Balanced Schneider's p-adic continued fractions

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Abstract

We study a balanced version of Schneider's p-adic continued fractions for odd primes p. We establish a folding lemma, prove bounds for the length of rational number expansions, and determine the possible lengths of expansions of rational integers. The folding construction is then applied to signed lacunary series, yielding their p-adic irrationality exponents. Finally, we explicitly construct a non-periodic balanced Schneider's 3-adic continued fraction whose full sequence of convergents also converges in the reals and whose real and 3-adic limits are transcendental.

1 Introduction

Schneider's p-adic continued fraction algorithm is one of the classical continued fraction algorithms in the field of p-adic numbers [12, 13]. In its usual form, the partial denominators are chosen from the least positive residue system $\{1, \dots, p-1\}$. For rational inputs the algorithm need not terminate; this phenomenon was already observed by Bundschuh [3] and has been studied further in several later works [5, 8, 9, 10].

Inspired by the use of symmetric representatives in Browkin's modification of Ruban's algorithm [1], we consider a balanced variant of Schneider's algorithm. Throughout, p is an odd prime and $\mathcal{R}_p = \{-(p-1)/2, \dots, -1, 1, \dots, (p-1)/2\}$. Thus the representative of a p-adic unit modulo p is chosen symmetrically around zero.

There are two main advantages compared with the usual form of Schneider's algorithm. First, every rational p-adic unit has a finite balanced expansion; in Section 3 this will follow from a height estimate. Second, the symmetric set of representatives gives a simple folding lemma, proved in Section 2. There are also some disadvantages. We must exclude $p = 2$, and the signs of the partial denominators obstruct the kind of lower bounds for the real sizes of the numerators and denominators of the convergents that are available in the positive setting, compare our Theorem 4.1.

For a reduced rational number $x = u/v$, we write $H(x) = \max\{|u|, |v|\}$. Since the balanced expansion of x is finite, we denote its length by $\ell_p(x)$. In Section 3 we prove that $\ell_p(x) \leq 1 + \log_\tau H(x)$, where $\tau > 1$ is the positive root of $\tau^2 + (p-1)\tau/2 = p$. We also give explicit examples with long expansion.

A natural question is to compare two notions of length for rational numbers: the length of the usual real continued fraction expansion and the length of the balanced Schneider's p-adic continued fraction expansion. The analogous comparison for the usual Schneider's p-adic continued fractions was studied in [11]. Several constructions from that setting transfer directly to the balanced algorithm, whenever all partial denominators used in the usual Schneider expansion already belong to $\mathcal{R}_p$. The case where the real continued fraction expansion has bounded length, whereas the p-adic continued fraction length is long, does not transfer in this way. In the usual setting it is obtained from finite expansions with all partial

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denominators equal to $p-1$, which are not allowed in $\mathcal{R}_p$. In Section 4 we prove a stronger balanced substitute. If $p>3$, then for every $L \geq 1$ there exists a positive integer z such that $\ell_p(z)=L$. Thus the ordinary real continued fraction expansion has length 1, while the balanced Schneider's length can be arbitrary. The proof uses two explicit infinite balanced expansions, producing integer convergents of all odd and all even lengths. For $p=3$, the possible lengths are precisely the positive integers not divisible by 3.

Section 5 applies the balanced folding lemma to lacunary p-adic series. Bugeaud and Pejković [2] proved that, if (c_n) is a sequence of positive integers such that $c_{n+1} \geq 2c_n$ for all sufficiently large n, then the p-adic irrationality exponent satisfies $\mu(\sum_{n\geq 0} p^{c_n}) = \limsup_{n\to\infty} c_{n+1}/c_n$. Using the balanced folding lemma, we give a direct p-adic continued fraction proof of the extension with coefficients ±1.

The purely periodic expansion $[1, p:1, p:1, \ldots]$ already gives a simple case in which the convergents also converge in the reals, with real limit $(1+\sqrt{1+4p})/2$. In Section 6 we give a non-periodic example by constructing a folded balanced Schneider's 3-adic continued fraction expansion whose partial numerators are all equal to 3. Its full sequence of convergents also converges in the reals. The real and 3-adic limits are explicitly described, and both are transcendental.

2 Balanced continued fractions and a folding lemma

Recall that $\mathcal{R}_p = \{-(p-1)/2, \ldots, -1, 1, \ldots, (p-1)/2\}$. Let $|\cdot|_p$ denote the p-adic absolute value, normalized by $|p|_p = p^{-1}$. Let α be a p-adic unit, that is $\alpha \in \mathbb{Z}_p^\times$, and put $\alpha_0 = \alpha$. Once $\alpha_n \in \mathbb{Z}_p^\times$ is defined, let b_n be the unique element of $\mathcal{R}_p$ congruent to α_n modulo p. If $\alpha_n = b_n$, the algorithm terminates. Otherwise let $a_{n+1} \geq 1$ be the unique integer such that $|\alpha_n - b_n|_p = p^{-a_{n+1}}$, and put $\alpha_{n+1} = p^{a_{n+1}}/(\alpha_n - b_n)$. This gives a finite or infinite expansion

$$\alpha = [b_0, p^{a_1} : b_1, p^{a_2} : b_2, \ldots],$$

which we call the balanced Schneider's p-adic continued fraction expansion of α. A finite expansion is evaluated by

$$\alpha = [b_0, p^{a_1} : b_1, \ldots, p^{a_n} : b_n] = b_0 + \cfrac{p^{a_1}}{b_1 + \cfrac{p^{a_2}}{\ddots + \cfrac{p^{a_n}}{b_n}}},$$

and its length is denoted by $\ell_p(\alpha) = n+1$. For such an expansion we define convergents P_j/Q_j by

$$P_{-1} = 1, \quad P_0 = b_0, \qquad Q_{-1} = 0, \quad Q_0 = 1, \tag{1}$$

and, for $j \geq 1$,

$$P_j = b_j P_{j-1} + p^{a_j} P_{j-2}, \qquad Q_j = b_j Q_{j-1} + p^{a_j} Q_{j-2}. \tag{2}$$

Then

$$[b_0, p^{a_1} : b_1, \ldots, p^{a_j} : b_j] = \frac{P_j}{Q_j}.$$

Every finite expression $[b_0, p^{a_1} : b_1, \ldots, p^{a_n} : b_n]$, with $b_j \in \mathcal{R}_p$ and $a_j \geq 1$, is the balanced expansion of its value. Indeed, each tail beginning with b_j is of the form $b_j + p^{a_{j+1}}/u$, where $u \in \mathbb{Z}_p^\times$, except for the last tail, which is equal to b_n. Hence the algorithm recovers the displayed partial denominators $(b_j)_{0\leq j\leq n}$ and exponents $(a_j)_{1\leq j\leq n}$.

As in the case of usual Schneider's p-adic continued fractions, the recurrence relations give

$$P_j Q_{j-1} - P_{j-1} Q_j = (-1)^{j-1} p^{a_1+\cdots+a_j} \qquad (j \geq 1), \tag{3}$$

see [8]. Also, $p \nmid P_j Q_j$ for every $j \geq 0$. This follows by induction from

$$P_j \equiv b_j P_{j-1} \pmod{p}, \qquad Q_j \equiv b_j Q_{j-1} \pmod{p},$$

since $b_j \in \mathcal{R}_p$.

For an infinite balanced expansion, the convergents converge in $\mathbb{Q}_p$. More precisely, if $\alpha = [b_0, p^{a_1} : b_1, \dots]$ and P_j/Q_j is its j-th convergent, then, using (3) as in [8], we obtain

$$\left| \alpha - \frac{P_j}{Q_j} \right|_p = p^{-(a_1+\cdots+a_{j+1})}. \tag{4}$$

Lemma 2.1 (Balanced folding lemma)**.** *Let $m \geq 1$, and let*

$$[b_0, p^{a_1} : b_1, \dots, p^{a_m} : b_m] = \frac{P_m}{Q_m},$$

where $b_i \in \mathcal{R}_p$. Then, for every positive integer t and every $c \in \mathcal{R}_p$,

$$\begin{aligned} &[b_0, p^{a_1} : b_1, \dots, p^{a_m} : b_m, p^t : c, p^t : (-b_m), p^{a_m} : (-b_{m-1}), \dots, p^{a_2} : (-b_1)] \\ &\quad = \frac{P_m}{Q_m} + (-1)^m \frac{p^{a_1+\cdots+a_m+t}}{cQ_m^2}. \end{aligned} \tag{5}$$

Proof. A direct induction using the denominator recurrence gives

$$[-b_m, p^{a_m} : (-b_{m-1}), \dots, p^{a_2} : (-b_1)] = -\frac{Q_m}{Q_{m-1}}.$$

Hence the complete quotient beginning with c is

$$[c, p^t : (-b_m), p^{a_m} : (-b_{m-1}), \dots, p^{a_2} : (-b_1)] = c - p^t \frac{Q_{m-1}}{Q_m}.$$

The recurrence formulas give the linear fractional identity

$$[b_0, \dots, p^{a_m} : b_m, p^t : x] = \frac{xP_m + p^t P_{m-1}}{xQ_m + p^t Q_{m-1}}.$$

Substituting $x = c - p^t Q_{m-1}/Q_m$, by (3), we obtain

$$\frac{cP_m + p^t(P_{m-1}Q_m - P_m Q_{m-1})/Q_m}{cQ_m} = \frac{P_m}{Q_m} + (-1)^m \frac{p^{a_1+\cdots+a_m+t}}{cQ_m^2}.$$

This gives precisely the claimed identity. □

3 Length and height

The finiteness of the balanced Schneider's continued fraction expansion for rational numbers can be proved by a direct descent argument, analogous to the one in Miller's dissertation [7, Theorem 19]. We shall instead deduce it from the stronger height estimate below.

Theorem 3.1. *Let $\tau > 1$ be the positive root of*

$$\tau^2 + \frac{p-1}{2}\tau = p. \tag{6}$$

Equivalently,

$$\tau = \frac{\sqrt{p^2+14p+1}-p+1}{4}.$$

Every $x \in \mathbb{Q} \cap \mathbb{Z}_p^\times$ has a finite balanced Schneider's continued fraction expansion. If $\ell_p(x)$ denotes its length, then

$$H(x) \geq \tau^{\ell_p(x)-1}.$$

Thus

$$\ell_p(x) \leq 1 + \log_\tau H(x) = 1 + \frac{\log p}{\log \tau} \log_p H(x).$$

Proof. Let $u/v \in \mathbb{Q} \cap \mathbb{Z}_p^\times$ be written in lowest terms, and put $W(u/v) = \max\{|u|, \tau|v|\}$. Suppose that the algorithm does not terminate at u/v, and let $b \in \mathcal{R}_p$ and $a \geq 1$ be the next partial denominator and exponent. Now $u - bv = p^a w$, where w is an integer not divisible by p and $\gcd(v, w) = 1$, so

$$\frac{u}{v} = b + \frac{p^a}{v/w}$$

and the next complete quotient is v/w, written in lowest terms.

We claim that

$$W(u/v) \geq \tau W(v/w). \tag{7}$$

If $|v| \geq \tau|w|$, then $W(v/w) = |v|$, and so $W(u/v) \geq \tau|v| = \tau W(v/w)$. If $|v| \leq \tau|w|$, then, using $|b| \leq (p-1)/2$ and $a \geq 1$,

$$|u| = |bv + p^a w| \geq p|w| - \frac{p-1}{2}|v| \geq \left(p - \frac{p-1}{2}\tau\right)|w| = \tau^2|w|.$$

Since $W(v/w) = \tau|w|$ in this case, this again gives $W(u/v) \geq \tau W(v/w)$.

Now let $x \in \mathbb{Q} \cap \mathbb{Z}_p^\times$. If the algorithm did not terminate, then starting with $\alpha_0 = x$ all complete quotients α_j would satisfy $W(\alpha_j) \leq \tau^{-j} W(\alpha_0)$ by (7). Since $W(\alpha_j) \geq \tau$, this is impossible for large j, so the expansion of x is finite.

Let its length be m. The last complete quotient is $b_{m-1} \in \mathcal{R}_p$, so $W(\alpha_{m-1}) = \max\{|b_{m-1}|, \tau\} \geq \tau$. Iterating (7) backwards and using $W(x) \leq \tau H(x)$, we obtain $\tau H(x) \geq W(x) \geq \tau^{m-1} W(\alpha_{m-1}) \geq \tau^m$, hence $H(x) \geq \tau^{m-1}$. This proves the theorem. ☐

The next result gives a family of rational numbers for which the length of the balanced Schneider's p-adic continued fraction expansion grows at least linearly in $\log_p H$.

Proposition 3.2. *For $k \geq 1$, let*

$$\xi_k = [-1, p : 1, p : -1, p : 1, \ldots, p : -1, p : 1],$$

where all partial numerators are equal to p, and the partial denominators -1 and 1 each occur k times. Then $\ell_p(\xi_k) = 2k$ and

$$H(\xi_k) \leq \frac{4p}{\sqrt{4p-1}} p^k.$$

Consequently,

$$\ell_p(\xi_k) \geq 2 \log_p H(\xi_k) - 2 \log_p \left(\frac{4p}{\sqrt{4p-1}}\right).$$

Proof. Write $\xi_k = A_k/B_k$ in lowest terms. The recurrence

$$\xi_{k+1} = -1 + \cfrac{p}{1 + \cfrac{p}{\xi_k}}$$

gives

$$A_{k+1} = (p-1)A_k - pB_k, \qquad B_{k+1} = A_k + pB_k,$$

with $A_1 = p-1$ and $B_1 = 1$. Hence

$$B_{k+2} = (2p-1)B_{k+1} - p^2 B_k.$$

The characteristic roots are

$$\lambda_\pm = \frac{2p-1 \pm i\sqrt{4p-1}}{2},$$

so $|\lambda_\pm| = p$ and $\lambda_+ - \lambda_- = i\sqrt{4p-1}$. A direct check gives

$$B_k = \frac{\lambda_+^k - \lambda_-^k}{\lambda_+ - \lambda_-},$$

whence

$$|B_k| \le \frac{2p^k}{\sqrt{4p-1}}.$$

Since $A_k = B_{k+1} - pB_k$, we obtain

$$|A_k| \le \frac{4p^{k+1}}{\sqrt{4p-1}}.$$

The stated height estimate follows. □

The following elementary estimate is included to make the constants in the bounds on length in Theorem 3.1 and Proposition 3.2 easier to compare.

Proposition 3.3. *If $p \ge 7$, then*

$$2 - \frac{6}{p} < \tau < 2.$$

Consequently,

$$\frac{\log p}{\log 2} < \frac{\log p}{\log \tau} < \frac{\log p}{\log(2-6/p)}.$$

For the two remaining small primes one has

$$\tau = \frac{\sqrt{13}-1}{2}, \qquad \frac{\log 3}{\log \tau} \approx 4.154$$

when $p = 3$, and

$$\tau = \sqrt{6} - 1, \qquad \frac{\log 5}{\log \tau} \approx 4.336$$

when $p = 5$.

Proof. The upper bound $\tau < 2$ follows from

$$\sqrt{p^2 + 14p + 1} < p + 7.$$

For $p \ge 7$ we have

$$p^2 + 14p + 1 - \left(p + 7 - \frac{24}{p}\right)^2 = \frac{48(7p-12)}{p^2} > 0.$$

Therefore

$$\sqrt{p^2 + 14p + 1} > p + 7 - \frac{24}{p},$$

and the explicit formula for τ gives $\tau > 2 - 6/p$. The displayed estimates for $p = 3$ and $p = 5$ follow by substitution in the same formula. □

4 Lengths of balanced expansions for integers

After the general height estimates, we turn to examples comparing two lengths attached to a rational number, namely the length of its balanced Schneider's p-adic continued fraction expansion and the length of its ordinary real continued fraction expansion. The analogous comparison for usual Schneider's p-adic continued fractions was studied in [11]. Some of the constructions from that paper transfer directly to the balanced setting, namely those in which all partial denominators already belong to $\mathcal{R}_p$. The case which is not covered by this transfer is the construction of rational numbers with long p-adic expansions and real continued fraction expansions of length at most 3. In the usual Schneider setting this is obtained by using partial denominators equal to $p-1$, which are not allowed in the balanced algorithm. We prove instead the stronger statement that, for $p>3$, every prescribed positive length occurs as the balanced Schneider's p-adic expansion length of a positive integer.

Theorem 4.1. *Let $p>3$ be a prime. For every $L \geq 1$ there exists a positive integer z, not divisible by p, such that $\ell_p(z) = L$.*

Proof. Put $c = (p-1)/2$. Since $p>3$, the elements $1, 2, -2, c, -c$ all lie in $\mathcal{R}_p$.

First consider the infinite balanced expansion

$$\alpha = [1, p : c, p : -2, p : c, p^2 : -2, p : c, p^3 : -2, \ldots],$$

where after $b_0 = 1$ the block $p : c, p^{n+1} : -2$ is inserted for $n = 0, 1, 2, \ldots$. Let P_j/Q_j be its convergents. We claim that, for every $n \geq 0$, one has $Q_{2n} = 1$ and $Q_{2n+1} = (p^{n+1}-1)/2$. The case $n=0$ is clear. If $Q_{2n} = 1$ and $Q_{2n-1} = (p^n-1)/2$, then

$$Q_{2n+1} = cQ_{2n} + pQ_{2n-1} = \frac{p^{n+1}-1}{2},$$

and

$$Q_{2n+2} = -2Q_{2n+1} + p^{n+1}Q_{2n} = 1.$$

Thus every convergent P_{2n}/Q_{2n} is an integer, and it has length $2n+1$. This realizes all odd lengths, at least up to a possible change of sign.

For even lengths, consider the infinite expansion

$$\beta = [1, p : 1, p : -c, p : -2, p : c, p^2 : 2, p : -c, p^3 : -2, p : c, p^4 : 2, \ldots].$$

Equivalently, $a_1 = 1$, $b_1 = 1$, and for $n \geq 1$ we take $a_{2n} = 1$, $b_{2n} = (-1)^n c$, $a_{2n+1} = n$, and $b_{2n+1} = 2(-1)^n$. Again, let P_j/Q_j be the convergents of β. We claim that, for every $n \geq 1$,

$$Q_{2n-1} = (-1)^{n-1}, \qquad Q_{2n} = \frac{p^n+1}{2}.$$

For $n=1$ this gives $Q_1 = 1$ and $Q_2 = -c + p = (p+1)/2$. Suppose that the assertion holds for some $n \geq 1$. Then

$$Q_{2n+1} = 2(-1)^n Q_{2n} + p^n Q_{2n-1} = (-1)^n,$$

and

$$Q_{2n+2} = (-1)^{n+1} c Q_{2n+1} + pQ_{2n} = \frac{p^{n+1}+1}{2}.$$

This proves the claim by induction. Hence every convergent P_{2n-1}/Q_{2n-1} is an integer, and it has length $2n$. This realizes all even lengths, again up to sign.

It remains only to obtain positive integers. If $z = [b_0, p^{a_1} : b_1, \dots, p^{a_m} : b_m]$, then

$$-z = [-b_0, p^{a_1} : (-b_1), \dots, p^{a_m} : (-b_m)]$$

is a balanced expansion of the same length. Thus, whenever one of the integers constructed above is negative, replacing all partial denominators by their negatives gives a positive integer of the same balanced length. Since all convergents of a balanced expansion are p-adic units, the resulting integer is not divisible by p. ☐

The proof gives a little more than the mere realization of lengths.

Proposition 4.2. *Let $p > 3$. There is an infinite balanced Schneider's p-adic continued fraction for which every second convergent is an integer. This is best possible, in the sense that, apart from the initial pair, two consecutive convergents of a balanced expansion cannot both be integers.*

Proof. The expansion α in the proof of Theorem 4.1 satisfies $Q_{2n} = 1$ for every $n \geq 0$, so each even-indexed convergent is an integer.

Conversely, suppose that two consecutive convergents P_j/Q_j and P_{j+1}/Q_{j+1}, with $j \geq 1$, are integers. By (3) and the fact that Q_i is a p-adic unit, we have $\gcd(P_i, Q_i) = 1$ for every i. Since P_j/Q_j and P_{j+1}/Q_{j+1} are integers, this forces $Q_j = \pm 1$ and $Q_{j+1} = \pm 1$. From the recurrence

$$Q_{j+1} = b_{j+1} Q_j + p^{a_{j+1}} Q_{j-1}$$

we get

$$p^{a_{j+1}} Q_{j-1} = Q_{j+1} - b_{j+1} Q_j.$$

The right-hand side is an integer of absolute value at most $1 + (p-1)/2 < p$. Since it is divisible by p, it is zero. Hence $Q_{j-1} = 0$, impossible for $j \geq 1$. Thus no two consecutive convergents from index 1 onward can both be integers. Hence the construction above is best possible. ☐

For $p = 3$, the statement of Theorem 4.1 no longer holds, but the set of possible lengths can still be described exactly.

Proposition 4.3. *Let $p = 3$.*

(i) *If a positive integer has a balanced Schneider's 3-adic continued fraction of length L, then $3 \nmid L$.*

(ii) *Conversely, for every positive integer L with $3 \nmid L$, there exists a positive integer z such that $\ell_3(z) = L$.*

Consequently, the set of lengths of balanced Schneider's 3-adic continued fractions of positive integers is exactly $\mathbb{N} \setminus 3\mathbb{N}$.

Proof. For $p = 3$ one has $\mathcal{R}_3 = \{-1, 1\}$.

Proof of (i). Let

$$z = [b_0, 3^{a_1} : b_1, \dots, 3^{a_n} : b_n]$$

be a positive integer. The case $n = 0$, i.e. $L = 1$, is immediate, so we may assume $n \geq 1$. The denominators satisfy $Q_j = b_j Q_{j-1} + 3^{a_j} Q_{j-2} \equiv Q_{j-1} + Q_{j-2} \pmod 2$, because b_j and 3^{a_j} are odd. Thus $Q_0, Q_1, Q_2, Q_3, \dots \equiv 1, 1, 0, 1, 1, 0, \dots \pmod 2$.

Since $z = P_n/Q_n$ is an integer, we have $Q_n \mid P_n$. By identity (3), every common divisor of P_n and Q_n divides a power of 3. Since Q_n is not divisible by 3, it follows that $Q_n = \pm 1$. Hence Q_n is odd, so $n \not\equiv 2 \pmod 3$. The length is $L = n + 1$, and therefore $3 \nmid L$.

Proof of (ii). The length 1 is realized by the integer 1. We shall construct integers whose balanced Schneider's 3-adic continued fraction expansions have lengths $2 + 3m$ and $4 + 3m$ for every $m \geq 0$.

We first record the denominator step used in both constructions. Suppose that a finite balanced Schneider's 3-adic continued fraction expansion has last two denominators $Q_{n-1} = v$ and $Q_n = 1$. If we continue this expansion by the three pairs $3 : (-1)$, $3^r : 1$, $3 : (-1)$, then the next denominators are

$$Q_{n+1} = 3v - 1, \qquad Q_{n+2} = 3v - 1 + 3^r, \qquad Q_{n+3} = 6v - 2 - 3^r.$$

Thus, if $3^r = 6v - 3$, then $Q_{n+2} = 9v - 4$ and $Q_{n+3} = 1$.

For lengths congruent to 2 modulo 3, start with

$$[-1, 3 : 1].$$

It has length 2 and last two denominators $Q_0 = 1$, $Q_1 = 1$. Suppose that, for some $m \geq 0$, we have constructed an expansion of length $2 + 3m$ whose last two denominators are

$$Q_{3m} = \frac{3^{2m} + 1}{2}, \qquad Q_{3m+1} = 1.$$

Continue it by $3 : (-1)$, $3^{2m+1} : 1$, $3 : (-1)$. Since $3^{2m+1} = 6(3^{2m} + 1)/2 - 3$, the step explained above gives an expansion of length $2 + 3(m + 1)$ whose last two denominators are

$$Q_{3m+3} = \frac{3^{2m+2} + 1}{2}, \qquad Q_{3m+4} = 1.$$

Hence every length $2 + 3m$ is obtained.

For lengths congruent to 1 modulo 3, apart from length 1, start with

$$[1, 3 : 1, 3 : (-1), 3 : (-1)].$$

It has length 4 and last two denominators $Q_2 = 2$, $Q_3 = 1$. Suppose that, for some $m \geq 0$, we have constructed an expansion of length $4 + 3m$ whose last two denominators are

$$Q_{3m+2} = \frac{3^{2m+1} + 1}{2}, \qquad Q_{3m+3} = 1.$$

Continue it by $3 : (-1)$, $3^{2m+2} : 1$, $3 : (-1)$. Since $3^{2m+2} = 6(3^{2m+1} + 1)/2 - 3$, the same step as before gives an expansion of length $4 + 3(m + 1)$ whose last two denominators are

$$Q_{3m+5} = \frac{3^{2m+3} + 1}{2}, \qquad Q_{3m+6} = 1.$$

Hence every length $4 + 3m$ is obtained.

In all these cases the last denominator is 1, so the value is an integer. Finally, changing all partial denominators to their negatives changes the value to its negative and preserves the length. Thus, if one of the constructed integers is negative, replacing all partial denominators by their negatives gives a positive integer of the same length. This completes the proof. $\square$

5 p-adic numbers with prescribed irrationality exponent

We now apply the folding lemma to a signed lacunary series

$$\xi_{\varepsilon,\mathbf{c}} = \sum_{n \geq 0} \varepsilon_n p^{c_n}, \qquad \varepsilon_n \in \{-1, 1\}.$$

For a p-adic irrational number ξ, its irrationality exponent is

$$\mu(\xi) = \sup\left\{\mu : \left|\xi - \frac{a}{b}\right|_p < H(a/b)^{-\mu} \text{ for infinitely many } a/b \in \mathbb{Q}\right\}.$$

Adding a rational number to ξ does not change $\mu(\xi)$.

We shall use the following elementary approximation lemma.

Lemma 5.1. *If x and y are distinct rational numbers, then*

$$|x - y|_p \geq \frac{1}{2H(x)H(y)}.$$

Proof. Write $x = a/b$ and $y = m/n$ in lowest terms. Then

$$|x - y|_p = \left|\frac{an - bm}{bn}\right|_p \geq |an - bm|_p \geq |an - bm|^{-1} \geq \frac{1}{2H(x)H(y)},$$

since $|bn|_p \leq 1$ and $an - bm$ is a nonzero integer satisfying $|an - bm| \leq 2H(x)H(y)$. □

The next criterion gives an upper bound for the p-adic irrationality exponent from controlled rational approximants.

Lemma 5.2. *Let $\xi \in \mathbb{Q}_p$, let (x_k) be a sequence of distinct rational numbers, and let (M_k) be a sequence of positive real numbers tending to $+\infty$. Assume that, for some $0 < \eta < 1$, one has*

$$M_k^{1-\eta} \leq H(x_k) \leq M_k^{1+\eta}$$

for all sufficiently large k, and that

$$|\xi - x_k|_p = M_k^{-\theta_k}$$

for a sequence (θ_k) satisfying $\liminf_{k\to\infty} \theta_k > 1 + \eta$. Then

$$\mu(\xi) \leq \max\left\{\frac{1}{1-\eta} \limsup_{k\to\infty} \theta_k,\ 1 + (1+\eta) \limsup_{k\to\infty} \frac{\log M_{k+1}}{(\theta_k - 1 - \eta)\log M_k}\right\}.$$

Proof. We use a standard gap argument. If the maximum on the right is infinite, there is nothing to prove. Otherwise, let τ be larger than this maximum. Then $(1-\eta)\tau > \limsup \theta_k$, and hence, for all sufficiently large k,

$$|\xi - x_k|_p = M_k^{-\theta_k} > M_k^{-(1-\eta)\tau} \geq H(x_k)^{-\tau}.$$

Thus only finitely many of the approximants x_k satisfy $|\xi - x_k|_p < H(x_k)^{-\tau}$.

It remains to exclude the rational approximants which do not occur in the sequence (x_k). Suppose, to the contrary, that there are infinitely many such rational numbers y with $|\xi - y|_p < H(y)^{-\tau}$. We may take y with arbitrarily large height. Let k be the largest index such that

$$(2M_k^{1+\eta})^{1/(\tau-1)} < H(y).$$

Then $k \to \infty$ as $H(y) \to \infty$, and the maximality of k gives

$$H(y) \leq (2M_{k+1}^{1+\eta})^{1/(\tau-1)}.$$

The defining inequality for k also gives

$$|\xi - y|_p < H(y)^{-\tau} < \frac{1}{2M_k^{1+\eta}H(y)} \leq \frac{1}{2H(x_k)H(y)} \leq |x_k - y|_p,$$

where the last inequality is by Lemma 5.1. By the ultrametric inequality,

$$\frac{1}{2H(x_k)H(y)} \leq |x_k - y|_p \leq \max\{|\xi - x_k|_p, |\xi - y|_p\} = |\xi - x_k|_p = M_k^{-\theta_k}.$$

Using $H(x_k) \leq M_k^{1+\eta}$ and the upper bound for $H(y)$, we get

$$\frac{1}{2} M_k^{\theta_k - 1 - \eta} \leq H(y) \leq (2M_{k+1}^{1+\eta})^{1/(\tau-1)}.$$

Taking logarithms gives

$$(\theta_k - 1 - \eta)\log M_k - \log 2 \leq \frac{\log 2 + (1+\eta)\log M_{k+1}}{\tau - 1}.$$

Equivalently,

$$\tau \leq 1 + \frac{(1+\eta)\log M_{k+1} + \tau \log 2}{(\theta_k - 1 - \eta)\log M_k}.$$

Taking the limsup along these indices k, using $\liminf \theta_k > 1 + \eta$ and $M_k \to +\infty$, gives

$$\tau \leq 1 + (1+\eta) \limsup_{k\to\infty} \frac{\log M_{k+1}}{(\theta_k - 1 - \eta)\log M_k},$$

contrary to the choice of τ. Therefore only finitely many rational numbers y satisfy $|\xi - y|_p < H(y)^{-\tau}$. Since this holds for every τ larger than the displayed maximum, the asserted upper bound follows. □

Theorem 5.3. *Let p be an odd prime, let $(\varepsilon_n)_{n\geq 0}$ be a sequence of elements in $\{-1, 1\}$, and let $(c_n)_{n\geq 0}$ be a sequence of positive integers such that*

$$c_{n+1} \geq 2c_n$$

for all sufficiently large n. Then

$$\mu\left(\sum_{n\geq 0} \varepsilon_n p^{c_n}\right) = \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}.$$

Proof. Let

$$L = \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}$$

and fix $0 < \eta < 1$. Deleting finitely many initial terms and adding a rational number do not change the irrationality exponent, nor the limsup L. We may therefore assume that c_0 is sufficiently large for the estimates below and that $c_{n+1} \geq 2c_n$ for all $n \geq 0$. It suffices to prove the result for

$$\xi = 1 + \sum_{n\geq 0} \varepsilon_n p^{c_n}.$$

For $n \geq 0$, we construct a finite balanced Schneider's p-adic continued fraction expansion Z_n such that

$$Z_n = 1 + \sum_{k=0}^{n} \varepsilon_k p^{c_k}.$$

Start with

$$Z_0 = [1, p^{c_0} : \varepsilon_0] = 1 + \varepsilon_0 p^{c_0}.$$

Suppose that

$$Z_n = [b_0, p^{a_1} : b_1, \dots, p^{a_m} : b_m] = 1 + \sum_{k=0}^{n} \varepsilon_k p^{c_k},$$

that $a_1 + \dots + a_m = 2c_n - c_0$, and that m is odd. Put $t = c_{n+1} - 2c_n + c_0$, so that $t \geq c_0 \geq 1$, and fold Z_n with central partial denominator $-\varepsilon_{n+1}$. Since Z_n is an integer, the denominator of its last convergent is ± 1, so the factor Q_m^2 appearing in Lemma 2.1 equals 1. As m is odd, the lemma shows that the extended expansion, which we denote by Z_{n+1}, satisfies

$$\begin{aligned} Z_{n+1} &= [b_0, p^{a_1} : b_1, \dots, p^{a_m} : b_m, p^t : -\varepsilon_{n+1}, p^t : -b_m, p^{a_m} : -b_{m-1}, \dots, p^{a_2} : -b_1] \\ &= 1 + \sum_{k=0}^{n} \varepsilon_k p^{c_k} + \varepsilon_{n+1} p^{2c_n - c_0 + t} = 1 + \sum_{k=0}^{n+1} \varepsilon_k p^{c_k}. \end{aligned}$$

The exponents of p in the partial numerators of Z_{n+1} sum to $2(2c_n - c_0) - c_0 + 2t = 2c_{n+1} - c_0$, because the first exponent remains c_0. Their number is $2m+1$, again odd. Thus the construction gives the required finite expansions Z_n for every n.

Since, at each step, the continued fraction for Z_{n+1} extends that for Z_n, these finite continued fractions are initial segments of an infinite balanced Schneider's p-adic continued fraction expansion. The convergents obtained at the ends of these segments are the rational numbers $Z_n = 1 + \sum_{k=0}^{n} \varepsilon_k p^{c_k}$, which tend to ξ in $\mathbb{Q}_p$. Hence the infinite continued fraction expansion represents ξ.

Let P_j/Q_j be its convergents, and define

$$\lambda_0 = 0, \qquad \lambda_{2r+1} = a_1 + a_3 + \dots + a_{2r+1}, \qquad \lambda_{2r} = a_2 + a_4 + \dots + a_{2r}.$$

With $\lambda_{-1} = 0$, we have $a_j = \lambda_j - \lambda_{j-2}$ and $a_1 + \dots + a_{j+1} = \lambda_j + \lambda_{j+1}$.

We next consider the initial part of (λ_j) up to and including the convergent equal to Z_n. Denote this finite list by Λ_n. For $n = 0$, we have $\Lambda_0 = (0, c_0)$. Suppose that, for some n, the corresponding list is

$$\Lambda_n = (\lambda_0, \lambda_1, \dots, \lambda_m), \qquad 0 = \lambda_0 < \lambda_1 \leq \dots \leq \lambda_m = c_n.$$

After applying the folding step from Z_n to Z_{n+1}, the new exponents are $t, t, a_m, a_{m-1}, \dots, a_2$. Since the sum of the exponents of p in the partial numerators of Z_n is $2c_n - c_0$ and $\lambda_m = c_n$, the other parity sum is $\lambda_{m-1} = c_n - c_0$. Hence the first two new entries of (λ_j) are

$$\lambda_{m+1} = \lambda_{m-1} + t = c_{n+1} - \lambda_m, \qquad \lambda_{m+2} = \lambda_m + t = c_{n+1} - \lambda_{m-1}.$$

For the remaining reflected exponents $a_m, a_{m-1}, \dots, a_2$, the relation $a_j = \lambda_j - \lambda_{j-2}$ gives the same pattern successively. Thus the entries added to this list, up to and including the convergent equal to Z_{n+1}, are

$$c_{n+1} - \lambda_m,\ c_{n+1} - \lambda_{m-1}, \dots, c_{n+1} - \lambda_1,\ c_{n+1}.$$

Thus

$$\Lambda_{n+1} = (\lambda_0, \lambda_1, \dots, \lambda_m, c_{n+1} - \lambda_m, c_{n+1} - \lambda_{m-1}, \dots, c_{n+1} - \lambda_1, c_{n+1}).$$

The added entries are nondecreasing in this order, and the first of them satisfies $c_{n+1} - \lambda_m = c_{n+1} - c_n \geq c_n = \lambda_m$. Hence the enlarged list is again nondecreasing and ends in c_{n+1}. This proves by induction that (λ_j) is nondecreasing and tends to infinity.

It remains to determine the limsup of the ratios λ_{j+1}/λ_j, with $j \geq 1$. We use the gaps between consecutive entries of the lists Λ_n. If u and $u+d$ are consecutive entries in some Λ_n, we call d the corresponding gap. For $u > 0$, the associated ratio is $(u+d)/u = 1 + d/u$.

From the displayed form of Λ_{n+1} in terms of Λ_n, the gaps in Λ_{n+1} are the old gaps from Λ_n, then the new junction gap $c_{n+1} - 2c_n$, and then the old gaps again, in reverse order. Hence every gap which occurs in the construction is either the initial gap c_0, or one of the gaps $d_k := c_{k+1} - 2c_k$, $k \geq 0$.

The gap $d_n = c_{n+1} - 2c_n$ first occurs with left endpoint c_n. It therefore gives the junction ratio

$$1 + \frac{d_n}{c_n} = \frac{c_{n+1} - c_n}{c_n} = \frac{c_{n+1}}{c_n} - 1.$$

These ratios occur in the sequence $(\lambda_{j+1}/\lambda_j)$. Hence

$$\limsup_{j\to\infty} \frac{\lambda_{j+1}}{\lambda_j} \geq L - 1.$$

We prove the reverse inequality. If $L = +\infty$, there is nothing to prove. Assume $L < +\infty$, and fix $\delta > 0$. Choose K such that

$$\frac{c_{k+1}}{c_k} - 1 \leq L - 1 + \delta$$

for all $k \geq K$.

First consider a ratio arising from a gap $d_k = c_{k+1} - 2c_k$ with $k \geq K$. Every occurrence of this gap has left endpoint at least c_k. Indeed, the first occurrence has left endpoint c_k, and later occurrences are obtained either by retaining an occurrence already present or by reflection; in the reflected case the left endpoint is at least $c_{n+1} - c_n \geq c_n \geq c_k$, if the occurrence is introduced in the passage from Λ_n to Λ_{n+1}. Therefore every ratio arising from this gap is at most

$$1 + \frac{d_k}{c_k} = \frac{c_{k+1}}{c_k} - 1 \leq L - 1 + \delta.$$

It remains to consider the finitely many gaps $c_0, d_0, d_1, \ldots, d_{K-1}$. The ratios arising from occurrences of these gaps already present in the fixed finite list Λ_K contribute only finitely many terms. Any occurrence of one of these gaps which is introduced after Λ_K appears in a reflected part. If it is introduced in the passage from Λ_n to Λ_{n+1}, then its left endpoint is at least $c_{n+1} - c_n \geq c_n$. Since the gap belongs to a fixed finite set and $c_n \to +\infty$, all such later ratios tend uniformly to 1. Since $L \geq 2$, we have $L - 1 \geq 1$, and therefore all sufficiently late ratios of this kind are at most $L - 1 + \delta$.

Thus, apart from finitely many ratios, every ratio λ_{j+1}/λ_j is at most $L - 1 + \delta$. Hence

$$\limsup_{j\to\infty} \frac{\lambda_{j+1}}{\lambda_j} \leq L - 1 + \delta.$$

Letting $\delta \to 0$, and combining this with the lower bound from the junction ratios, gives

$$\limsup_{j\to\infty} \frac{\lambda_{j+1}}{\lambda_j} = \limsup_{n\to\infty} \left(\frac{c_{n+1}}{c_n} - 1 \right) = L - 1. \tag{8}$$

We next estimate the heights of the convergents. Put $R_j = \max\{|P_j|, |Q_j|\}$. By (3), and because neither P_j nor Q_j is divisible by p, the integers P_j and Q_j are coprime, so $H(P_j/Q_j) = R_j$. We claim that

$$R_j \leq F_{j+2} p^{\lambda_j}, \tag{9}$$

where $F_0 = 0$, $F_1 = 1$, and $F_{j+2} = F_{j+1} + F_j$. This is clear for $j = 0, 1$. For $j \geq 2$, the recurrences, $|b_j| = 1$, $a_j = \lambda_j - \lambda_{j-2}$, and the monotonicity of (λ_j) give

$$R_j \leq R_{j-1} + p^{a_j} R_{j-2} \leq F_{j+1} p^{\lambda_{j-1}} + F_j p^{a_j + \lambda_{j-2}} \leq F_{j+2} p^{\lambda_j}.$$

All exponents of p in the partial numerators of the constructed continued fraction expansion of ξ are at least c_0, and hence $\lambda_j \geq jc_0/2$ for $j \geq 1$. Taking c_0 sufficiently large, (9) gives

$$H\left(\frac{P_j}{Q_j}\right) = R_j \leq p^{(1+\eta/2)\lambda_j} \tag{10}$$

for all $j \geq 1$.

Conversely, identity (3) gives

$$p^{\lambda_{j-1}+\lambda_j} = |P_jQ_{j-1} - P_{j-1}Q_j| \leq 2R_jR_{j-1}.$$

Using (10) for R_{j-1}, and then $\lambda_{j-1} \leq \lambda_j$, we obtain

$$R_j \geq \frac{1}{2}p^{\lambda_j-(\eta/2)\lambda_{j-1}} \geq \frac{1}{2}p^{(1-\eta/2)\lambda_j}.$$

Hence, for all sufficiently large j,

$$p^{(1-\eta)\lambda_j} \leq H\left(\frac{P_j}{Q_j}\right) \leq p^{(1+\eta)\lambda_j}. \tag{11}$$

By (4),

$$\left|\xi - \frac{P_j}{Q_j}\right|_p = p^{-(\lambda_j+\lambda_{j+1})} = (p^{\lambda_j})^{-(1+\lambda_{j+1}/\lambda_j)}.$$

Apply Lemma 5.2 with $x_j = P_j/Q_j$, $M_j = p^{\lambda_j}$, and $\theta_j = 1 + \lambda_{j+1}/\lambda_j$, after discarding the initial index with $\lambda_j = 0$. From (8) we get $\limsup \theta_j = L$. Moreover, since $\lambda_{j+1}/\lambda_j \geq 1$,

$$1 + (1+\eta)\limsup_{j\to\infty} \frac{\log M_{j+1}}{(\theta_j - 1 - \eta)\log M_j} \leq 1 + \frac{1+\eta}{1-\eta}.$$

Lemma 5.2 therefore gives

$$\mu(\xi) \leq \max\left\{\frac{L}{1-\eta}, 1 + \frac{1+\eta}{1-\eta}\right\}.$$

If $L < +\infty$, then letting η tend to 0 gives $\mu(\xi) \leq L$, since $L \geq 2$. If $L = +\infty$, the desired upper bound is void.

For the reverse inequality, use the rational integers $1 + \sum_{k=0}^{n} \varepsilon_k p^{c_k}$. Since $c_{n+1} \geq 2c_n$, the last real term dominates the preceding ones, so for every $\eta > 0$ and all sufficiently large n,

$$p^{(1-\eta)c_n} \leq H\left(1 + \sum_{k=0}^{n} \varepsilon_k p^{c_k}\right) \leq p^{(1+\eta)c_n}.$$

Also

$$\left|\xi - \left(1 + \sum_{k=0}^{n} \varepsilon_k p^{c_k}\right)\right|_p = p^{-c_{n+1}}.$$

It follows that

$$\mu(\xi) \geq \frac{1}{1+\eta}\limsup_{n\to\infty} \frac{c_{n+1}}{c_n}.$$

Letting η tend to 0 gives $\mu(\xi) \geq L$. This proves the theorem. □

The case $\varepsilon_n = 1$ for all n is the theorem of Bugeaud and Pejković [2]. Thus Theorem 5.3 gives its signed extension, and the preceding proof obtains this extension directly from the balanced folding construction, without passing through continued fractions in the field of formal Laurent series.

6 An explicit 3-adic construction with real convergence

The purely periodic expansion $[1, p : 1, p : 1, \ldots]$ has real limit $(1 + \sqrt{1 + 4p})/2$. The following example is different: it is non-periodic, arises from repeated folding, and has all partial numerators equal to 3 and all partial denominators equal to ± 1. We prove that its full sequence of convergents converges in the reals and then record an explicit formula for the real and 3-adic limits, both of which are transcendental.

Proposition 6.1. *Let $C_0 = 1^7$, where 1^7 denotes the word $(1, 1, 1, 1, 1, 1, 1)$, and define finite words recursively by $C_{m+1} = C_m, -1, -\overleftarrow{C_m}$ for $m \geq 0$, where $\overleftarrow{C_m}$ denotes the reversal of C_m and the minus sign changes all signs in the word. Since each C_m is a prefix of C_{m+1}, let C_∞ be the unique infinite word having C_m as a prefix for every m. Consider the balanced Schneider's 3-adic continued fraction whose initial partial denominator is 1, whose subsequent partial denominators are the letters of C_∞, and whose partial numerators are all equal to 3. Then all its convergents converge in $\mathbb{R}$.*

Proof. We first record a continuant estimate for finite blocks of partial denominators, all partial numerators being equal to 3. For a finite word $B = (b_1, \ldots, b_\ell)$ with $b_i \in \{-1, 1\}$, define the 3-continuant by $K(\emptyset) = 1$, $K(b_1) = b_1$, and, for $\ell \geq 2$, by

$$K(b_1, \ldots, b_\ell) = b_1 K(b_2, \ldots, b_\ell) + 3K(b_3, \ldots, b_\ell).$$

Thus, if Q_j denotes the denominator of the j-th convergent whose partial denominator sequence is $1, b_1, \ldots, b_j$ and whose partial numerators are all equal to 3, then $Q_j = K(b_j, \ldots, b_1)$. We shall also use the elementary symmetry $K(b_1, \ldots, b_\ell) = K(b_\ell, \ldots, b_1)$, which follows from the same recurrence. For $\ell \geq 1$ put $R(b_1, \ldots, b_\ell) = K(b_2, \ldots, b_\ell)/K(b_1, \ldots, b_\ell)$, with the convention $K(\emptyset) = 1$. Thus $R(1) = 1$ and $R(-1) = -1$.

Let $I_+ = [1/4, 4/7]$ and $I_- = [-4/7, -1/4]$. For $\varepsilon \in \{-1, 1\}$ set $\phi_\varepsilon(x) = 1/(\varepsilon + 3x)$. Then $R(b_1, \ldots, b_\ell) = \phi_{b_1}(R(b_2, \ldots, b_\ell))$. Since $\phi_+(1/4) = 4/7$ and $\phi_+(4/7) = 7/19$, we have $\phi_+(I_+) \subset I_+$. The identity $\phi_-(x) = -\phi_+(-x)$ gives $\phi_-(I_-) \subset I_-$. Additionally, $\phi_+(1) = 1/4 \in I_+$ and $\phi_-(-1) = -1/4 \in I_-$.

We shall also use a whole block of seven equal signs. A direct calculation gives $\phi_+^7(x) = (120x + 97)/(291x + 217)$. On I_- the denominator is positive, since $291x + 217 \geq 291(-4/7) + 217 = 355/7 > 0$. Hence

$$\phi_+^7(I_-) = \left[\frac{268}{577}, \frac{199}{355}\right] \subset I_+, \qquad \phi_+^7(-1) = \frac{23}{74} \in I_+.$$

Thus $\phi_+^7(I_- \cup \{-1\}) \subset I_+$. By symmetry, $\phi_-^7(I_+ \cup \{1\}) \subset I_-$. Since I_+ and I_- are invariant under further applications of ϕ_+ and ϕ_-, respectively, it follows that, for every $r \geq 7$,

$$\phi_+^r(I_- \cup \{-1\}) \subset I_+, \qquad \phi_-^r(I_+ \cup \{1\}) \subset I_-.$$

Consequently, if a word B has length at least 2, is written as alternating blocks of 1's and -1's, and all complete blocks except possibly the last one have length at least 7, then $R(B) \in I_+$ if the first block of B is positive, and $R(B) \in I_-$ if the first block of B is negative. Indeed, one reads the word from right to left. The last, possibly incomplete, block either has length 1, giving the initial value 1 or -1, or has length at least 2, in which case the corresponding value already lies in I_+ or I_-. Each preceding complete block has length at least 7, and the inclusions above apply.

We now return to the words C_m. Their lengths are $n_m := |C_m| = 2^{m+3} - 1$. For $m \geq 1$, the word C_m begins with a positive block, ends with a negative block, and all its blocks have length at least 7. This follows by induction from $C_{m+1} = C_m, -1, -\overleftarrow{C_m}$: the assertion is clear for $C_1 = 1^7(-1)^8$, and in the induction step the inserted -1 extends the final negative block of C_m, while $-\overleftarrow{C_m}$ begins with a positive block.

Let $T_m = P_{n_m}/Q_{n_m}$ be the convergent whose partial denominator sequence is 1 followed by the word C_m, all partial numerators being equal to 3. Lemma 2.1 gives $T_{m+1} = T_m + 3^{n_m+1}/Q_{n_m}^2$. Put $\delta_m = T_{m+1} - T_m$. Then $\delta_m = 3^{n_m+1}/Q_{n_m}^2$. Moreover, $Q_{n_{m+1}}^2 = Q_{n_m}^4$ and $n_{m+1} + 1 = 2(n_m + 1)$, so $\delta_{m+1} = \delta_m^2$. For the initial word $C_0 = 1^7$ one has $Q_{n_0} = Q_7 = 217$, and hence $\delta_0 = 3^8/217^2 = 6561/47089 < 1$. It follows that the subsequence (T_m) converges in $\mathbb{R}$, and we denote its limit by x_∞.

It remains to control the convergents obtained while passing from T_m to T_{m+1}. Fix $m \geq 1$. The new part added after C_m is $w_1, w_2, \ldots, w_{n_m+1}$, where $w_1 = -1$ and $(w_2, \ldots, w_{n_m+1}) = -\overleftarrow{C_m}$. Let $D_{m,t}$ be the convergent obtained after appending the prefix $w_1, \ldots, w_t$ to C_m.

We claim that

$$D_{m,t} = \frac{K(w_1, \ldots, w_t)P_{n_m} + 3K(w_2, \ldots, w_t)P_{n_m-1}}{K(w_1, \ldots, w_t)Q_{n_m} + 3K(w_2, \ldots, w_t)Q_{n_m-1}}.$$

Put $n = n_m$. After appending $w_1, \ldots, w_t$, the convergent recurrences are

$$P_{n+s} = w_s P_{n+s-1} + 3P_{n+s-2}, \qquad Q_{n+s} = w_s Q_{n+s-1} + 3Q_{n+s-2}$$

for $1 \leq s \leq t$. The corresponding identities for P_{n+t} and Q_{n+t} are checked directly for $t = 1$ and $t = 2$. If these identities hold for $t-1$ and $t-2$, then the recurrence for P_{n+t}, together with the symmetry of the continuants, gives

$$P_{n+t} = K(w_1, \ldots, w_t)P_n + 3K(w_2, \ldots, w_t)P_{n-1}.$$

The proof for Q_{n+t} is identical, and the claim follows.

Now put

$$u_t = K(w_1, \ldots, w_t), \qquad v_t = 3K(w_2, \ldots, w_t),$$

where for $t = 1$ the second formula means $v_1 = 3K(\emptyset) = 3$. Then the preceding formula becomes

$$D_{m,t} = \frac{u_t P_{n_m} + v_t P_{n_m-1}}{u_t Q_{n_m} + v_t Q_{n_m-1}}.$$

We shall use that $v_t \neq 0$. Indeed, $v_1 = 3$, and $v_2 = 3$ since $w_2 = 1$. For $t \geq 3$, the word $(w_2, \ldots, w_t)$ is a prefix of $-\overleftarrow{C_m}$ of length at least 2; it begins with a positive block, and all its complete blocks except possibly the last one have length at least 7. The continuant estimate above gives $R(w_2, \ldots, w_t) \in I_+$, so $K(w_2, \ldots, w_t) \neq 0$. Thus $v_t \neq 0$.

Using identity (3) in the form $|P_{n_m}Q_{n_m-1} - P_{n_m-1}Q_{n_m}| = 3^{n_m}$, we obtain

$$|D_{m,t} - T_m| = \frac{|v_t| 3^{n_m}}{|Q_{n_m}|\,|u_t Q_{n_m} + v_t Q_{n_m-1}|}.$$

Set $\rho_m = Q_{n_m-1}/Q_{n_m}$. Comparing with the formula for δ_m, we get

$$\frac{|D_{m,t} - T_m|}{\delta_m} = \frac{1}{3\,|u_t/v_t + \rho_m|}. \tag{12}$$

We now estimate the two quantities in the denominator. For $t = 1$ one has $-u_1/v_1 = 1/3$. For $t = 2$, since $w_1 = -1$ and $w_2 = 1$, one has $-u_2/v_2 = -2/3$. For $t \geq 3$, using $w_1 = -1$, we have $K(w_1, \ldots, w_t) = -K(w_2, \ldots, w_t) + 3K(w_3, \ldots, w_t)$, and therefore

$$-\frac{u_t}{v_t} = \frac{1}{3} - \frac{K(w_3, \ldots, w_t)}{K(w_2, \ldots, w_t)}.$$

The word $(w_2, \dots, w_t)$ is a prefix of $-\overleftarrow{C_m}$. Since $m \geq 1$, this word begins with a positive block, and all its complete blocks have length at least 7. By the continuant estimate proved above, $K(w_3, \dots, w_t)/K(w_2, \dots, w_t) \in [1/4, 4/7]$. Thus, for $t \geq 3$, $-u_t/v_t \in [-5/21, 1/12]$. Altogether,

$$-\frac{u_t}{v_t} \in \left\{\frac{1}{3}, -\frac{2}{3}\right\} \cup \left[-\frac{5}{21}, \frac{1}{12}\right].$$

Write $C_m = (b_1, \dots, b_{n_m})$. From the denominator recurrence, we have $Q_j = K(b_j, \dots, b_1)$ for $1 \leq j \leq n_m$, and hence $\rho_m = Q_{n_m - 1}/Q_{n_m} = R(\overleftarrow{C_m})$. For $m \geq 1$, the word C_m ends with a negative block; hence $\overleftarrow{C_m}$ begins with a negative block. All its complete blocks have length at least 7. Therefore the continuant estimate gives $\rho_m \in I_- = [-4/7, -1/4]$.

The interval $[-4/7, -1/4]$ is separated from $\{1/3, -2/3\} \cup [-5/21, 1/12]$. The minimal distance is $(-5/21) - (-1/4) = 1/84$. Hence $|u_t/v_t + \rho_m| \geq 1/84$. By (12), every convergent $D_{m,t}$ obtained by appending a nonempty prefix of the new block satisfies $|D_{m,t} - T_m| \leq 28\delta_m$.

Let D_N be an arbitrary convergent of our infinite continued fraction with N sufficiently large. Then, for some $m = m(N)$ tending to infinity with N, either $D_N = T_m$, or $D_N = D_{m,t}$ for some $1 \leq t \leq n_m$. In the first case $|D_N - T_m| = 0$, while in the second case the preceding estimate gives $|D_N - T_m| \leq 28\delta_m$. Hence, in all cases,

$$|D_N - T_m| \leq 28\delta_m.$$

We have

$$|D_N - x_\infty| \leq |D_N - T_m| + |T_m - x_\infty| \leq 28\delta_m + |T_m - x_\infty|.$$

Since $m = m(N) \to \infty$ as $N \to \infty$, and since $\delta_m \to 0$ and $T_m \to x_\infty$ in the usual real absolute value, the right-hand side tends to 0 as $N \to \infty$. Therefore the full sequence of convergents converges to x_∞ in $\mathbb{R}$. □

In the example above the increments $\delta_m = T_{m+1} - T_m$ satisfy $\delta_{m+1} = \delta_m^2$. Since $T_0 = 508/217$ and $\delta_0 = 6561/47089$, the real limit is

$$x_\infty = \frac{508}{217} + \sum_{m \geq 0} \left(\frac{6561}{47089}\right)^{2^m}.$$

The same expression converges in $\mathbb{Q}_3$, since $|6561/47089|_3 = 3^{-8}$, and gives the corresponding 3-adic limit of the folded expansion. Both limits are transcendental. This follows from the lacunary series criterion of Corvaja and Zannier [4, Corollary 5], which is valid for arbitrary valuations and contains the corresponding Mahler-type transcendence results as special cases. The usual real transcendence also follows from Mahler's theorem for the Fredholm series $\sum_{m \geq 0} z^{2^m}$, as stated for example in [6, Theorem 11.3].

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