$.

We first show that $|P_n(1/p)| \leqslant n^2$. More precisely, we prove by induction on $k \geqslant 0$ that for $3\cdot 2^k - 2 \leqslant n < 3\cdot 2^{k+1} - 2$ the inequality $|P_n(1/p)| \leqslant 2^{2k}$ holds.

This can be easily checked for $k = 0$. Suppose the statement holds for all $k < K$, where $K \geqslant 1$, and let n with

$$3\cdot 2^K - 2 \leqslant n < 3\cdot 2^{K+1} - 2.$$

If $n = 3\cdot 2^K - 2$, we obtain a much stronger result directly from (6)

$$|P_n(1/p)| < \frac{p}{p-1} \leqslant 2, \qquad |Q_n(1/p)| = 1/p^{c_K} \leqslant 1/2.$$

For $n = 3\cdot 2^K - 1$, we have $a_n(1/p) = p^{2c_K - c_{K+1}} - 1$, so

$$|P_n(1/p)| \leqslant |a_n(1/p)P_{n-1}(1/p)| + |P_{n-2}(1/p)| \leqslant 2 + 2^{2K-2} \leqslant 2^{2K}.$$

Finally, if $3\cdot 2^K \leqslant n < 3\cdot 2^{K+1} - 2$, we use Proposition 3 which gives

$$|P_n(1/p)| < p^{2c_K - c_{K+1}} 2\Big(2\cdot 2^{2K-2} + \frac{1}{2}2^{2K-2}\Big) + 2^{2K-2} < 2^{2K}.$$

Analogously, we prove that $|Q_n(1/p)| \leqslant n^2$.

Furthermore, (8) shows that

$$\max\{|P_n(1/p)Q_{n-1}(1/p)|, |P_{n-1}(1/p)Q_n(1/p)|\} \geqslant 1/2,$$

thus

$$\max\{|P_n(1/p)|, |Q_n(1/p)|\} \geqslant \frac{1}{2\max\{|P_{n-1}(1/p)|, |Q_{n-1}(1/p)|\}} \geqslant \frac{1}{2n^2}.$$

□

We are now armed to complete the proof of Theorem 1. Since

$$\frac{\log|\xi_{\mathbf{c}} - A_n/B_n|_p^{-1}}{\log \mathrm{H}(A_n/B_n)} \leqslant \frac{(\deg Q_n + \deg Q_{n+1})\log p}{-3\log n + \deg Q_n \log p} = \frac{1 + \frac{\deg Q_{n+1}}{\deg Q_n}}{-\frac{\log n}{\deg Q_n}\cdot\frac{3}{\log p} + 1},$$

and

$$\frac{\log|\xi_{\mathbf{c}} - A_n/B_n|_p^{-1}}{\log \mathrm{H}(A_n/B_n)} \geqslant \frac{(\deg Q_n + \deg Q_{n+1})\log p}{2\log n + \deg Q_n \log p} = \frac{1 + \frac{\deg Q_{n+1}}{\deg Q_n}}{\frac{\log n}{\deg Q_n}\cdot\frac{2}{\log p} + 1},$$

Proposition 2 gives

$$\limsup_{n\to+\infty} \frac{\log|\xi_{\mathbf{c}} - A_n/B_n|_p^{-1}}{\log \mathrm{H}(A_n/B_n)} = c \quad \text{and} \quad \liminf_{n\to+\infty} \frac{\log|\xi_{\mathbf{c}} - A_n/B_n|_p^{-1}}{\log \mathrm{H}(A_n/B_n)} = 2.$$

In order to apply Lemma 2 to $\xi_{\mathbf{c}}$ and $(A_n/B_n)_{n\geqslant 0}$, we need to evaluate the right hand side of (2). Using the results of Proposition 4, we get

$$\begin{aligned}
&\limsup_{n\to+\infty} \frac{\log \mathrm{H}(A_{n+1}/B_{n+1})}{\log|\xi_{\mathbf{c}} - A_n/B_n|_p^{-1} - \log \mathrm{H}(A_n/B_n)} \\
&\qquad \leqslant \limsup_{n\to+\infty} \frac{2\log(n+1) + \deg Q_{n+1}\log p}{(\deg Q_n + \deg Q_{n+1})\log p - 2\log n - \deg Q_n \log p} \\
&\qquad \leqslant \limsup_{n\to+\infty} \frac{2\frac{\log(n+1)}{\deg Q_{n+1}} + \log p}{-2\frac{\log n}{\deg Q_{n+1}} + \log p} = 1,
\end{aligned}$$

where we used (11) in the last step.

Bound (4) from Lemma 2 finally implies that $\mu(\xi_{\mathbf{c}}) \leqslant c$ and thus $\mu(\xi_{\mathbf{c}}) = c$. This finishes the proof of Theorem 1.

As a final remark, we show that not only the best approximations to ξ_c, but also the second best ones, can be written explicitly as series. The best approximations $P_{3\cdot 2^k-2}/Q_{3\cdot 2^k-2}$ to the associated power series $\hat{\xi_c}$ are obtained by cutting off its continued fraction expansion before the first appearance of large partial quotients, or alternatively, by deleting the terms that follow $X^{-c_k} = X^{-\lfloor c^k \rfloor}$ in $\sum_{i=0}^{+\infty} X^{-c_i}$. These approximations of $\hat{\xi_c}$ in $\mathbb{Q}((X^{-1}))$ correspond to the approximations β_k of $\xi_c \in \mathbb{Q}_p$ defined in Example 1. The next best sequence of approximations is $(P_{9\cdot 2^{k-1}-1}/Q_{9\cdot 2^{k-1}-1})_{k\geqslant 2}$ and, as the following lemma shows, it actually corresponds to the sequence $(\gamma_k)_{k\geqslant 2}$ of p-adic numbers introduced in Example 1.

Proposition 5. *For $k \geqslant 2$, we have*

$$\frac{P_{9\cdot 2^{k-1}-1}}{Q_{9\cdot 2^{k-1}-1}} = \sum_{i=0}^{k-1} X^{-c_i} + \frac{X^{-c_k}}{1 - X^{c_k - c_{k+1}}}. \tag{21}$$

Proof. This is an immediate consequence of Proposition 3 where $n = 9 \cdot 2^{k-1} - 1$, so that $m = 3 \cdot 2^k - 2$ and $l = 3 \cdot 2^{k-1} - 2$. Since P_m, Q_m, P_l, Q_l are now explicitly given by (6), inserting their values into (15) gives (21). □

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