NONPERIODIC QUADRATIC BROWKIN CONTINUED FRACTIONS

TOMISLAV PEJKOVIĆ

Abstract

We give explicit quadratic irrationals whose continued fraction expansions under Browkin's first algorithm are not eventually periodic. In particular, this holds for $(13+\sqrt{1394})/35$ in $\mathbb{Q}_7$, with $\sqrt{1394}\equiv 6 \pmod 7$, and $(19+\sqrt{6290})/77$ in $\mathbb{Q}_{11}$, with $\sqrt{6290}\equiv 8 \pmod{11}$. The proof uses two involutions preserving the discriminant and the insolubility of negative Pell equations over $\mathbb{Z}[1/p]$.

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1. Introduction

Let p be an odd prime. For $x \in \mathbb{Q}_p$, write

$$x = \sum_{j=k}^{\infty} c_j p^j, \qquad c_j \in \left\{-\frac{p-1}{2}, \dots, \frac{p-1}{2}\right\}.$$

Browkin's first continued fraction algorithm [3] defines the partial quotient of x by $a(x) = \sum_{j=k}^{0} c_j p^j$, with an empty sum equal to zero. Equivalently, $a(x)$ is the unique element satisfying

$$a(x) \in \mathbb{Z}[1/p] \cap (-p/2, p/2), \qquad x - a(x) \in p\mathbb{Z}_p. \tag{1}$$

Starting with $x_0 = x$, the algorithm defines

$$a_n = a(x_n), \qquad x_{n+1} = T(x_n) = \frac{1}{x_n - a_n}.$$

The numbers x_n are the complete quotients. For an irrational input this process does not terminate. Throughout the paper, periodicity means eventual periodicity of the sequence $(a_n)_{n\geq 0}$ unless pure periodicity is specified.

The analogue of Lagrange's theorem for this algorithm asks whether every quadratic irrational in $\mathbb{Q}_p$ has an eventually periodic expansion. For Ruban continued fractions, an effective periodicity criterion for quadratic irrationals was obtained by Capuano,

Veneziano and Zannier [5]. For Browkin's algorithm, Bedocchi [2, Propositions 3.1 and 3.2] proved that an eventually periodic expansion is purely periodic precisely when the initial quadratic irrational is regular, in the sense recalled below. Sufficient conditions for periodicity and constructions with prescribed periods were studied by Capuano, Murru and Terracini [4]. Romeo [7] investigated the relation between periodicity and real convergence. Further periodicity criteria and period-length results were obtained by Barbero, Murru and Urani [1]. The existence of a quadratic irrational whose Browkin I expansion is not eventually periodic was still open in [1]. We give explicit examples for $p = 7$ and $p = 11$.

Theorem 1. *Let $s_7 \in \mathbb{Q}_7$ satisfy $s_7^2 = 1394$ and $s_7 \equiv 6 \pmod 7$. Let $s_{11} \in \mathbb{Q}_{11}$ satisfy $s_{11}^2 = 6290$ and $s_{11} \equiv 8 \pmod{11}$. The Browkin I expansions of*

$$\frac{13 + s_7}{35} \quad \text{in } \mathbb{Q}_7, \qquad \frac{19 + s_{11}}{77} \quad \text{in } \mathbb{Q}_{11}$$

are not eventually periodic.

We prove Theorem 1 by applying the general nonperiodicity criterion in Proposition 4. Its hypotheses concern representations of integers as sums of two squares and the insolubility of negative Pell equations over $\mathbb{Z}[1/p]$. The proof of the criterion uses two involutions whose composition is T and a parity argument based on their fixed points.

2. Regular complete quotients

The odd prime p is fixed throughout this section. For a quadratic irrational $x \in \mathbb{Q}_p$, write x' for its algebraic conjugate. Following [4, Section 3], call x *regular* if $|x|_p > 1$ and $|x'|_p < 1$. We take minimal integer polynomials to be primitive, with positive leading coefficient. The discriminant of a quadratic irrational with minimal integer polynomial $AX^2 + BX + C$ is $B^2 - 4AC$. For a positive nonsquare integer D, let $\mathcal{X}_D$ be the set of regular quadratic irrationals in $\mathbb{Q}_p$ with discriminant $4D$. We formulate the reversal identity from [4, Section 3] in terms of two involutions and verify that they preserve the discriminant.

Lemma 2. *The maps*

$$R(x) = -\frac{1}{x'}, \qquad S(x) = a(x) - x'$$

are involutions of $\mathcal{X}_D$ satisfying $R(-x) = -R(x)$ and $S(-x) = -S(x)$. Moreover,

$$T = RS, \qquad RTR = T^{-1}. \tag{2}$$

Consequently, T is a bijection of $\mathcal{X}_D$.

Proof. Let $F(X) = AX^2 + BX + C$ be the minimal integer polynomial of $x \in \mathcal{X}_D$, with $A > 0$. Write $|x|_p = p^k$ and $|x'|_p = p^{-\ell}$, where $k, \ell \geq 1$. Since $B = -A(x + x')$ and $C = Axx'$, we have

$$|B|_p = p^k |A|_p, \qquad |C|_p = p^{k-\ell} |A|_p.$$

If p divided B, it would divide A and C as well. Primitivity therefore gives

$$|A|_p = p^{-k}, \qquad |B|_p = 1, \qquad |C|_p = p^{-\ell}. \tag{3}$$

Put $a = a(x)$. Then $|a|_p = p^k$, so $Aa \in \mathbb{Z}[1/p] \cap \mathbb{Z}_p = \mathbb{Z}$. Also $F(a) \in \mathbb{Z}[1/p]$ and

$$|F(a)|_p = |A(a-x)(a-x')|_p = |a-x|_p < 1,$$

hence $F(a) \in p\mathbb{Z}$. Moreover, $B + Aa = A(a - x - x') \in p\mathbb{Z}$, and thus

$$B + 2Aa \equiv -B \not\equiv 0 \pmod{p}. \tag{4}$$

The numbers $R(x)$ and $S(x)$ are roots of the respective polynomials

$$\begin{aligned} X^2 F(-1/X) &= CX^2 - BX + A, \\ F(a - X) &= AX^2 - (B + 2Aa)X + F(a). \end{aligned} \tag{5}$$

Both have integer coefficients by the preceding calculations. The first is primitive because $\gcd(A, B, C) = 1$.

For $F(a - X)$, (4) shows that p does not divide the coefficient of X. Suppose that a prime $q \neq p$ divides all three coefficients. Since the denominator of a is a power of p, we may reduce a modulo q. The leading coefficient gives $q \mid A$. Since $q \mid B + 2Aa$, we obtain $B \equiv -2Aa \equiv 0 \pmod{q}$. It follows that $F(a) = Aa^2 + Ba + C \equiv C \pmod{q}$, and $q \mid F(a)$ therefore gives $q \mid C$. This contradicts $\gcd(A, B, C) = 1$, so $F(a - X)$ is primitive.

Both polynomials in (5) have discriminant $B^2 - 4AC = 4D$. Since $R(x)$ and $S(x)$ are quadratic irrationals, these are their minimal integer polynomials up to sign.

The conjugates of $R(x)$ and $S(x)$ are $-1/x$ and $a - x$, respectively. Hence $|R(x)|_p = |x'|_p^{-1} > 1$ and $|(R(x))'|_p = |x|_p^{-1} < 1$. Also, $|a|_p > 1 > |x'|_p$ gives $|S(x)|_p = |a|_p > 1$, while $|(S(x))'|_p = |a - x|_p < 1$. Thus $R(x), S(x) \in \mathcal{X}_D$.

We have $R(R(x)) = x$. Since $S(x) - a = -x' \in p\mathbb{Z}_p$, (1) gives $a(S(x)) = a$, and therefore $S(S(x)) = a - (a - x) = x$. Moreover, $R(S(x)) = -1/(a - x) = T(x)$. Thus $T = RS$ is bijective and $RTR = SR = T^{-1}$. Finally, (1) gives $a(-x) = -a(x)$, from which $R(-x) = -R(x)$ and $S(-x) = -S(x)$ follow. □

Lemma 3. *If $x \in \mathcal{X}_D$ has an eventually periodic Browkin expansion, then $T^L x = x$ for some $L \geq 1$.*

Proof. By eventual periodicity, there exist $n \geq 0$ and $L \geq 1$ such that $T^{n+L}x$ and $T^n x$ have the same Browkin expansion. Their convergents coincide and converge in $\mathbb{Q}_p$ to the respective numbers, so $T^{n+L}x = T^n x$. Since T is a bijection of $\mathcal{X}_D$ by Lemma 2, applying T^{-n} gives $T^L x = x$. □

3. A nonperiodicity criterion

A finite T-orbit through a fixed point of R is invariant under both involutions R and S. In the criterion below, uniqueness of a representation as a sum of two squares gives exactly one fixed class for R after identifying each number with its negative. Insolubility of a negative Pell equation excludes fixed classes for S on the same orbit. The two involutions would therefore force the number of classes to be both odd and even.

Proposition 4. *Let p be an odd prime and let $D > 0$ be a nonsquare integer. Suppose that D has a unique representation*

$$D = u^2 + A^2, \qquad p \mid A, \qquad \gcd(A, 2u) = 1, \tag{6}$$

with u, A positive integers. Assume also that

$$U^2 - DV^2 = -1 \tag{7}$$

has no solution in $\mathbb{Z}[1/p]$. Let $s \in \mathbb{Q}_p$ satisfy $s^2 = D$ and $s \equiv u \pmod p$. Then the Browkin expansion of $z = (u + s)/A$ is not eventually periodic.

Proof. We have $p \nmid u$, so the specified root s exists by Hensel's lemma. Since $u + s \equiv 2u \not\equiv 0 \pmod p$, we have $|z|_p = |A|_p^{-1} > 1$. Also $zz' = -1$, so $|z'|_p = |A|_p < 1$ and the number z is regular. The minimal integer polynomial of z is $AX^2 - 2uX - A$ and it is indeed primitive because $\gcd(A, 2u) = 1$, and its discriminant is $4D$. Thus $z \in X_D$ and $R(z) = -1/z' = z$.

We first show that $T^m z = z$ is impossible for odd $m \geq 1$. Suppose that such a return exists, and put $x_j = T^j z$ and $a_j = a(x_j)$. The relation $x_j = a_j + 1/x_{j+1}$ corresponds to the matrix

$$\begin{pmatrix} a_j & 1 \\ 1 & 0 \end{pmatrix},$$

which acts by $w \mapsto (a_j w + 1)/w$ and sends x_{j+1} to x_j. Hence the product

$$H = (h_{ij}) = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{m-1} & 1 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{Z}[1/p])$$

has determinant $(-1)^m = -1$ and satisfies

$$z = \frac{h_{11} z + h_{12}}{h_{21} z + h_{22}}.$$

The denominator is nonzero, since z is irrational and $\det H \neq 0$. Therefore

$$h_{21} z^2 + (h_{22} - h_{11}) z - h_{12} = 0.$$

Comparison with the minimal integer polynomial of z gives

$$h_{21} = h_{12} = At, \qquad h_{11} - h_{22} = 2ut$$

for some $t \in \mathbb{Q}$, with $t = 0$ allowed. Choose $c, d \in \mathbb{Z}$ such that $cA + 2du = 1$. Then

$$t = c h_{21} + d(h_{11} - h_{22}) \in \mathbb{Z}[1/p].$$

Consequently,

$$-1 = \det H = h_{22}(h_{22} + 2ut) - A^2 t^2 = (h_{22} + ut)^2 - Dt^2.$$

Since $h_{22} + ut$ and t both belong to $\mathbb{Z}[1/p]$, this contradicts the assumption on (7).

Suppose now that the expansion of z is eventually periodic. By Lemma 3, we have $T^L z = z$ for some $L \geq 1$. Write $[x] = \{x, -x\}$ and consider the finite set

$$\mathcal{O} = \{[T^j z] : 0 \leq j < L\}.$$

By Lemma 2, the maps R, S, T are well defined on these classes. Equation (2) gives $RT^j = T^{-j}R$. Together with $R(z) = z$, this implies $R(T^j z) = T^{-j}z$, so R preserves $\mathcal{O}$. Since $S = RT$, the same holds for S.

We show that $[z]$ is the only fixed point of R on $\mathcal{O}$. If $R([x]) = [x]$, then $R(x) = \pm x$, so $xx' = \pm 1$. Write the minimal integer polynomial of x as $A_1X^2 + 2rX + C_1$, with $A_1 > 0$. Its middle coefficient is even because its discriminant is $4D$, and $p \mid A_1$ by (3).

If $xx' = 1$, then $C_1 = A_1$. Primitivity implies that A_1 is odd, and A is odd because $\gcd(A, 2u) = 1$. The identities $D = r^2 - A_1^2 = u^2 + A^2$ would therefore give

$$r^2 - u^2 = A_1^2 + A^2 \equiv 2 \pmod 4,$$

which is impossible.

If $xx' = -1$, then $C_1 = -A_1$ and $D = r^2 + A_1^2$. Here $r \neq 0$ because D is not a square, and primitivity gives $\gcd(A_1, 2r) = 1$. The uniqueness in (6) implies $A_1 = A$ and $|r| = u$. Thus x is a regular root of one of the polynomials $AX^2 - 2uX - A$ and $AX^2 + 2uX - A$. Their regular roots are z and $-z$, respectively, so $[x] = [z]$.

It remains to show that S has no fixed point on $\mathcal{O}$. Such a fixed point would give $S(y) = y$ or $S(y) = -y$ for some $y = T^n z$, with $n \geq 0$. The second possibility gives $a(y) = y' - y$. Since $a(y)$ is rational, conjugation gives $a(y) = y - y'$, forcing $y = y'$, a contradiction. In the first case, $S = RT$ and $R(z) = z$ give

$$T^n z = S(T^n z) = RT^{n+1}z = T^{-n-1}z.$$

Hence $T^{2n+1}z = z$, contrary to the exclusion of odd returns above.

The involution R has exactly one fixed point on $\mathcal{O}$, and all its other elements occur in pairs. Thus $|\mathcal{O}|$ is odd. The involution S has no fixed point on $\mathcal{O}$, so all its elements occur in pairs and $|\mathcal{O}|$ is even. This contradiction completes the proof. □

The uniqueness in (6) holds whenever $p^2 > \sqrt{D}+u$. Indeed, suppose that $D = r^2+B^2$ with integers r, B satisfying $B > 0$ and $p \mid B$. Then $r^2 \equiv u^2 \pmod{p^2}$. Since $p \nmid 2u$, we may change the sign of r so that $r \equiv u \pmod{p^2}$. Then $|r - u| \leq |r| + u < \sqrt{D} + u < p^2$, so $r = u$ and $B = A$.

The identity $D = u^2 + A^2$ gives the rational solution $(U, V) = (u/A, 1/A)$ of (7). This pair lies in $\mathbb{Z}[1/p]^2$ if and only if A is a power of p. Thus the hypotheses of Proposition 4 force A to have a prime divisor other than p.

4. Explicit examples

We use an elementary descent to deduce insolubility of Pellian equations over $\mathbb{Z}[1/p]$ from insolubility over $\mathbb{Z}$. The examples in Theorem 1 are chosen to keep the arithmetic verifications short.

Lemma 5. *Let $D > 0$ be a nonsquare integer and let p be an odd prime with $p \nmid D$. Suppose that there are integers h, j with $p \nmid j$ and*

$$h^2 - Dj^2 = p^2.$$

If $U^2 - DV^2 = -1$ has no integer solution, it has no solution in $\mathbb{Z}[1/p]$.

Proof. Suppose otherwise. Choose a solution $(U/p^k, V/p^k)$, with $U, V \in \mathbb{Z}$ and $k \geq 0$, for which k is minimal. Then $k \geq 1$ and

$$U^2 - DV^2 = -p^{2k},$$

where U, V are not both divisible by p. Reduction modulo p shows that both are units. Also h and j are units modulo p.

The identity

$$(jU - hV)(jU + hV) = -p^{2k} j^2 - p^2 V^2$$

shows that the product is divisible by p^2. The two factors cannot both be divisible by p, since their sum is $2jU$. Hence one factor is divisible by p^2. Choose $\delta \in \{1, -1\}$ so that $jU \equiv \delta hV \pmod{p^2}$. Put

$$U_1 = \frac{hU - \delta DjV}{p^2}, \qquad V_1 = \frac{hV - \delta jU}{p^2}.$$

The second numerator is divisible by p^2. For the first,

$$j(hU - \delta DjV) \equiv \delta(h^2 - Dj^2)V \equiv 0 \pmod{p^2}.$$

Since j is invertible modulo p^2, both U_1 and V_1 are integers. Taking norms in

$$U_1 + V_1\sqrt{D} = \frac{(h - \delta j\sqrt{D})(U + V\sqrt{D})}{p^2}$$

gives $U_1^2 - DV_1^2 = -p^{2k-2}$. Dividing U_1, V_1 by p^{k-1} produces a solution of norm -1 with a smaller common denominator, a contradiction. □

Proof of Theorem 1. We apply Proposition 4 to

$$(p, D, u, A) = (7, 1394, 13, 35), \qquad (11, 6290, 19, 77).$$

Both values of D are congruent to 2 modulo 4 and thus not squares. Directly,

$$1394 = 13^2 + 35^2, \quad \gcd(35, 26) = 1, \qquad 6290 = 19^2 + 77^2, \quad \gcd(77, 38) = 1.$$

The required residues of s are $13 \equiv 6 \pmod 7$ and $19 \equiv 8 \pmod{11}$.

The ordinary continued fraction expansions of the positive real square roots are

$$\sqrt{1394} = [37; \overline{2, 1, 36, 1, 2, 74}], \qquad \sqrt{6290} = [79; \overline{3, 4, 3, 158}].$$

Their periods have lengths 6 and 4, respectively. Since both lengths are even, the equations $U^2 - 1394V^2 = -1$ and $U^2 - 6290V^2 = -1$ have no integer solutions (see e.g. [6, Theorem 10.20]).

Moreover,

$$75^2 - 1394 \cdot 2^2 = 7^2, \qquad 159^2 - 6290 \cdot 2^2 = 11^2.$$

Since $D \equiv u^2 \not\equiv 0 \pmod{p}$ and p is odd, Lemma 5 applies with $j = 2$ and excludes solutions over $\mathbb{Z}[1/7]$ and $\mathbb{Z}[1/11]$ in the corresponding cases.

It remains to check uniqueness in (6). For $D = 1394$, suppose that $1394 = r^2 + B^2$ with r, B positive integers and $7 \mid B$. Then $r^2 \equiv 13^2 \pmod{49}$ and $r < 38$, so $r = 13$ or $r = 36$. The latter would give $B^2 = 98$, which is not a square. Hence $r = 13$ and $B = 35$. For $D = 6290$, uniqueness follows from the observation after Proposition 4, since $\sqrt{6290} + 19 < 80 + 19 < 121$. All hypotheses of Proposition 4 are satisfied. □

Remark 6. *Candidates for Proposition 4 can be obtained from an odd prime p and a positive odd integer m by setting*

$$D = \left(m^2 + \frac{(p-1)^2}{4}\right)\left(m^2 + \frac{(p+1)^2}{4}\right),$$
$$u = \left|m^2 - \frac{p^2-1}{4}\right|, \qquad A = mp, \qquad h = 2m^2 + \frac{p^2+1}{2}.$$

Direct expansion gives $D = u^2 + A^2$ and $h^2 - 4D = p^2$. Also $D \equiv 2 \pmod 4$, since m is odd and $(p-1)/2, (p+1)/2$ are consecutive integers. Thus D is nonsquare and the identity required in Lemma 5 holds with $j = 2$. It remains to check $\gcd(A, 2u) = 1$, the uniqueness in (6), and the absence of integer solutions of (7). Here $\gcd(A, 2u) = 1$ is equivalent to $\gcd(m, (p^2-1)/4) = 1$ and $p \nmid (4m^2+1)$. For $p \equiv 3 \pmod 4$, the second condition is automatic. The coprimality condition also gives $p \nmid D$. The choices $(p, m) = (7, 5)$ and $(11, 7)$ recover the two examples of Theorem 1. Under the coprimality condition, $2m^2 < p^2$ is sufficient for uniqueness in (6). Indeed, this inequality gives $u \le (p^2-1)/4$, and hence $\sqrt{D} + u < h/2 + u \le m^2 + p^2/2 < p^2$. The condition $2m^2 < p^2$ holds for $(11, 7)$. For $(7, 5)$ it fails, and uniqueness is verified directly in the proof of Theorem 1.

For $(p, m) = (23, 5)$, this construction gives

$$D = 24674 = 13^2 \cdot 146, \qquad u = 107, \qquad A = 115, \qquad h = 315.$$

Here $\gcd(115, 214) = 1$. The ordinary continued fraction $\sqrt{146} = [12; \overline{12, 24}]$ has period length 2. Hence $X^2 - 146Y^2 = -1$ has no integer solution (see e.g. [6, Theorem 10.20*]). An integer solution of $U^2 - 24674V^2 = -1$ would give one of $X^2 - 146Y^2 = -1$ by taking $X = U$ and $Y = 13V$. Thus $U^2 - 24674V^2 = -1$ has no integer solution. Since $315^2 - 24674 \cdot 2^2 = 23^2$, Lemma 5 also excludes solutions over $\mathbb{Z}[1/23]$.*

The uniqueness in (6) follows from the observation after Proposition 4, since $\sqrt{24674} + 107 < 158 + 107 < 529$. Let $\sigma \in \mathbb{Q}_{23}$ satisfy $\sigma^2 = 24674$ and $\sigma \equiv 15 \pmod{23}$. Proposition 4 shows that the Browkin expansion of $(107 + \sigma)/115$ is not eventually periodic. Its minimal integer polynomial is $115X^2 - 214X - 115$, of discriminant 98696, although D is not squarefree.

Each example also yields infinitely many distinct quadratic irrationals in the same field with a common nonperiodic tail. For any of the examples z above and each $n \geq 1$, put $\xi_n = p^{-n} + 1/z$. Since $1/z \in p\mathbb{Z}_p$, we have $a(\xi_n) = p^{-n}$ and $T(\xi_n) = z$.

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Tomislav Pejković, Department of Mathematics, Faculty of Science, University of Zagreb, Bijenička cesta 30, 10000 Zagreb, Croatia
e-mail: pejkovic@math.hr